Locally identifying colorings of graphs

Aline Parreau

Joined work with:
Louis Esperet, Sylvain Gravier, Mickaël Montassier, Pascal Ochem

and:
Florent Foucaud, Iiro Honkala, Tero Laihonen, Guillem Perarnau

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Outline

Identification in graphs

$G = (V, E)$ graph. How to identify the vertices of G ?

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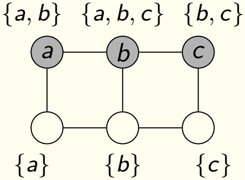

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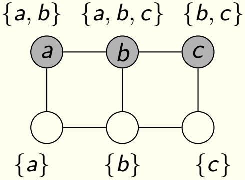

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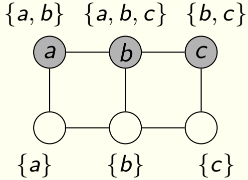

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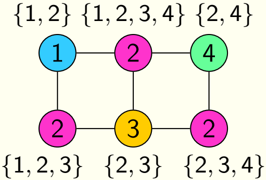

Coloring $c : V \rightarrow \mathbb{N}$
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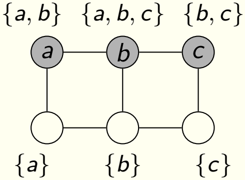

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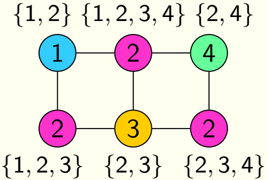

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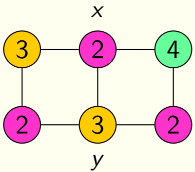
$\rightarrow$ Locally Identifying coloring: only adjacent vertices are separated.

Locally identifying coloring (lid-coloring)

- $c : V \to \mathbb{N}$ proper coloring: $xy \in E \Rightarrow c(x) \neq c(y)$
- $c(S)$: set of colors in S: $c(S) = \{c(x), x \in S\}$
- color c_0 separates x and y if $c_0 \in c(N[x]) \Delta c(N[y])$



x and y are separated by color 4

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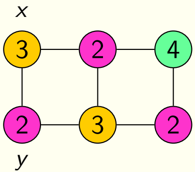

x and y are not separated:
$c(N[x]) = c(N[y])$

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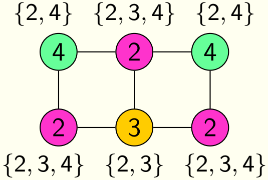

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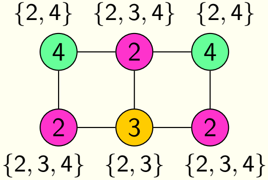

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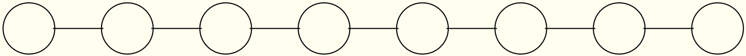
- $\chi_{lid}(G)$: min. number of colors needed in a lid-coloring G.

Coloring the path

With 4 colors :



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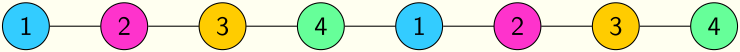

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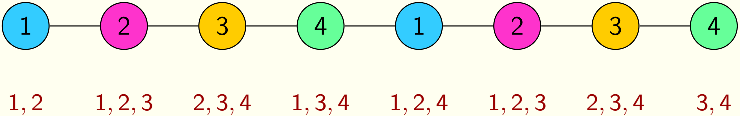

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$$\chi_{lid}(P_k) \leq 4$$

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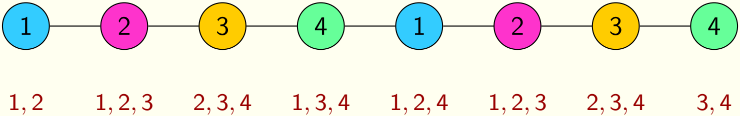

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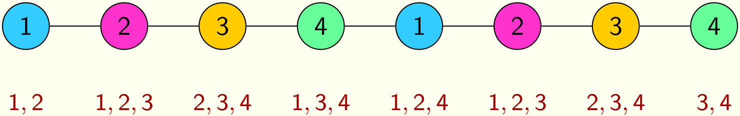

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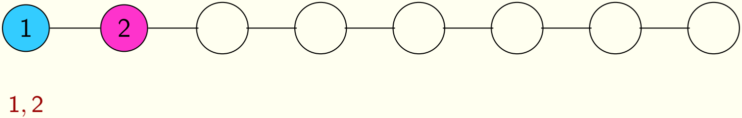

