Generalized Sierpiński Graphs

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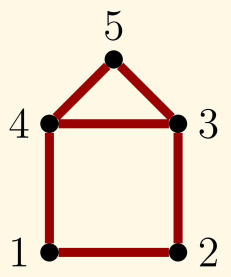
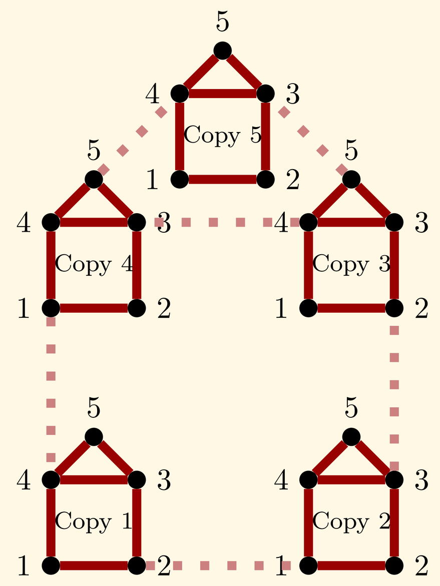
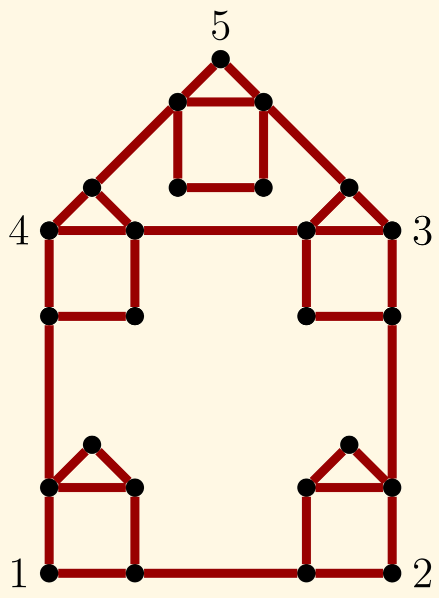
Construction of $S(n,G)$



Start with a labelled graph G.

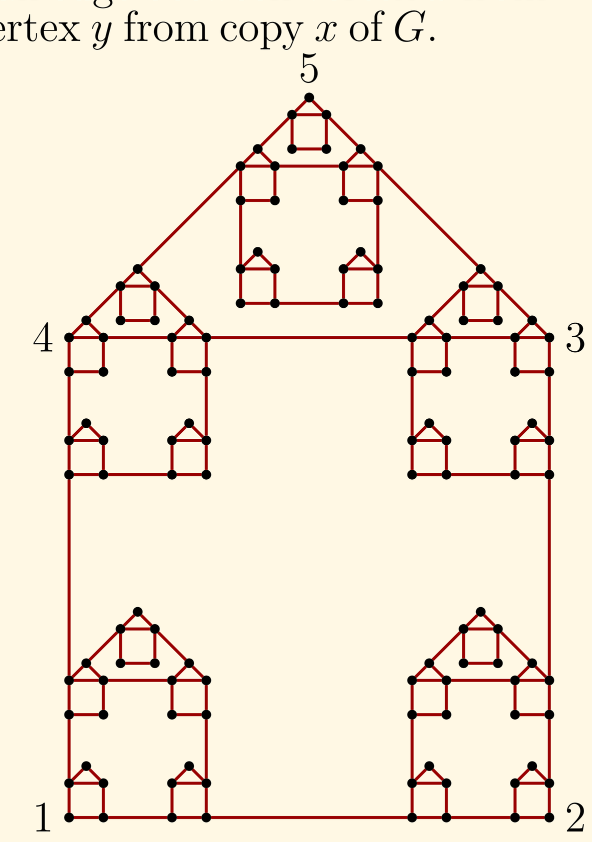


Take $|V(G)|$ copies of G, and for each edge xy of G, add an edge between vertex x from copy y and vertex y from copy x of G.



