Locally identifying colorings of graphs

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Joint work with Louis Esperet, Sylvain Gravier, Mickaël Montassier, Pascal Ochem

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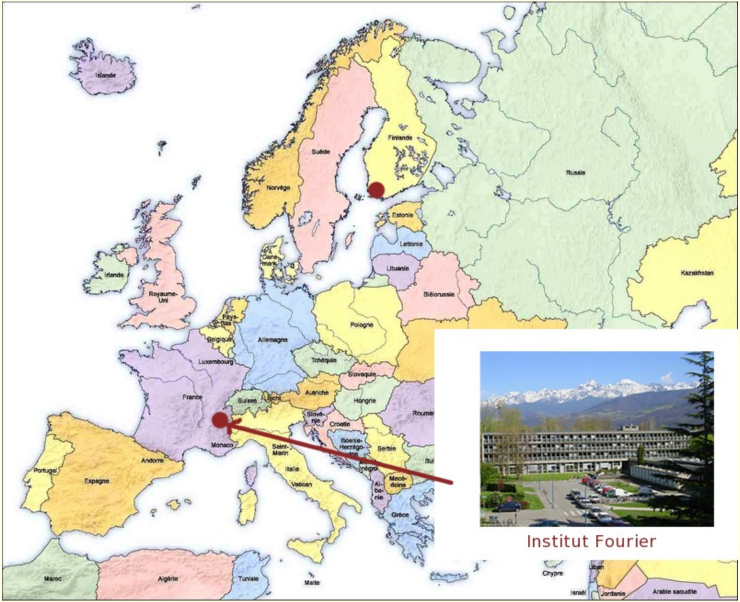
ANR IDEA

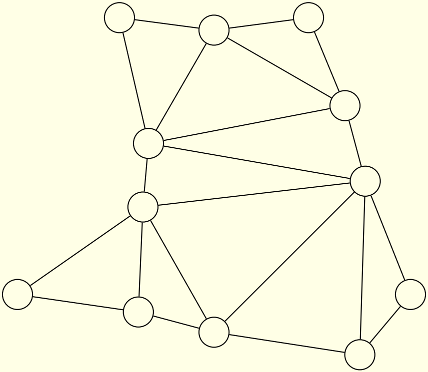
Grenoble-Turku



Outline

Proper coloring

Graph $G = (V, E)$
Proper coloring : $c : V \to \mathbb{N}$ s.t. $uv \in E \Rightarrow c(u) \neq c(v)$



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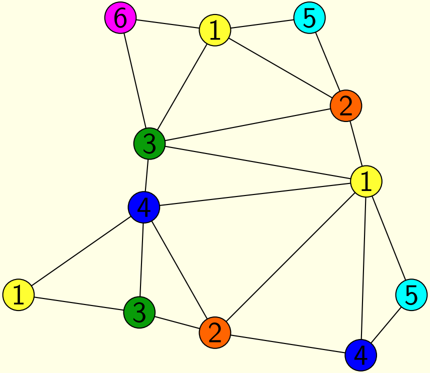

A well-known example

4-Colour Theorem :

But also : frequency allocation, scheduling,...

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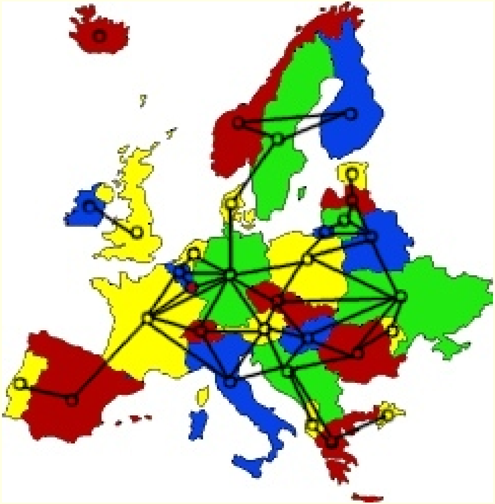

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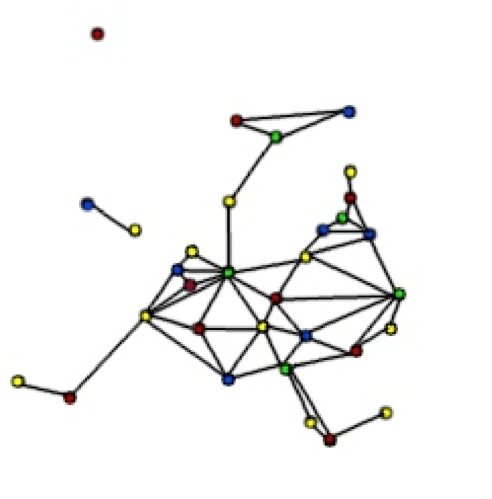
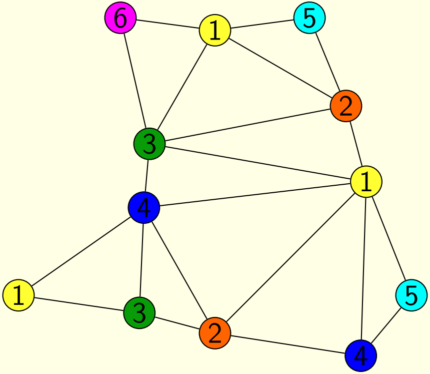


But also : frequency allocation, scheduling,...