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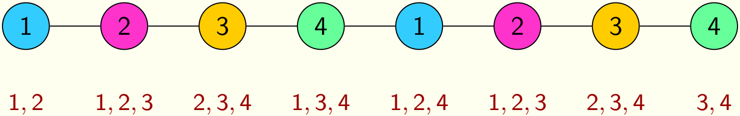

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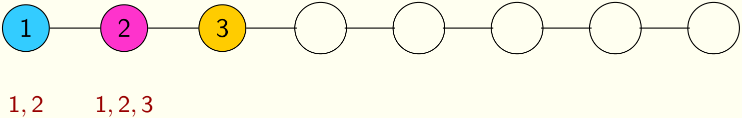

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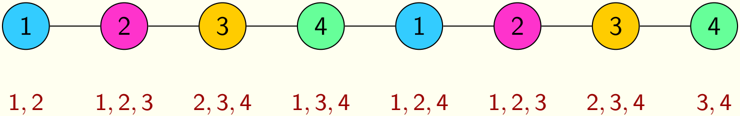

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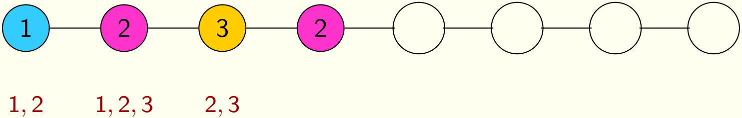

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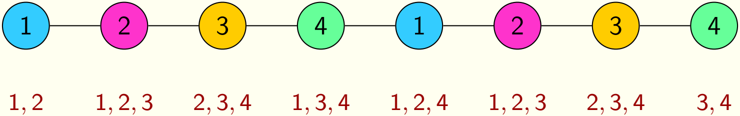

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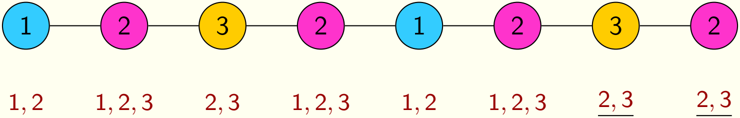

$$\chi_{lid}(P_k) = 3 \Leftrightarrow k \text{ is odd}$$

Related works

With edge colorings:

- Vertex-distinguishing edge colorings (Observability of a graph) (Hornak et al, 95'),
- Adjacent vertex-distinguishing edge colorings (Zhang et al, 02')

With total colorings:

- Adjacent vertex-distinguishing total colorings (Zhang, 05')

What are we doing next?

Remarks:

- Refinment of proper colorings: $\chi(G) \leq \chi_{lid}(G)$
- χ_{lid} is not heriditary: $\chi_{lid}(P_4) \geq \chi_{lid}(P_5)$

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Upperbound using χ?

Link with maximum degree?

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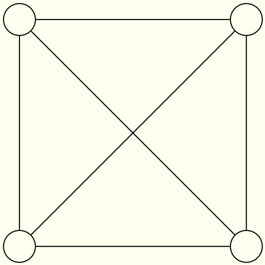
In perfect graphs?

In planar graphs?

Upperbound using chromatic number χ?

Def: $\forall xy \in E$, $c(x) \neq c(y)$ and $c(N[x]) \neq c(N[y])$



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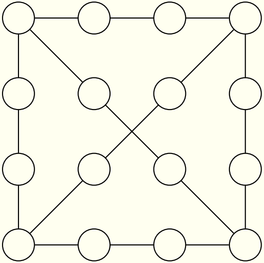

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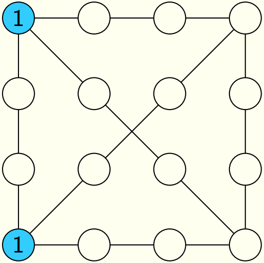

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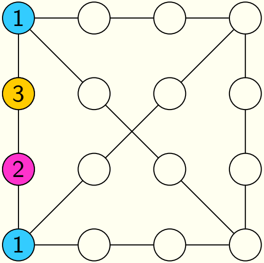

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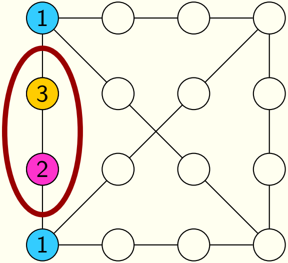

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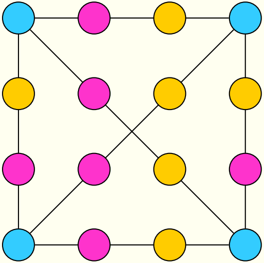

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$\rightarrow$ $\chi_{lid}(G_k) \geq k$

$\rightarrow$ But $\chi(G_k) \leq 3$

For each k, there exists graph G_k with $\chi(G_k) \leq 3$ and $\chi_{lid}(G_k) \geq k$

No upper bound of χ_{lid} using χ !

