Spanning Trees in Grid Graphs With Fixed Leaves

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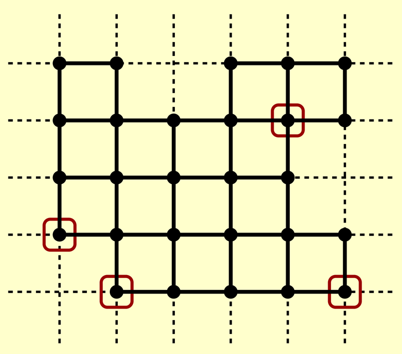
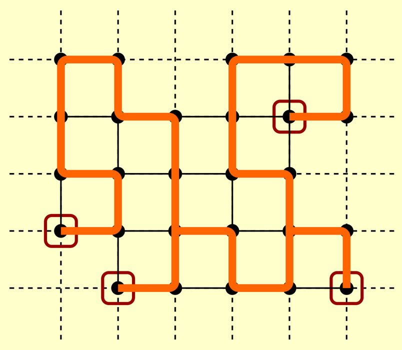
What is the problem ?

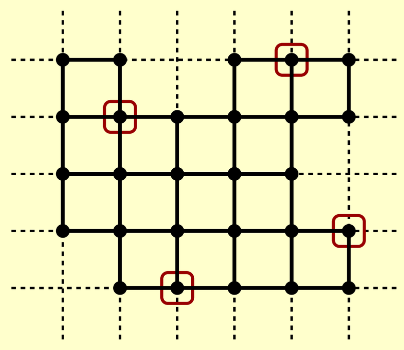


This is a grid graph G with a set L of marked vertices.

Is there a spanning tree of G whose set of leaves is exactly L ?

Yes !





Is it always the case ?

Basic case

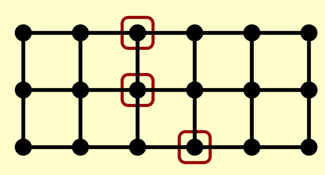
Itaï,Papadimitriou and Szwarcfiter study in [1] the case of $|L| = 2$, they show that it is :

- NP-complete for general grid graphs,
- Polynomial for rectangular grid graphs.

Is there some easy conditions ?

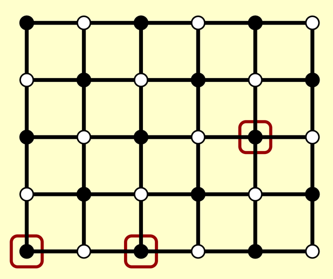
Connectivity condition

If $G \setminus L$ is not connected there is no solution.



No solution !

Color condition



No solution !

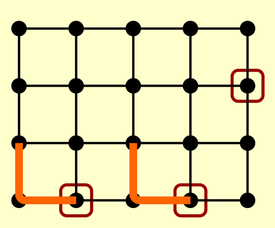
One can color a grid graph like a checkerboard. This implies a *color condition.*

In rectangular grid graphs, it is :

- if $|G|$ is even, L must have vertices of both colors,
- if $|G|$ is odd, L must have at least 2 vertices of the main color.

Freedom condition

If $G \setminus L$ has too many vertices of degree 1, then (G, L) has no solution.

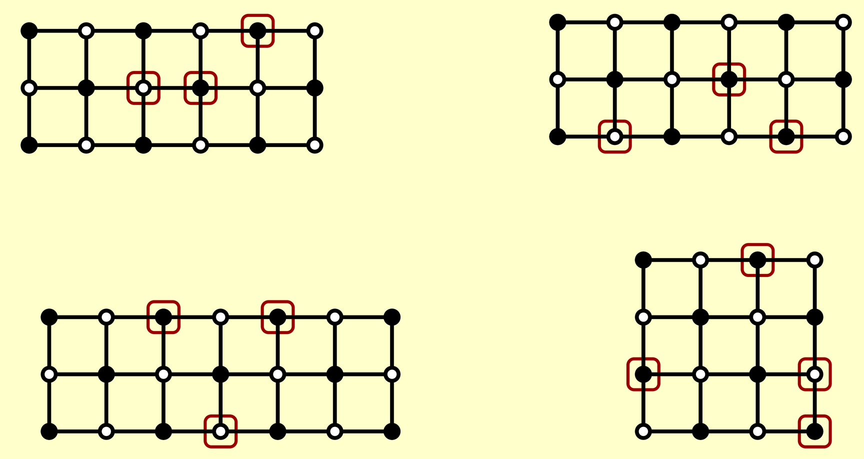


No solution !