Chromatic number

Chromatic number $\chi(G)$: minimum number of colors to have a porper coloring :



$$\chi(G) \leq 6$$

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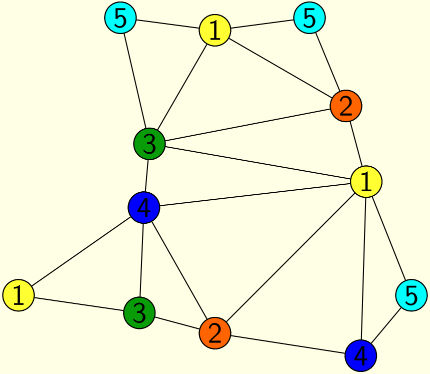

$$\chi(G) \leq 5$$

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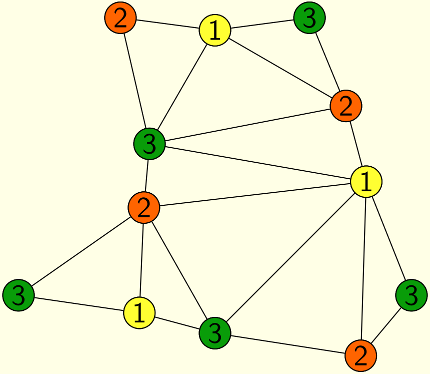
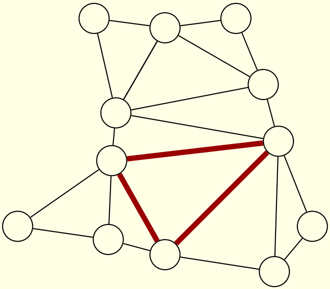


$$\chi(G) = 3$$

A lower bound...

Clique number $\omega(G)$: max k such that there are k vertices in G that are all connected to each other



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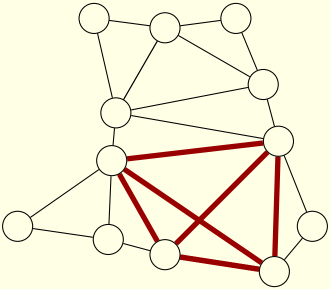

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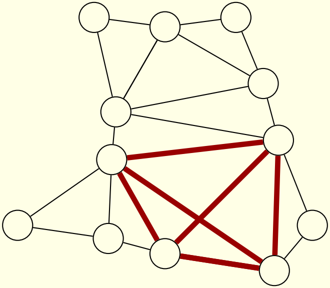
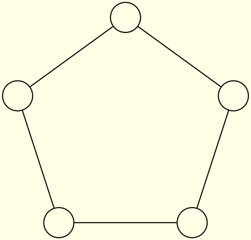


$$\omega(G) = 4$$

For any graph G, $\chi(G) \geq \omega(G)$

... that is not always reached



$\chi(C_5) = 3$ but $\omega(C_5) = 2$

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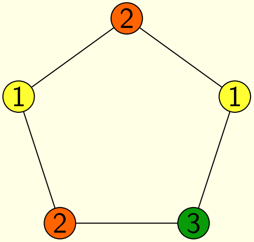

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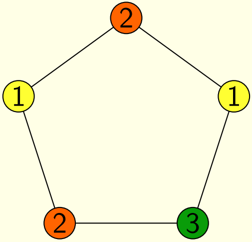


$$\chi(C_5) = 3 \text{ but } \omega(C_5) = 2$$

Mycielski graphs M_k such that $\chi(M_k) = k$ but $\omega(M_k) = 2$

An upper bound

$\Delta(G)$: maximum degree of G

Upper bound with greedy algorithm :

$$\chi(G) \leq \Delta(G) + 1$$

Tight for : complete graphs, odd cycles

Brook's theorem (1941) :

$$\chi(G) \leq \Delta(G) \text{ if } G \text{ is not a complete graph or an odd cycle}$$

It's hard to guess !

k-Coloring is NP-complete for any $k \geq 3$

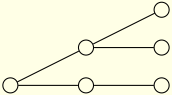
Even for $k = 3$ and planar graphs with maximum degree 4 !

Good graphs for coloring

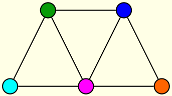
Perfect graph (1963) : G is *perfect* if $\omega(H) = \chi(H)$ for any induced subgraph H of G

Examples :

- trees (and bipartite graphs) :



- interval graphs :



Strong Perfect Graph Theorem

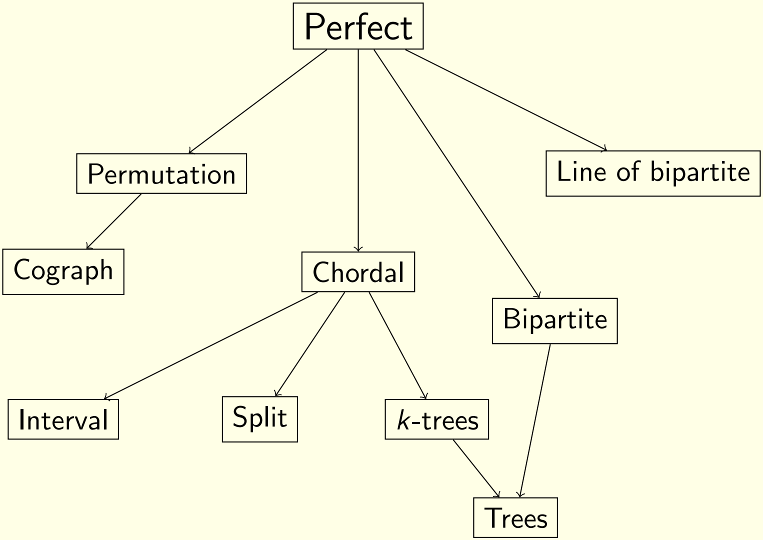
Remark : perfect graphs are stable by induced subgraphs

Smallest non perfect graphs ?
$\rightarrow$ odd cycles of size ≥ 5, complement of odd cycles of size ≥ 5

Strong Perfect Graph Theorem (Chudnovsky, Robertson, Seymour, Thomas 2002) :

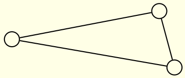
> G is perfect if and only if it has no induced odd cycle or complement of odd cycle with more than 5 vertices

A part of the big family of perfect graphs



k-trees

Construction of 2-trees :