Upper bound on a graph with n vertices ?

Proper colorings: $\chi(G) = |V(G)| = n \Leftrightarrow G = K_n$

Lid-colorings: for which graphs $\chi_{lid}(G) = |V(G)|$?

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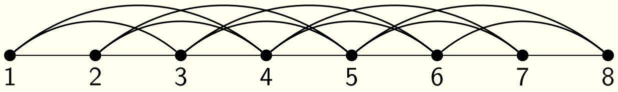

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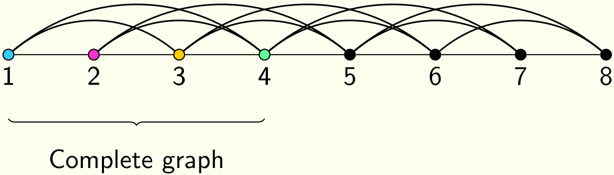

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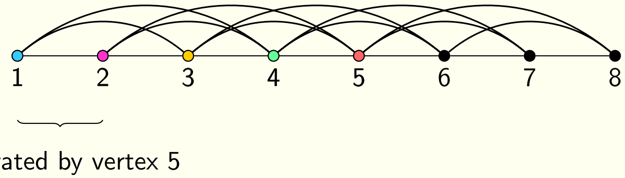

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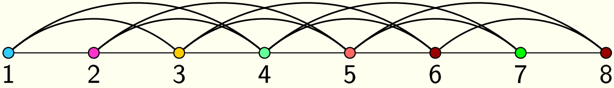

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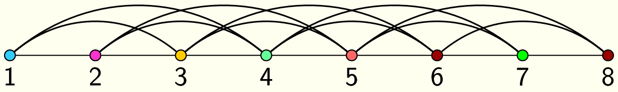

- ... ?

Open question

Caracterize graphs G such that $\chi_{lid}(G) = n$.

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Proper colorings: $\chi(G) \leq \Delta + 1$, tight for odd cycles and complete graphs

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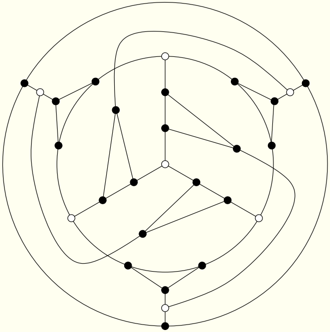
- Graphs with $\chi_{lid}(G) \geq \Delta^2 - \Delta + 1$ using projective plane



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Theorem (Foucaud,Honkala,Laihonen,P.,Perarnau, 2012)

For any graph G with $\Delta \geq 3$: $\chi_{lid}(G) \leq 2\Delta^2 - 3\Delta + 3$

Open question

Do we have $\chi_{lid}(G) \leq \Delta^2 + O(\Delta)$?

Idea of the proof

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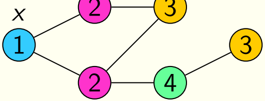

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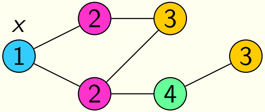

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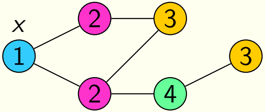

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Recoloring Lemma: *there are at most* $2d(x)(\Delta - 1)$ *forbidden colors for* x

- Proof by induction
- When we add a new vertex v, change $c(N(v))$ in a good way with recoloring lemma.
- Give a completetely new colour to v, c remains lid-coloring.
- Use recoloring lemma on v.

What about good classes for proper coloring of graphs ?

Perfect graphs

- $\omega(G)$: maximum size of a clique of G
- A graph G is perfect if for all subgraphs H of G, $\chi(H) = \omega(H)$

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Example: Bipartite graphs are perfect, C_5 is not perfect: $\omega(C_5) = 2$, $\chi(C_5) = 3$

Theorem (Chudnovsky, Robertson, Seymour, Thomas, 2002)

G is perfect if and only if it has no induced odd cycle or complement of odd cycle with more than 5 vertices.