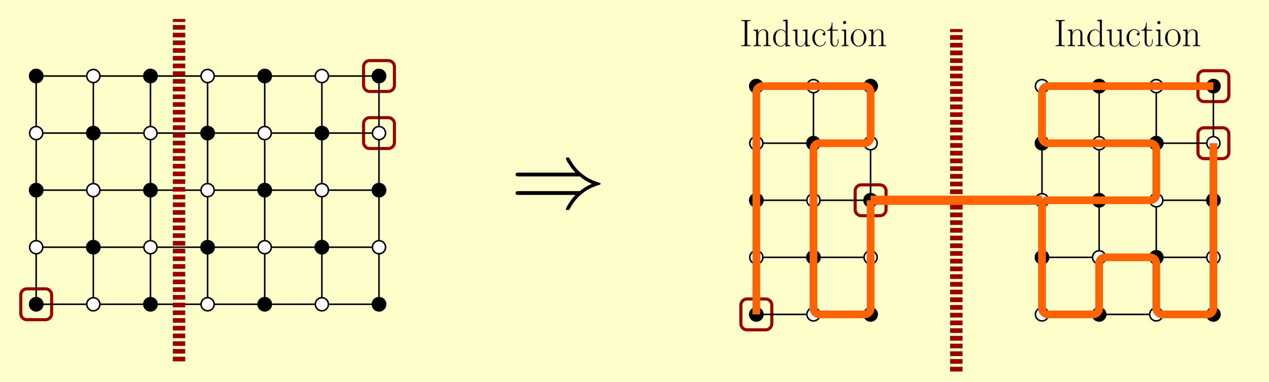
Forbidden cases

The previous conditions are not sufficient !

These instances all satisfy the three conditions but have no solution, they are *forbidden cases.*



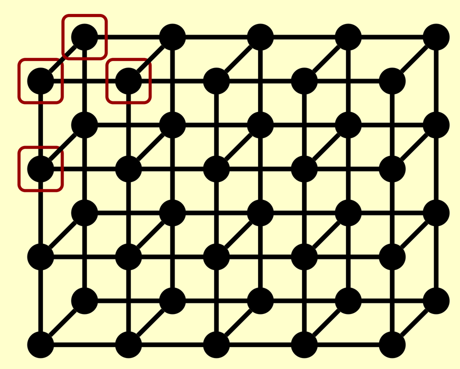
Division lemma and result for 3 vertices



Theorem : If $|L| = 3$ and G is a rectangular grid graph, (G, L) has a solution iff the three conditions are satisfied and if it does not belong to a polynomial class of forbidden cases.

What happens in higher dimension ?

Theorem : Let G be a rectangular grid graph of dimension d and L be a set of at most d vertices of G. G has a spanning tree with set of leaves L iff color condition is satisfied.



This condition is sharp as demonstrates the figure on the left

Future work

- In dimension 2, division methods are too tedious when L is too big and so, the problem is still open for $|L| \geq 4$,
- In dimension d, we don't know anything if there is more than d vertices,
- What about the number of spanning trees whose set of leaves is fixed ?

References

[1] A. Itai, C.H. Papadimitriou, and J.L. Szwarcfiter. Hamilton paths in grid graphs. *SIAM Journal on Computing.* Vol. 11(4).1982.

[2] A. Parreau. Arbres couvrants dans la grille. Mémoire de stage de recherche. M2 ROCO. 2009

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