- Start with a triangle

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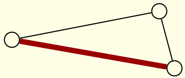

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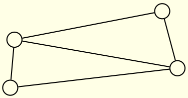

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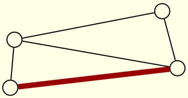

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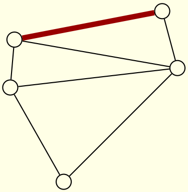

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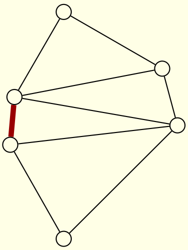

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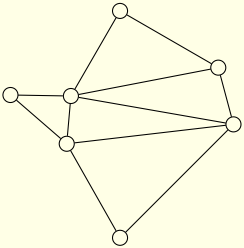

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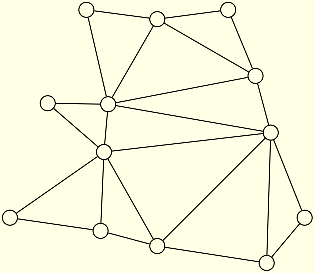

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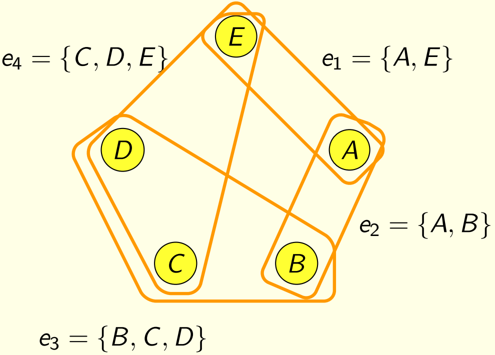
Hypergraph

Hypergraph $\mathcal{H} = (V, \mathcal{E})$ where an edge $e \in \mathcal{E}$ is a subset of vertices



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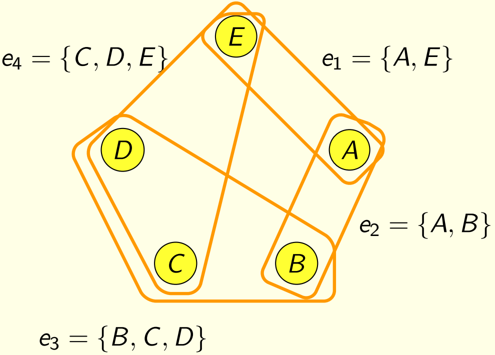

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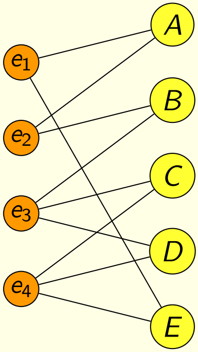
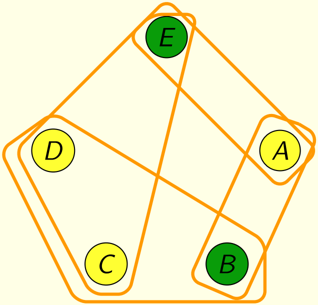


Corresponding bipartite graph

Colorings of Hypergraph

$c : V \to \mathbb{N}$ is a proper coloring of $\mathcal{H}$ if no edge is unicolor.



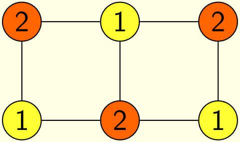
A 2-proper coloring

2-Hypergraph-Coloring is NP-complete

Locally identifying coloring (lid-coloring)

In proper coloring :

Two adjacent vertices have distinct colors .



$B_t(u) = \{v \mid d(u, v) \leq t\}$

For any edge uv, $c(B_0(u)) \neq c(B_0(v))$

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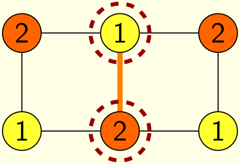

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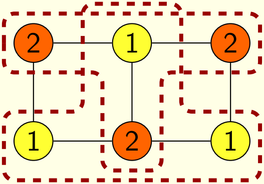

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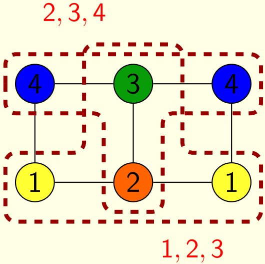

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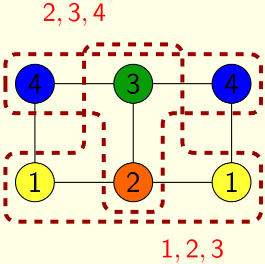

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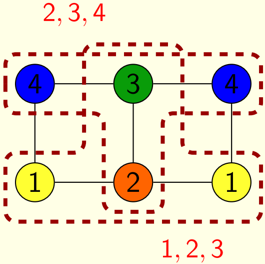

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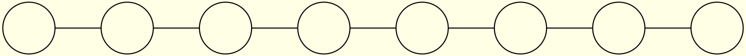
$\chi_{lid}(G)$: lid-chromatic number

An example : the path

With 4 colors :



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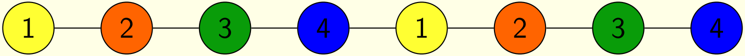

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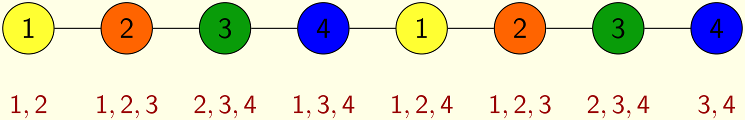

So :

$$\chi_{lid}(P_k) \leq 4$$

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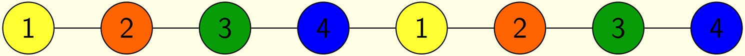