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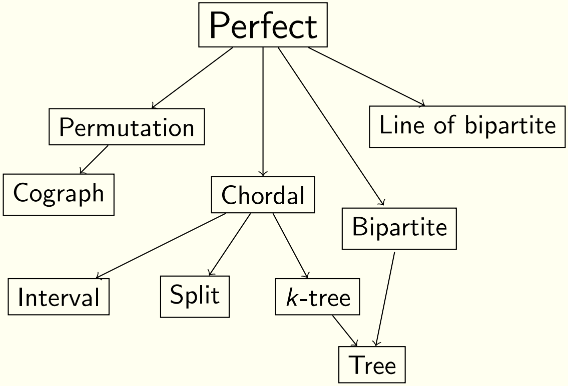

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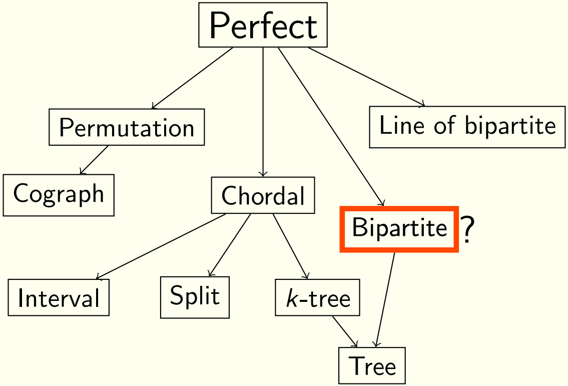

Bipartite graphs

G connected graph:

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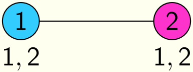
- $\chi_{lid}(G) = 1 \Rightarrow G$ is a single vertex
- $\chi_{lid}(G) = 2 \Rightarrow G$ is just an edge



Bipartite graphs

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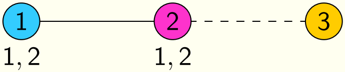

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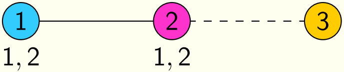
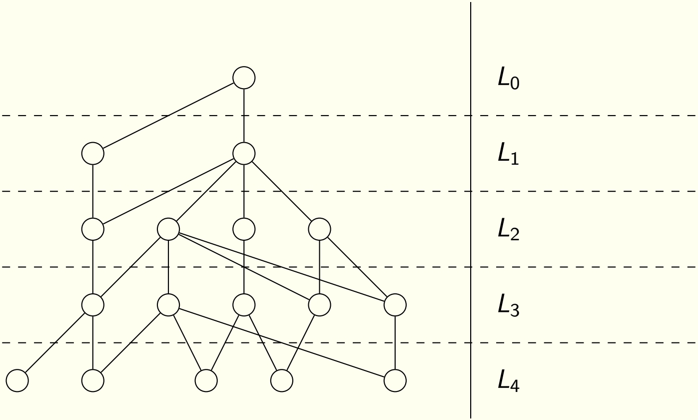


- $\chi_{lid}(G) = 3 \Rightarrow G$ is a triangle or a bipartite graph:
  $\rightarrow$ Partition vertices with the number of colors in $c(N[x])$

Bipartite graphs are 4-lid-colorable



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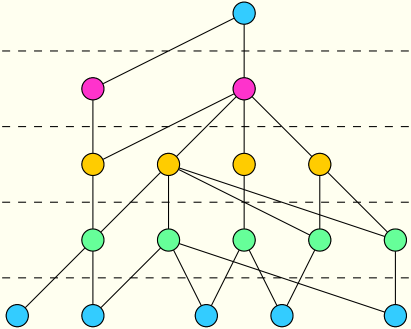

L_0	$\rightarrow$	1	$1, 2$
L_1	$\rightarrow$	2	$1, 2, 3$
L_2	$\rightarrow$	3	$2, 3, 4$ or $2, 3$
L_3	$\rightarrow$	4	$1, 3, 4$ or $3, 4$
L_4	$\rightarrow$	1	$1, 4$

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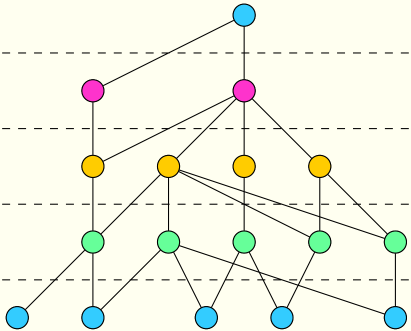

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L_4	$\rightarrow$	1	$1, 4$

Theorem (Esperet, Gravier, Montassier, Ochem, P., 2012)

If G is bipartite, $\chi_{lid}(G) \leq 4$.