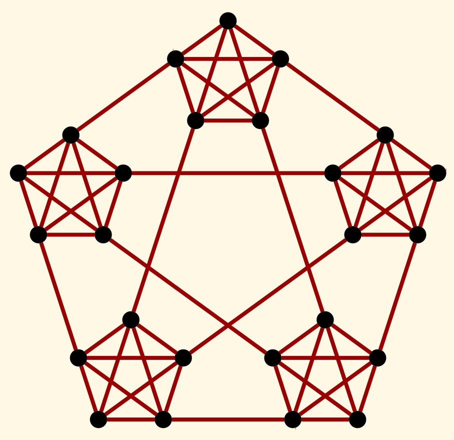
This is $S(2,G)$. New vertex x is vertex x from copy x.

Repeat the operation to obtain $S(3,G)$.

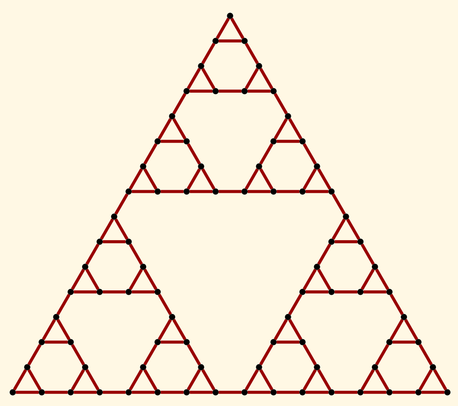


Classical Sierpiński Graphs

When G is a complete graph, $S(n,G)$ is a Sierpiński graph as defined in [1]. If $G=K_3$, it corresponds to the Tower of Hanoï graphs.



$S(2,K_5)$



$S(4,K_3)$ is the graph of Tower of Hanoï with 4 disks

A graph on words

$S(n,G)$ can be seen as a graph on vertex set $\{1,\dots,k\}^n$, with $k=|V(G)|$. Then $\{u,v\}$ is an edge of $S(n,G)$ if there is an edge $\{x,y\}$ of G such that:

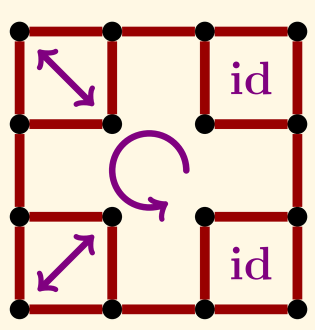
$u=$	w	x	$y\dots y$
$v=$	w	y	$x\dots x$

The graph induced by vertices starting by w is isomorphic to $S(n-|w|,G)$. For $x\in V(G)$, vertex with label $x\dots x$ is called *extreme vertex x*.

Automorphism group

Let $\tau\in Aut(S(n,G))$. Each copy of $S(n-1,G)$ is sent by τ to a copy of $S(n-1,G)$. τ can be decomposed to:

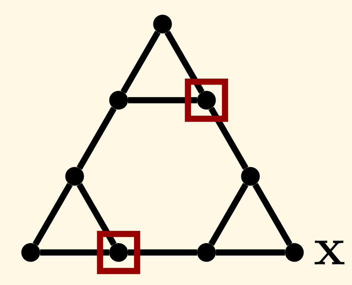
- one automorphism of G,
- k automorphisms of $S(n-1,G)$, one for each copy, that fix the vertices connected to the other copies.



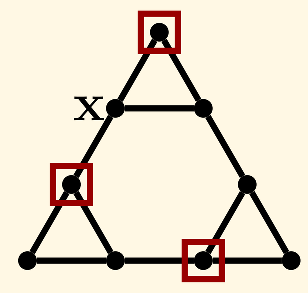
$|Aut(S(2,\mathcal{C}_4))| = 8\times 2^4$

Perfect code

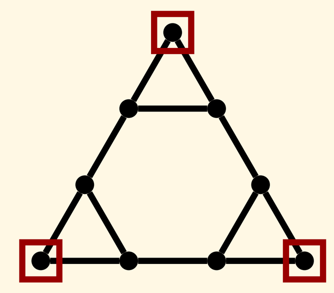
A *perfect code* is a subset of vertices that is both a *dominating set* (every vertex is dominated) and a *packing* (no vertex is dominated twice).