$1, 2$ $\quad$ $1, 2, 3$ $\quad$ $2, 3, 4$ $\quad$ $1, 3, 4$ $\quad$ $1, 2, 4$ $\quad$ $1, 2, 3$ $\quad$ $2, 3, 4$ $\quad$ $3, 4$

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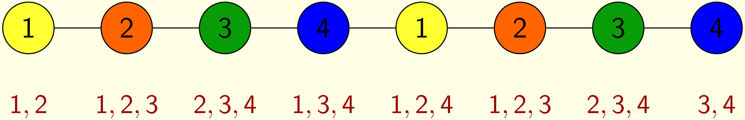

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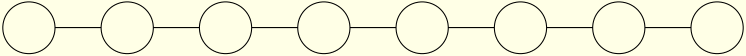

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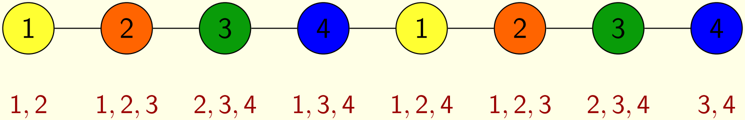

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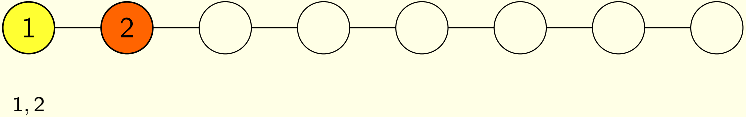

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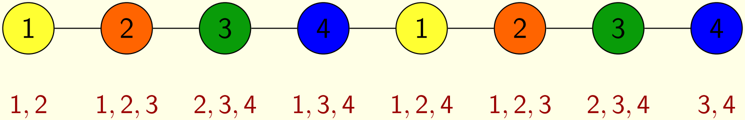

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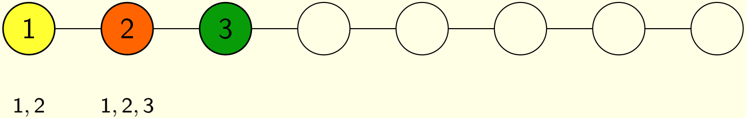

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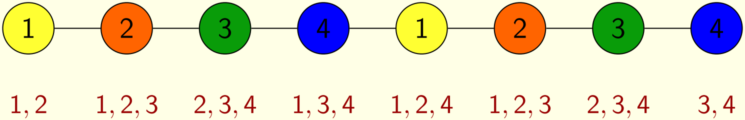

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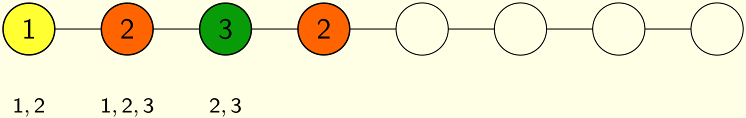

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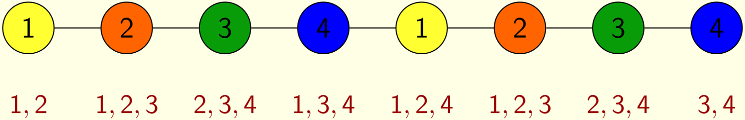

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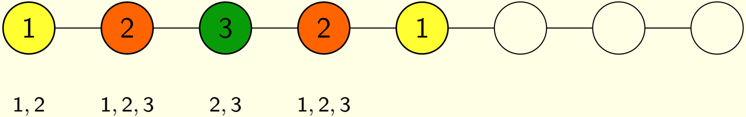

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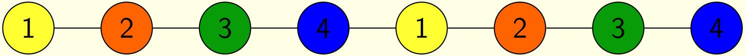

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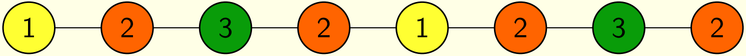

1, 2   1, 2, 3   2, 3   1, 2, 3   1, 2   1, 2, 3 $\underline{2, 3}$ $\underline{2, 3}$

$$\chi_{lid}(P_k) = 3 \Leftrightarrow k \text{ is odd}$$

Related works

With edge colorings :

- Vertex-distinguishing edge colorings (Observability of a graph) (Hornak et al, 95'),
- Adjacent vertex-distinguishing edge colorings (Zhang et al, 02')

With total colorings :

- Adjacent vertex-distinguishing total colorings (Zhang, 05')

Link with chromatic number

$\chi_{lid}(G) \geq \chi(G)$
Do we need much more than $\chi(G)$ colors ?

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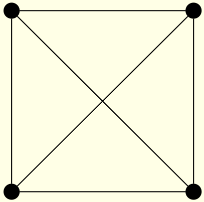

Link with chromatic number

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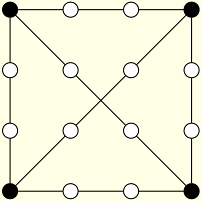

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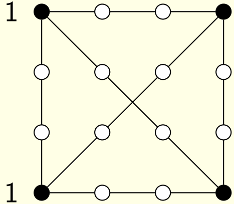

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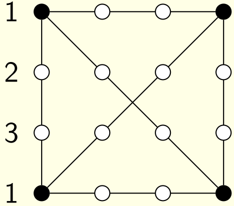

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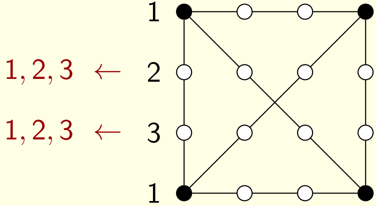

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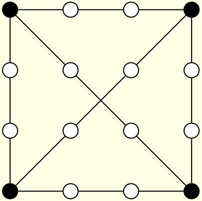

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χ_{lid} is not bounded by a function of χ
But...