Bipartite graphs

General bounds: $3 \leq \chi_{lid}(B) \leq 4$

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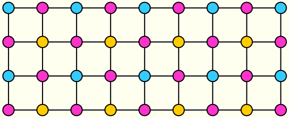

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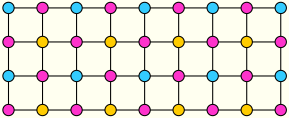

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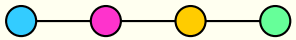

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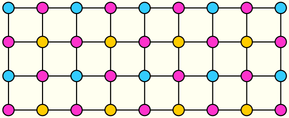

$\leftarrow ? \rightarrow$

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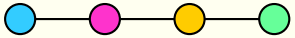
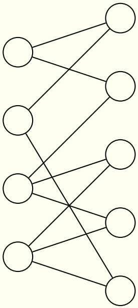


In general... 3-Lid-Coloring is NP-complete in bipartite graphs

Link with 2-coloring of hypergraph

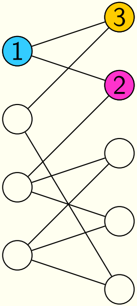
Try to color a graph with 3 colors



Link with 2-coloring of hypergraph

Try to color a graph with 3 colors

$1, 2, 3$



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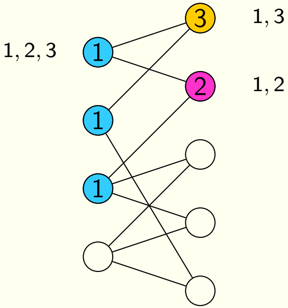

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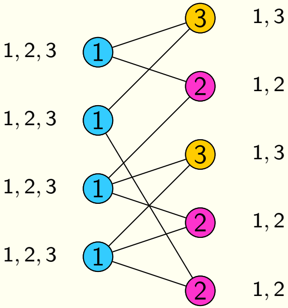

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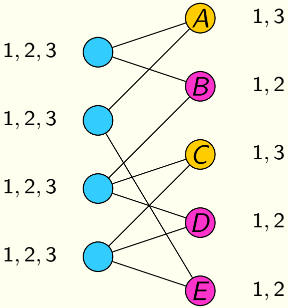
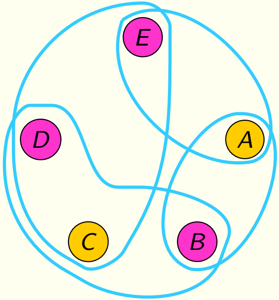
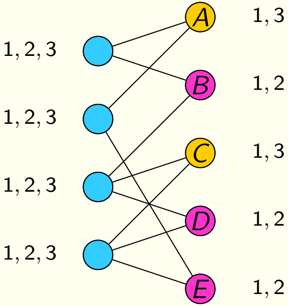
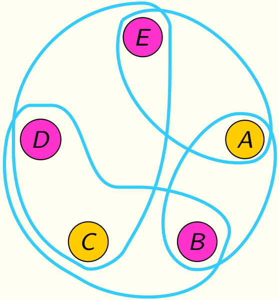




3-lid-coloring in bipartite graph $\Leftrightarrow$ 2-coloring in hypergraph

Link with 2-coloring of hypergraph

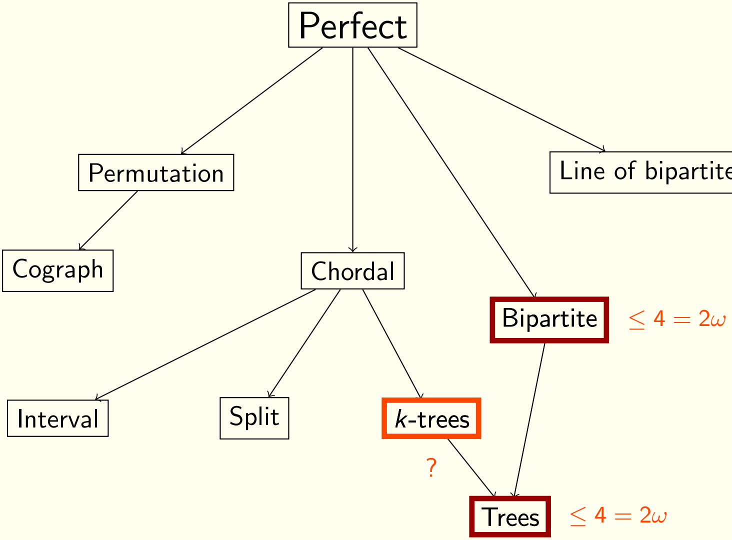




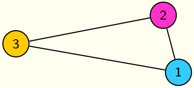
3-lid-coloring in bipartite graph $\Leftrightarrow$ 2-coloring in hypergraph

- 3-Lid-Coloring in bipartite graph is NP-Complete
- Polynomial in regular graphs, in planar graphs with maximum degree 3, in trees.