Packing but not perfect



Dominating but not perfect



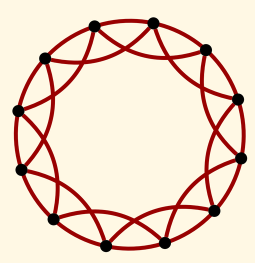
Perfect

For which graph G and which n, has $S(n,G)$ a perfect code ?

Case 1: G has no perfect code

General Case:
If G has no perfect code, the following are equivalent:

(i) $\exists n>1$ s.t. $S(n,G)$ has a perfect code,
(ii) $\forall n>1$, $S(n,G)$ has a perfect code,
(iii) $S(2,G)$ has a perfect code,
(iv) there is an oriented 2-factor in G, s.t. for each vertex x, with outgoing neighbor $s(x)$ and ingoing neighbor $p(x)$, there is a packing of G containing $s(x)$ where $p(x)$ is the only vertex not dominated.



$\mathcal{C}_{12}^2$

Power of cycles:
If $k\not\equiv 0 \mod (2r+1)$, there is a perfect code in $S(2,\mathcal{C}_k^r)$ if and only if $k\equiv 1 \mod (2r+1)$ and r is odd.

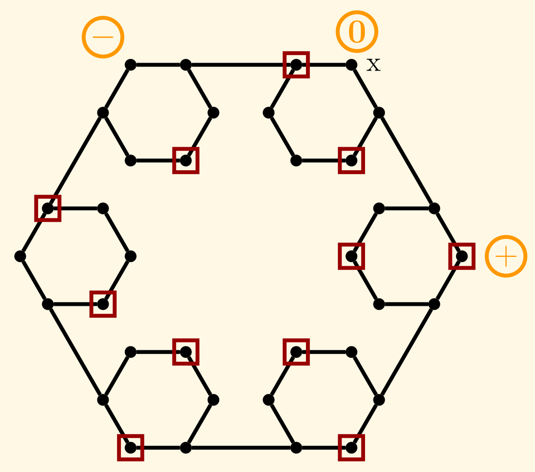
Case 2: G has a perfect code

A *weak perfect code within x* of $S(n,G)$ is a packing of $S(n,G)$ such that the only vertices not dominated are extreme vertices y with $d_G(x,y)\le 1$.

We give marks to the extreme vertices:

- ⊕ : vertex in the code
- ⊖ : vertex not dominated
- ⓪ : vertex not in the code but dominated

A weak code is characterized by the marks of the extreme vertices.



A weak perfect code of $S(2,\mathcal{C}_6)$ within x

By combining the weak perfect codes of $S(n,G)$ we obtain the weak perfect codes of $S(n+1,G)$. If $S(n,G)$ has no weak perfect code, $S(n',G)$ has no perfect codes for $n'\ge n$.

If $k\equiv 0 \mod 2k+1$, there is a perfect code in $S(n,\mathcal{C}_k^r)$ if and only if:

- $r=1$ or $r\ge\frac{k-1}{2}$, or
- $k\equiv 0 \mod (2r+1)$ and $n=2$.

References

[1] S. Klavžar and U. Milutinović, Graphs $S(n,k)$ and a variant of the Tower of Hanoï problem, *Czechoslovak Math. J.* 47(122) (1997) 95-104.
[2] S. Klavžar, U. Milutinović and C. Petr, 1-Perfect codes in Sierpiński graphs, *Bull. Austral. Math. Soc.*, 66 (2002), 369-384.

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