Link with maximum degree

We have :

$$\chi_{lid}(G) \leq \chi(G^3)$$

This implies :

$$\chi_{lid}(G) \leq \Delta(G)^3 - \Delta(G)^2 + \Delta(G) + 1$$

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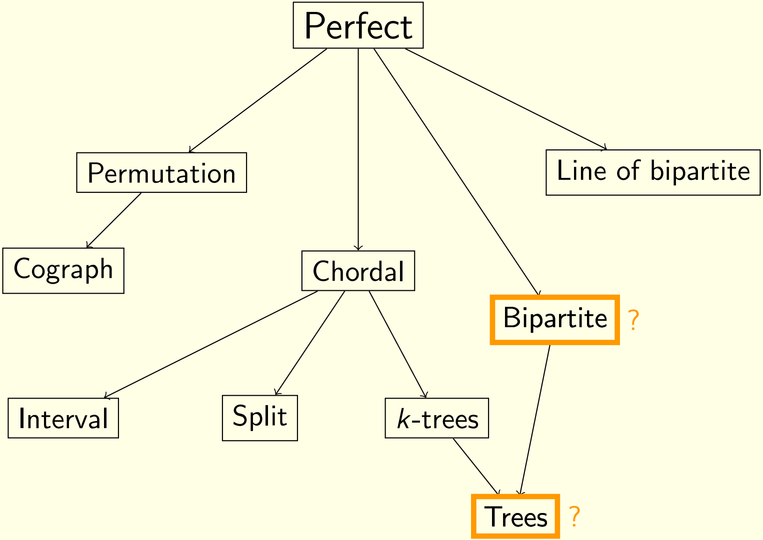
We know only graph that needs $\Delta(G)^2 + \Delta(G) + 1$

Transition Slide

No bounds with χ for general graphs..

What about “good classes” for proper colorings ?

Perfect graphs



An amazing fact about bipartite graphs

G connected graph :

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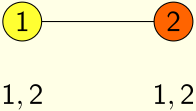

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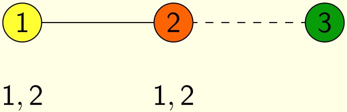

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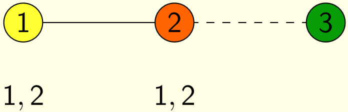
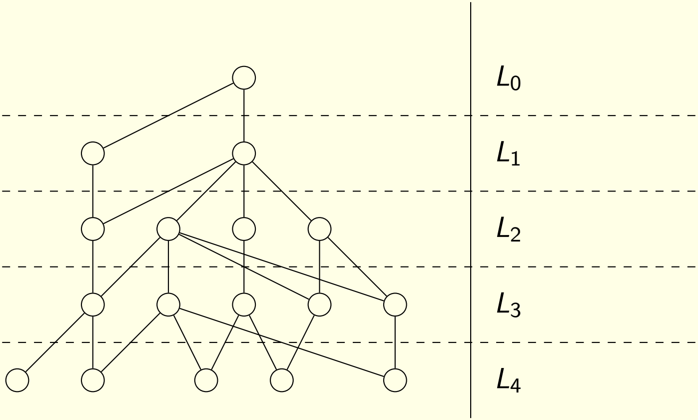


- $\chi_{lid}(G) = 3 \Rightarrow G$ is a triangle or a bipartite graph :
  $\rightarrow$ Partition vertices with the number of colors they see

Bipartite graphs



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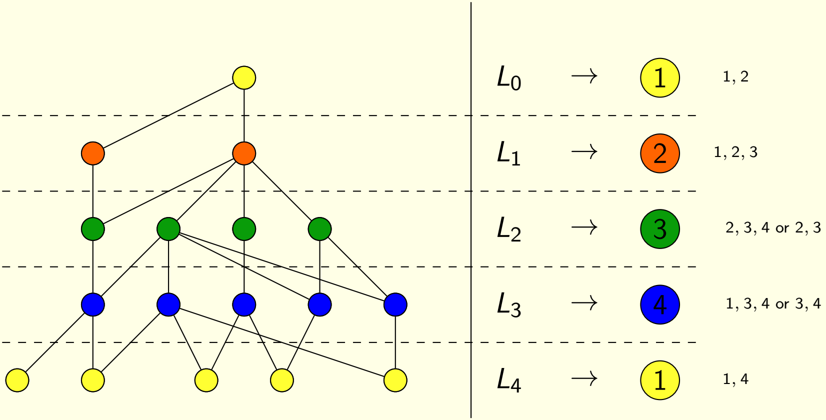

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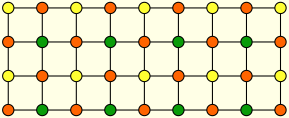

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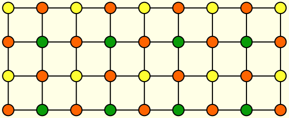

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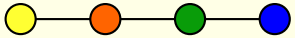

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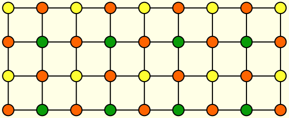

$\leftarrow ? \rightarrow$

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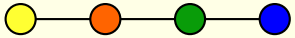
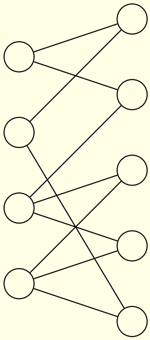


In general... 3-Lid-Coloring is NP-complete in bipartite graphs

Link with 2-coloring of hypergraph

Try to color a graph with 3 colors



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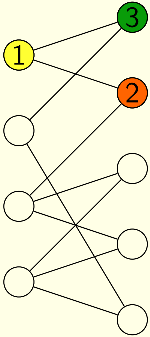

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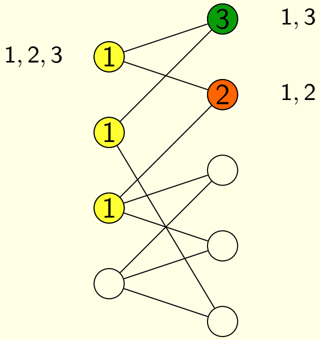