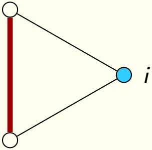
Perfect graphs



Lid-coloring of 2-trees with 6 colors



- Color the triangle with colors $1, 2, 3$
- Step:



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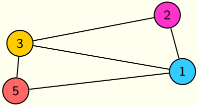

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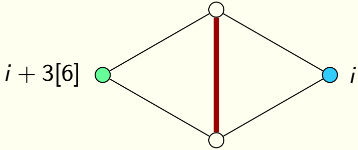

- We always have:
  - proper coloring
  - no edge $(i, i + 3)$

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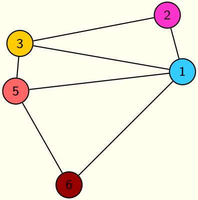

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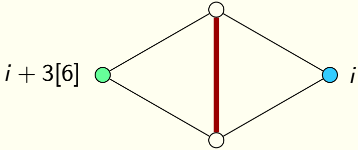

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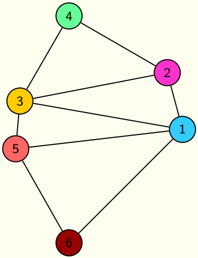

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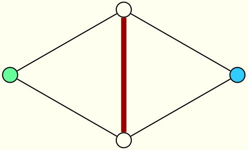

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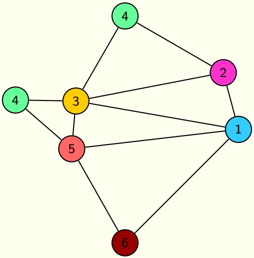

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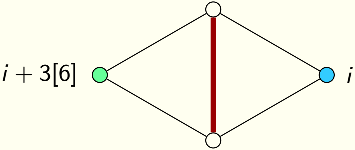

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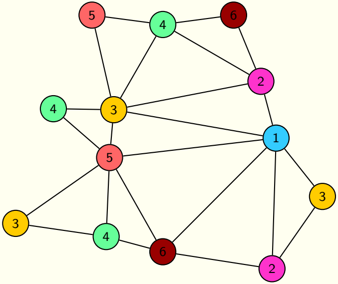

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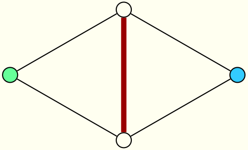

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Theorem (Esperet, Gravier, Montassier, Ochem, P., 2012)

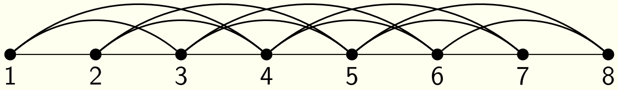
A 2-tree is 6-lid-colorable.

To perfect graph: k-trees

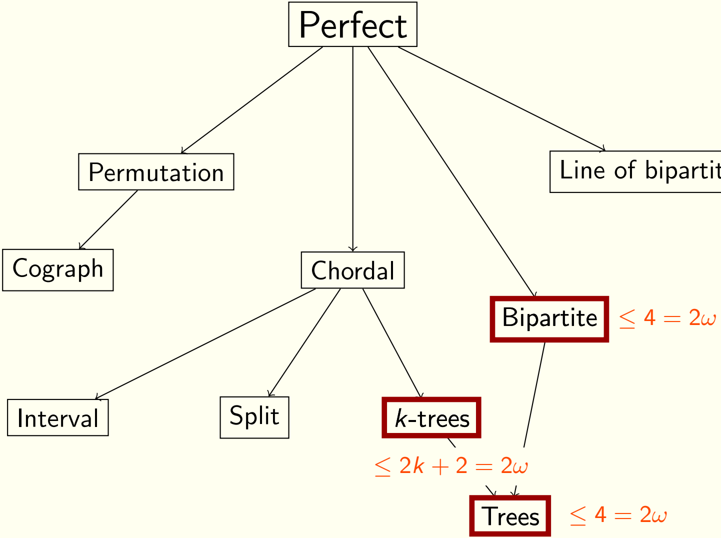
We can extend the construction to k-trees:

$\rightarrow$ A k-tree G has lid-chromatic number at most $2k+2 = 2\omega(G)$

This bound is sharp: P^k_{2k+2} is a $k+1$-tree and $\chi_{lid}(P^k_{2k+2}) = 2k+2$



Perfect graphs



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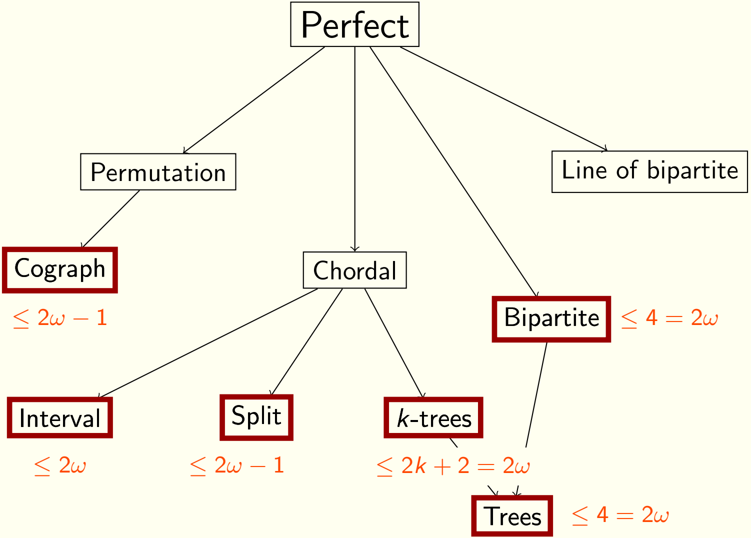

