Identifying codes and VC-dimension

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Joint work with:
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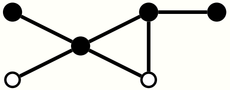
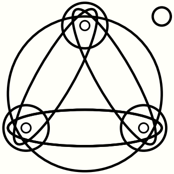
Combgraph Seminar - October 3, 2013

Contents

Identifying codes



VC-dimension



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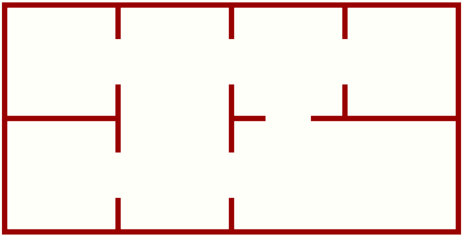
Part I

Identifying codes

Fire detection in a museum?



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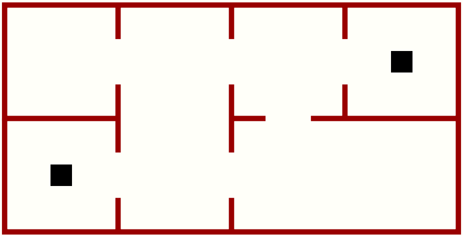

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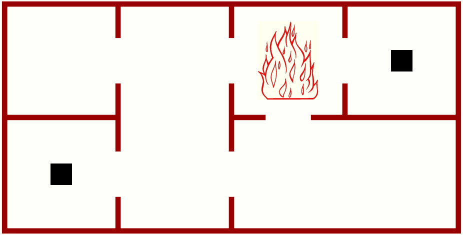

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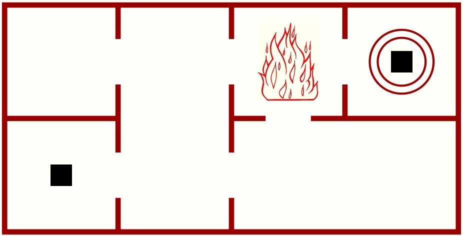

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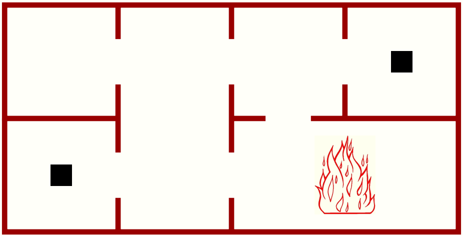

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- Each room must contain a detector or have a detector in a neighboring room.

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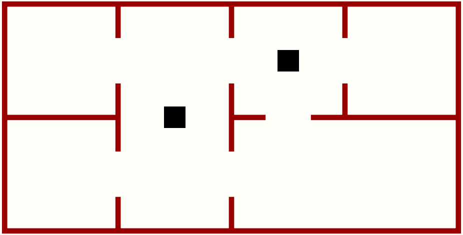

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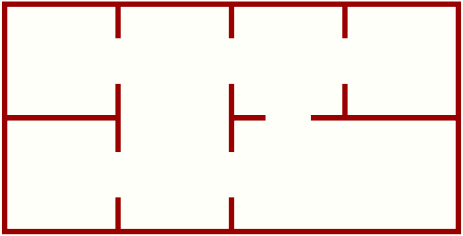
Modelization with a graph



- Vertices V: rooms
- Edges E: between two neighboring rooms

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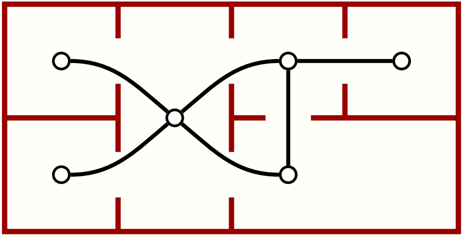

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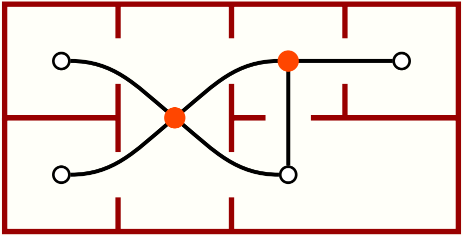

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$$\forall u \in V, N[u] \cap S \neq \emptyset$$

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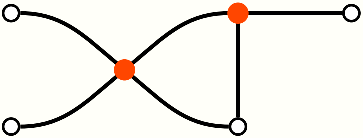

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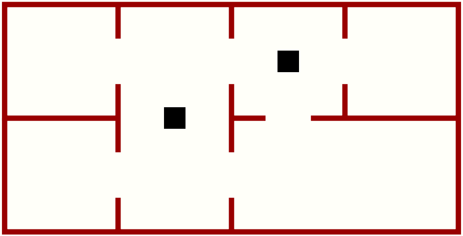

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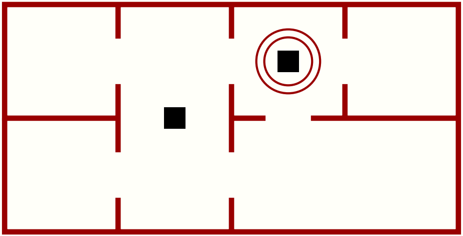

Where is the fire ?