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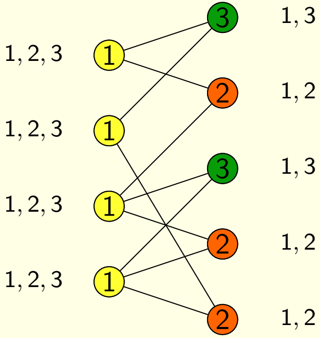

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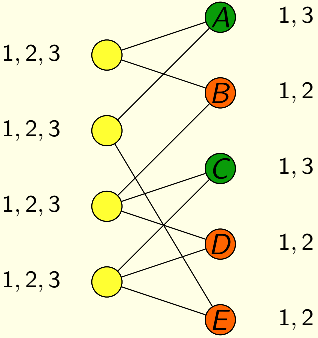

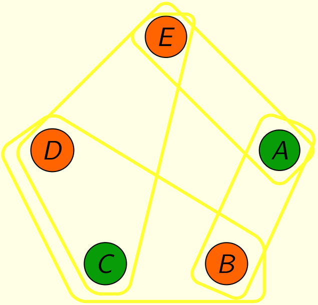

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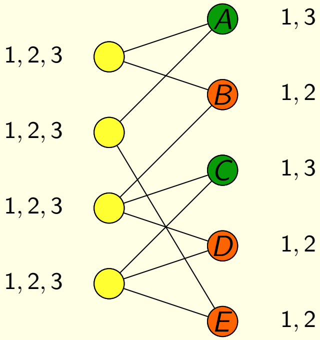
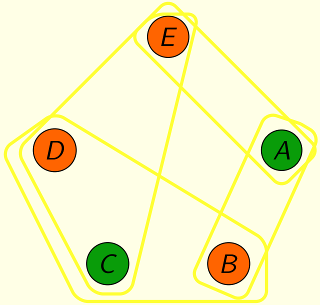
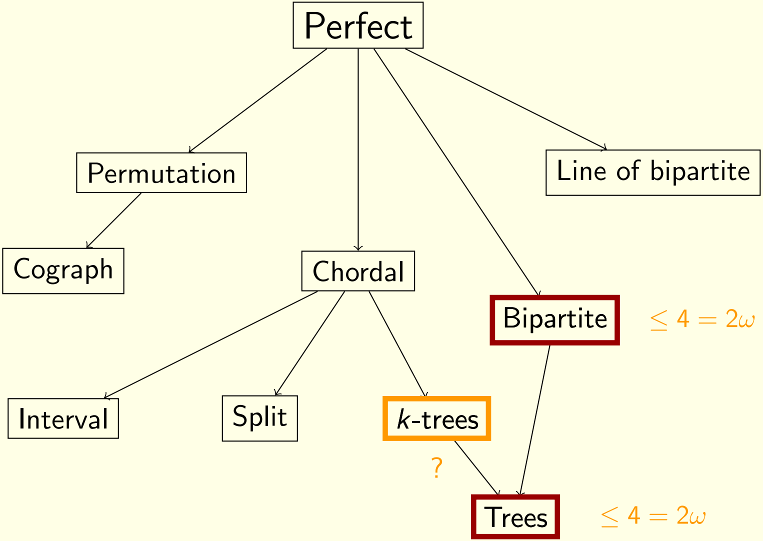




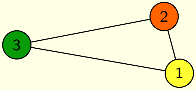
- 3-LID-COLORING in bipartite graph is NP-Complete
- Polynomial if B regular, if B is planar with maximum degree 3, if B is a tree.

Perfect graphs



To perfect graph : k-trees

Lid-coloring of 2-trees with 6 colors :



- Color the triangle with colors $1, 2, 3$
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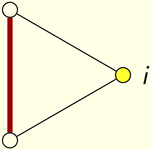

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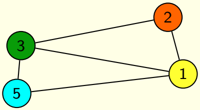

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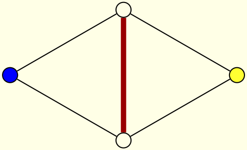

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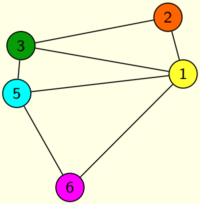

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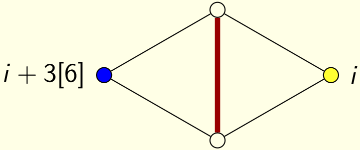

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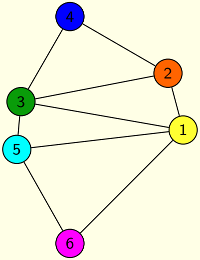

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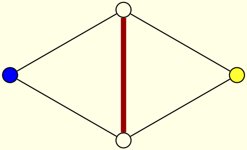

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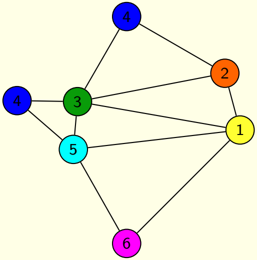

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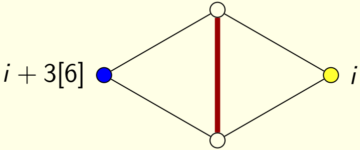

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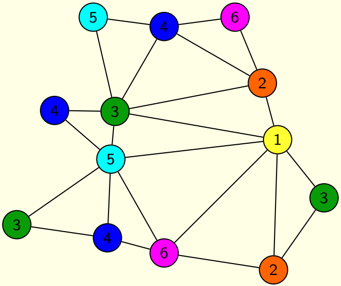

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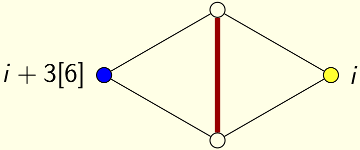