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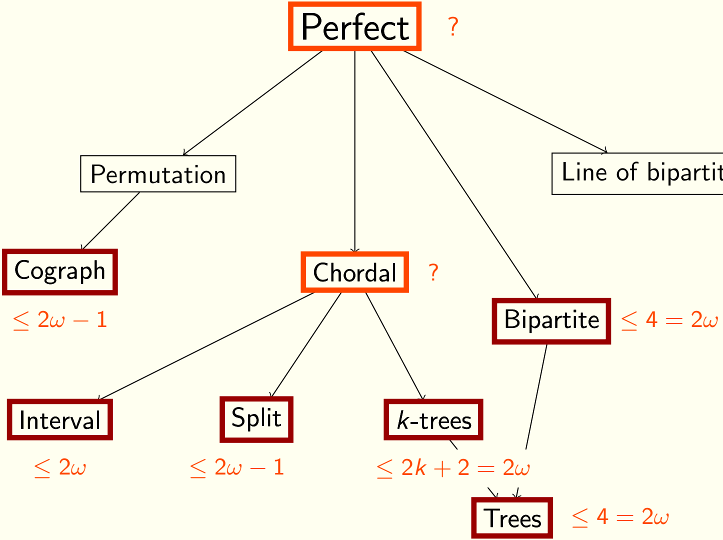

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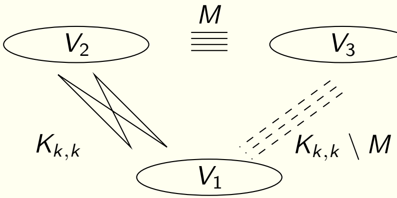

V_1, V_2, V_3 stable sets of size k

$$\chi_{lid} \geq k + 2 \text{ but } \omega = 3$$

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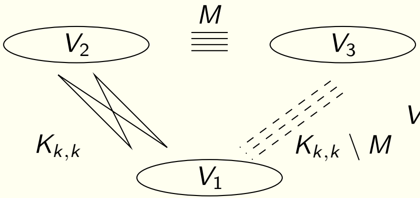


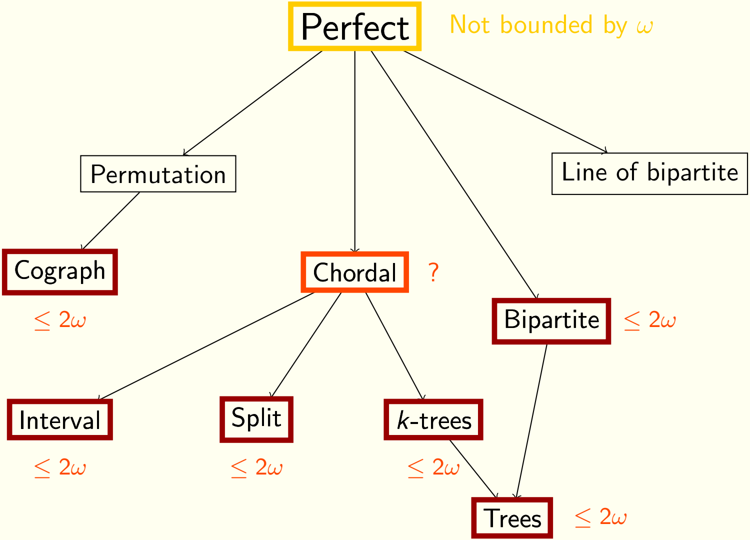
V_1, V_2, V_3 stable sets of size k

$\chi_{lid} \geq k + 2$ but $\omega = 3$

Conjecture

We can color any chordal graph G with $2\omega(G)$ colors

Perfect Graphs



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Chordal graph: constructed like k-trees but the size of the clique can change

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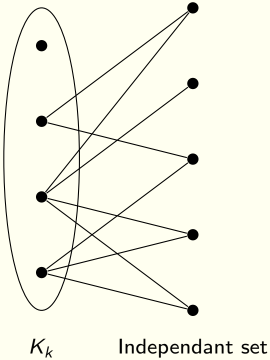