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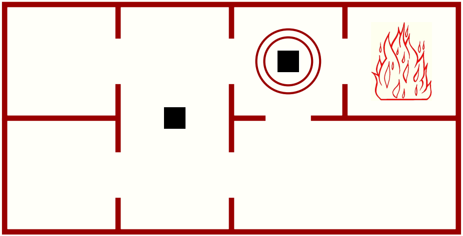

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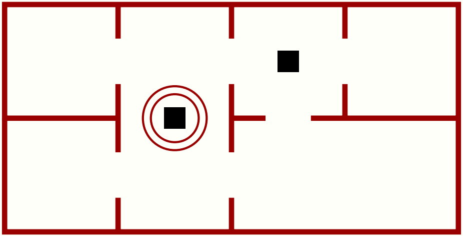

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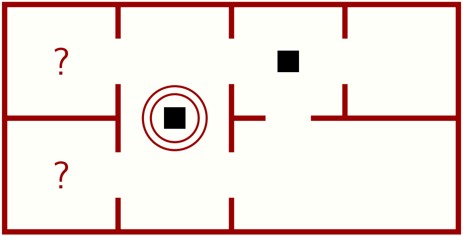
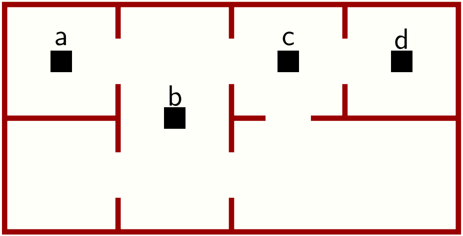


Where is the fire ?

To locate the fire, we need more detectors.

Identifying where is the fire



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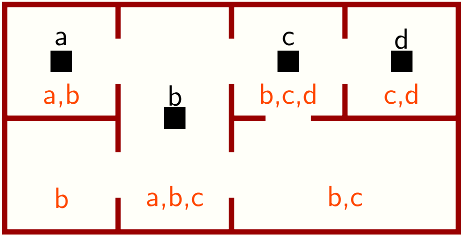
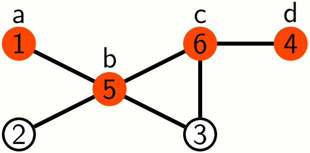


In each room, the set of detectors in the neighborhood is unique.

Modelization with a graph

Identifying code C = subset of vertices which is

- dominating : $\forall u \in V, N[u] \cap C \neq \emptyset$,
- separating : $\forall u, v \in V, N[u] \cap C \neq N[v] \cap C$.



V \ C	a	b	c	d
1	•	•	-	-
2	-	•	-	-
3	-	•	•	-
4	-	-	•	•
5	•	•	•	-
6	-	•	•	•

Given a graph G, what is the size $\gamma^{ID}(G)$ of minimum identifying code ?

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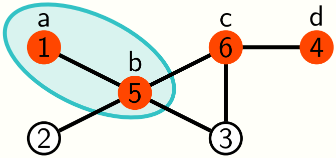

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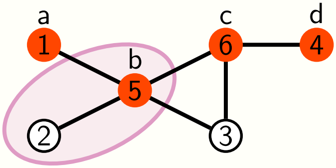

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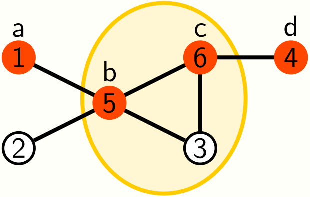

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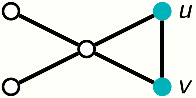
Some facts about identifying codes

- Introduced in 1998 by Karpvosky, Chakrabarty and Levitin
- Exists iff there is no twins



Twins: $N[u] = N[v]$

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  $\rightarrow$ A vertex is identified by a nonempty subset of $C \Rightarrow |V| \leq 2^{\gamma^{ID}(G)} - 1$

$$\gamma^{ID}(G) \geq \log(|V| + 1)$$

Tight example:

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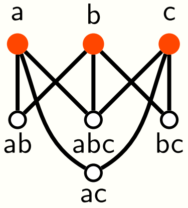
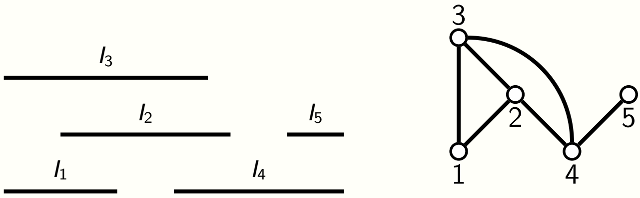


In restricted classes of graphs?

Example 1: Class of interval graphs



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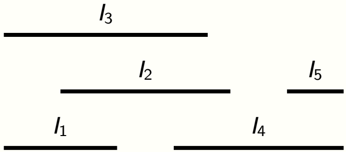
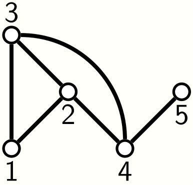


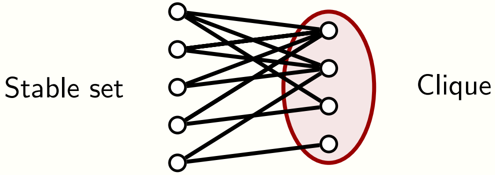


Proposition Foucaud, Naserasr, P., Valicov, 2012+

If G is an interval graph, $\gamma^{ID}(G) \geq \sqrt{2|V|}$.

In restricted classes of graphs?

Example 2: Class of split graphs



In restricted classes of graphs?

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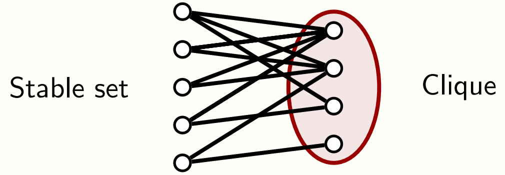


Proposition

For infinitely many split graphs G, $\gamma^{ID}(G) = \log(|V| + 1)$.

Proposition Foucaud, 2013

Min-Id-Code is log-APX-hard for split graphs.

Some known results in restricted classes of graphs

Restriction: classes of graphs closed by induced subgraphs

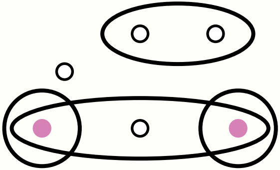
Graph class	lower bound (order)	Approximation
All	$\log n$	log APX-h
Split	$\log n$	log APX-h
Interval	$n^{1/2}$	open
Unit Interval	n	2
Bipartite	$\log n$	log APX-h
Line graphs	$n^{1/2}$	4
Chordal	$\log n$	log APX-h
Planar	n	7
Cograph	n	1

Part II

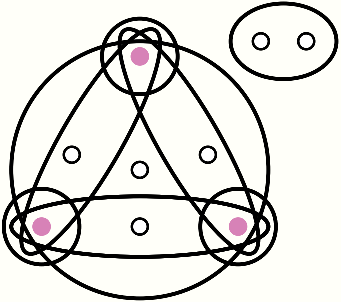
VC-dimension

Shattered set

- $\mathcal{H} = (V, \mathcal{E})$ an hypergraph
- A set $X \subseteq V$ is shattered if for all $Y \subseteq X$, there exists $e \in \mathcal{E}$, s.t $e \cap X = Y$.



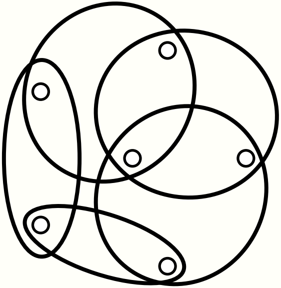
A 2-shattered set



A 3-shattered set

Vapnik Chervonenkis (VC) dimension of an hypergraph

- A set X is shattered if $\forall Y \subseteq X$, $\exists e \in \mathcal{E}$, s.t $e \cap X = Y$.
- VC-dimension of $\mathcal{H}$: largest size of a shattered set.