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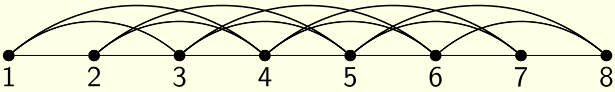
To perfect graph : k-trees

We can extend the construction to k-trees :

$\rightarrow$ A k-tree has lid-chromatic number at most $2k + 2$

This bound is sharp : P^k_{2k+2}



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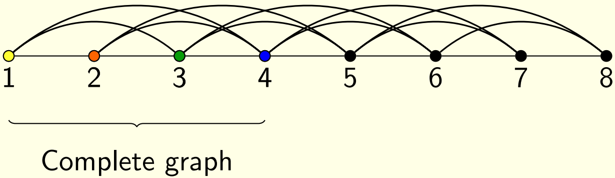

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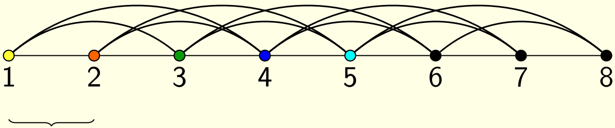

Separated by vertex 5

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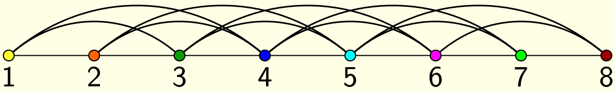
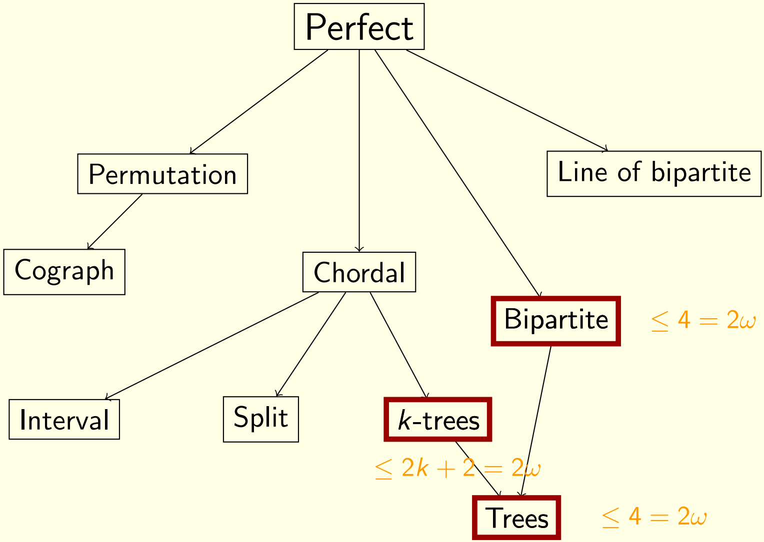


Perfect Graphs



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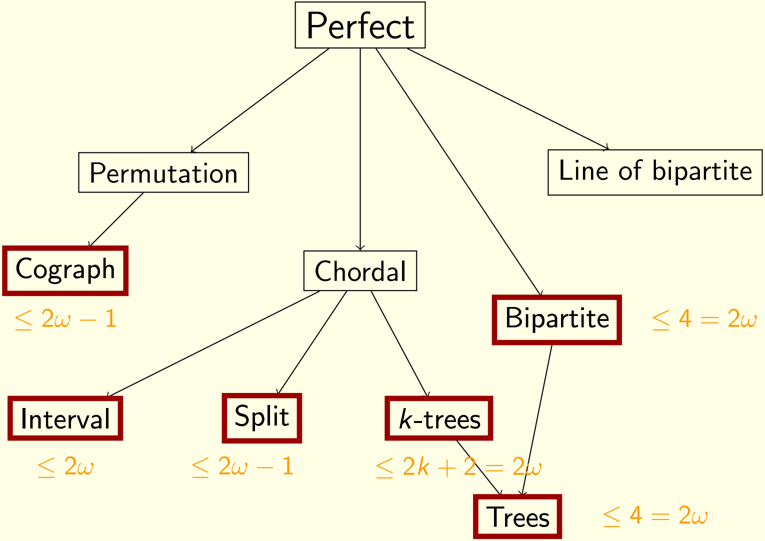

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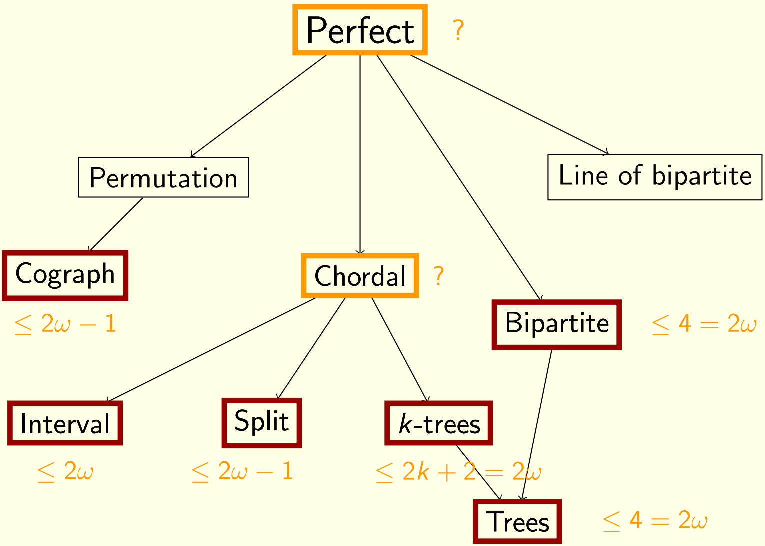

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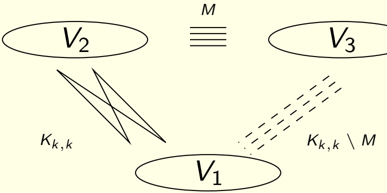