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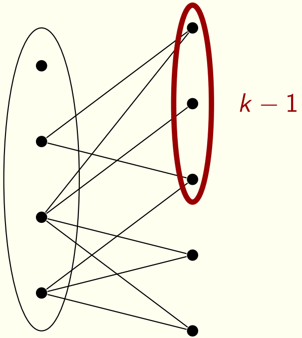
Split graphs:

- Bondy's theorem : $k-1$ vertices of the stable set are enough to separate the clique vertices



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Chordal graph: constructed like k-trees but the size of the clique can change

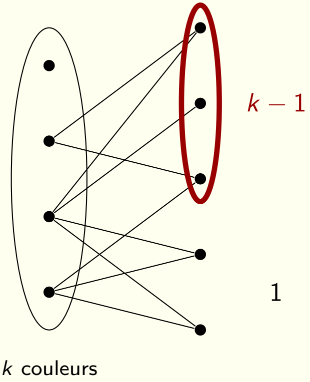
Split graphs:

- Bondy's theorem : $k-1$ vertices of the stable set are enough to separate the clique vertices

$\rightarrow$ We can color with $2k$ colors

$\rightarrow$ Possible with $2k-1$ colors

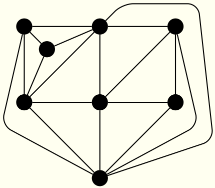
$\rightarrow$ It's sharp



What about planar graphs ?

Planar graphs

Is lid-chromatic number bounded for planar graphs ?



- Worse example : 8 colors,

Planar graphs

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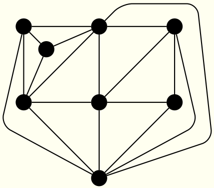

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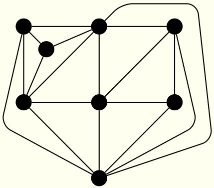

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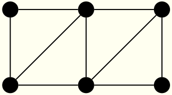
Outerplanar graphs:

- General bound: 20 colors,
- Max outerplanar graphs: ≤ 6 colors,
- Without triangles: ≤ 8 colors,



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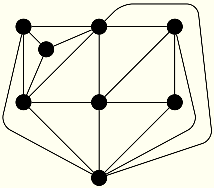

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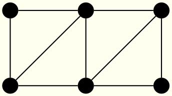
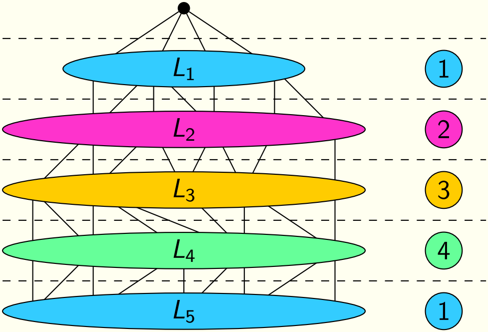


- General bound: 20 colors,
- Max outerplanar graphs: ≤ 6 colors,
- Without triangles: ≤ 8 colors,
- Examples with at most 6 colors

A bound for outerplanar graphs



- a layer = union of paths,
- 5 colors in a layer,
- $4 \times 5 = 20$

Bound for planar graphs ?

Really large bound by Gonzales and Pinlou (2012)

More general result :

Theorem (Gonzales, Pinlou)

Any family of graph closed by minor has lid-chromatic bounded

A remark

- For some subclasses of perfect graphs : $\chi_{lid}(G) \leq 2\omega(G) = 2\chi(G)$
- For planar graphs, worse example : $\chi_{lid}(G) \leq 8 = 2\chi(G)$
- For outerplanar graphs, worse example : $\chi_{lid}(G) \leq 6 = 2\chi(G)$
- ...

Open question

For which graphs do we have $\chi_{lid}(G) \leq 2\chi(G)$?

Another remark

- $\chi_{lid}(G) = 2 \Leftrightarrow G = K_2$
- $\chi_{lid}(G) = 3 \Rightarrow G = K_3$ or G is bipartite
- $\chi_{lid}(G) = 3$ and $\chi(G) = 3 \Leftrightarrow G = K_3$

Open question

Caracterize graphs G such that $\chi_{lid}(G) = \chi(G)$. Are they only the complete graphs ?

Conclusion

Lot of open questions:

- Graphs with $\chi_{lid} = n$?
- Graphs with $\chi_{lid} = \chi$?
- Do we have $\chi_{lid}(G) \leq \Delta^2 + O(\Delta)$?
- Do we have $\chi_{lid}(G) \leq 2\chi(G)$ for chordal graphs? for planar graphs? for which graphs?
- Find a good bound for planar graphs.

Thanks !