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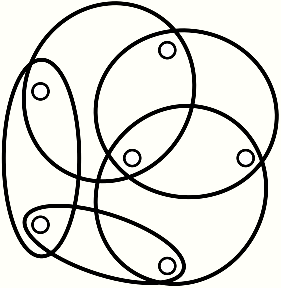

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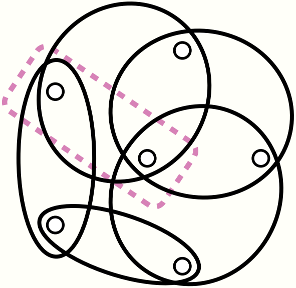

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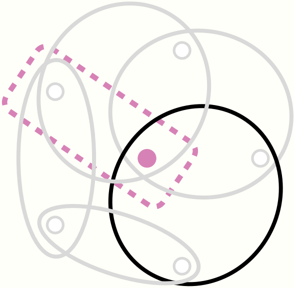

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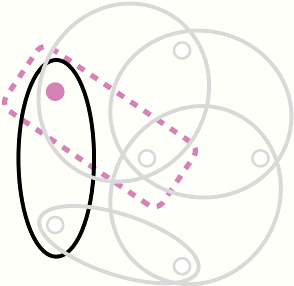

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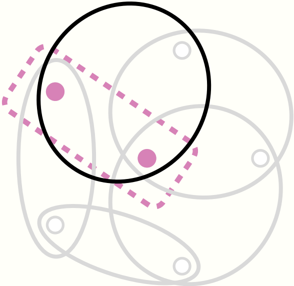

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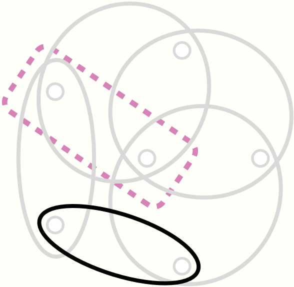

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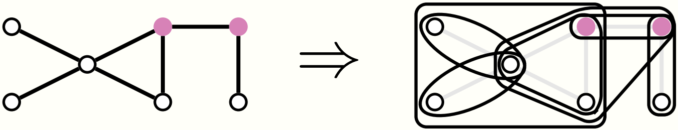
VC dimension of a graph / of a class of graph

- VC-dimension of G: VC-dim of the hypergraph of closed neighborhoods



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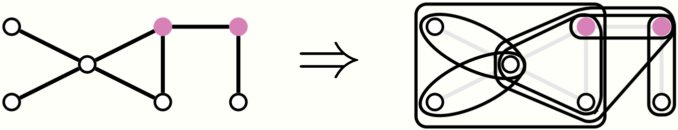
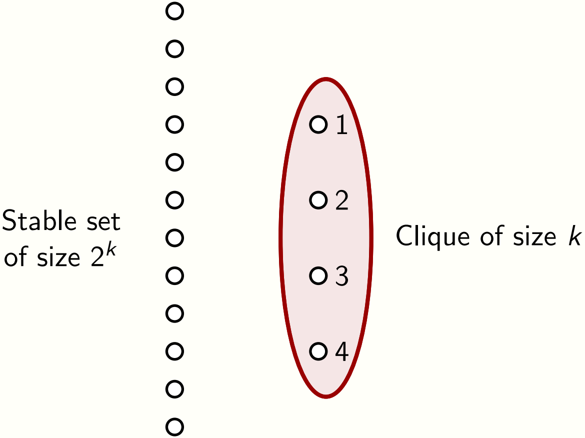


$$\text{VC-dim}(G) = 2$$

- VC-dimension of a class $\mathcal{C}$: maximal VC-dimension over $\mathcal{C}$
  - Class of interval graphs has VC-dimension 2.
  - Class of split graphs has infinite VC-dimension.

Split graphs have infinite VC-dimension

For any k, there is a split graph with VC-dimension k.



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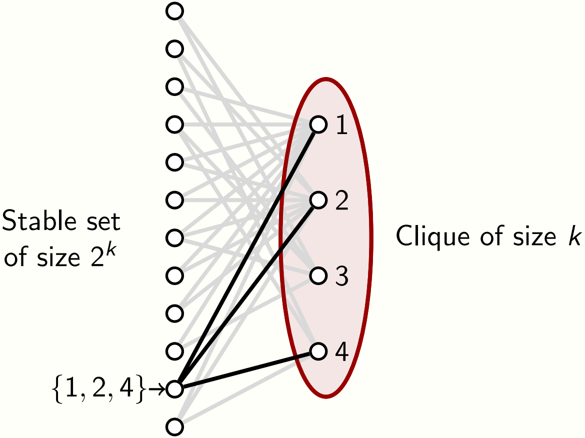
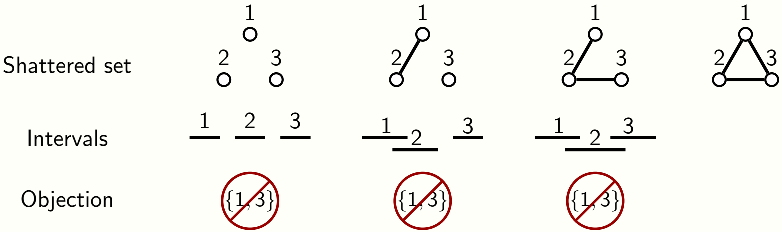


Intervals have finite VC-dimension

There is no interval graph with VC-dimension 3.