V_1, V_2, V_3 stable sets of size k

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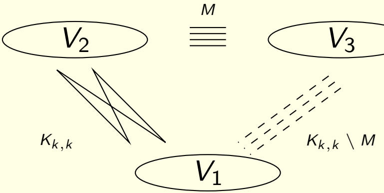

V_1, V_2, V_3 stable sets of size k

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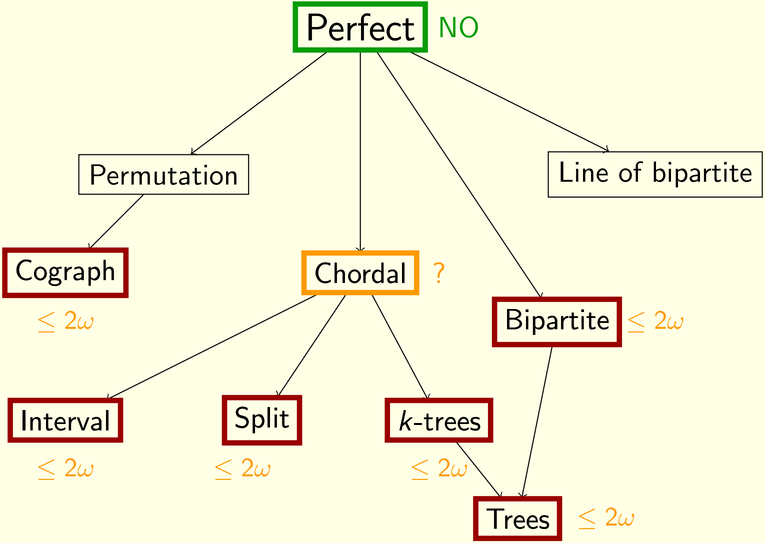
Conjecture : We can color any chordal graph G with $2\omega(G)$ colors

Perfect Graphs



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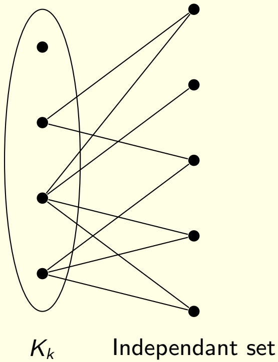

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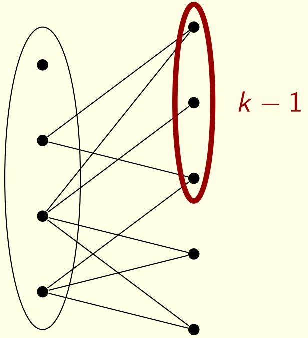
Split graphs :

- Bondy's theorem : $k-1$ vertices of the stable set are enough to separate the clique vertices



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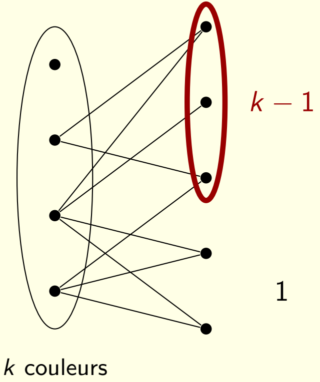
$\rightarrow$ We can color with $2k$ colors

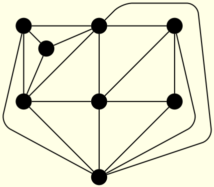
$\rightarrow$ Possible with $2k-1$ colors

$\rightarrow$ It's sharp



Planar graphs

Is lid-chromatic number bounded for planar graphs ?



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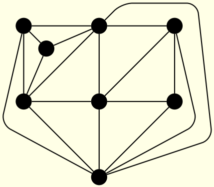

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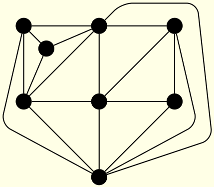

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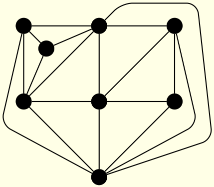

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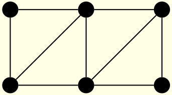
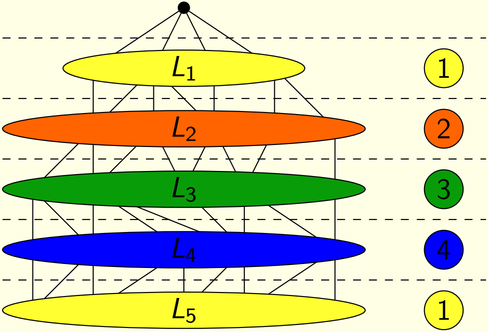
- With large girth (36) bounded by 5

Outerplanar graphs :



- General bound : 20 colors,
- Max outerplanar graphs : ≤ 6 colors,
- Without triangles : ≤ 8 colors,
- Examples with at most 6 colors

A bound for outerplanar graphs



- a layer = union of paths,
- 5 colors in a layer,
- $4 \times 5 = 20$

Bound for planar graphs ?

Really large bound by Gonzcales and Pinlou (2010)

More general result :

Any family of graph closed by minor has lid-chromatic bounded

A remark

- For some subclasses of perfect graphs : $\chi_{lid}(G) \leq 2\omega(G) = 2\chi(G)$
- For planar graphs, worse example : $\chi_{lid}(G) \leq 8 = 2\chi(G)$
- For outerplanar graphs, worse example : $\chi_{lid}(G) \leq 6 = 2\chi(G)$
- ...

A remark

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For which graphs do we have $\chi_{lid}(G) \leq 2\chi(G)$? Is it true for planar graphs ?

Some open problems

- Find a good bound for χ_{lid} in planar graphs
- Prove (or disprove) conjecture for chordal graphs
- For which graphs $\chi_{lid} = \chi$?
- Better bound with maximum degree Δ ?
- What about a global version ?

Kiitos !