Assume there is a shattered set $\{1, 2, 3\}$.



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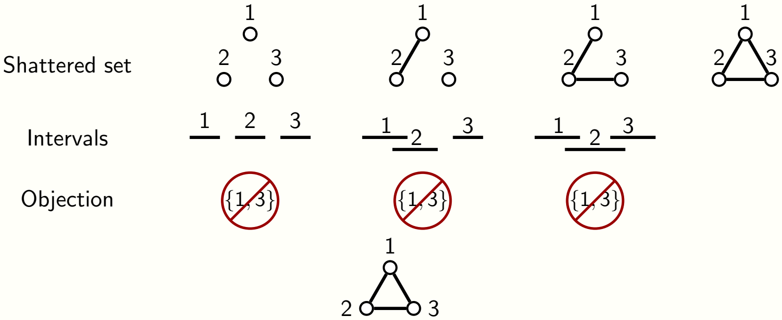

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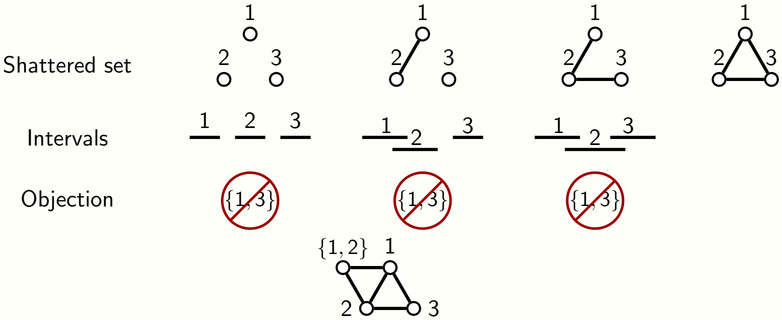

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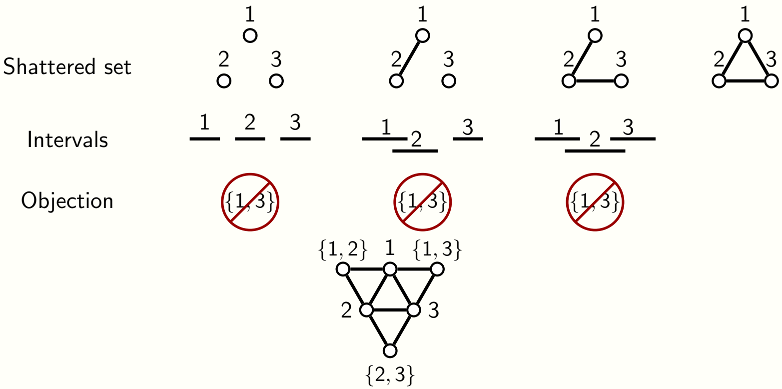

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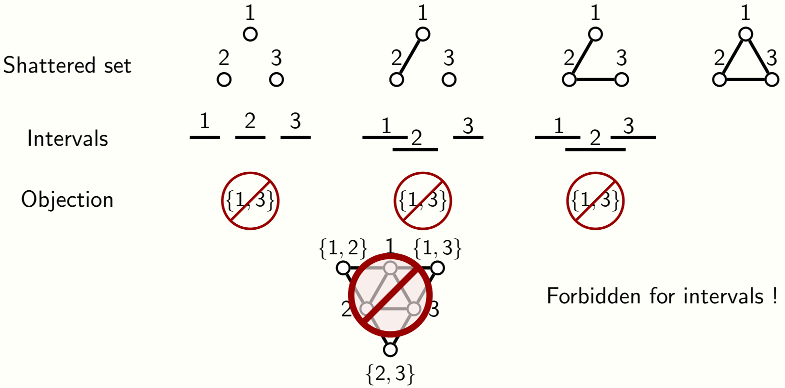

Interval graphs have VC-dimension at most 2.

Part III

Identifying codes and VC-dimension

Back to identifying codes

Graph class	Lower bound (order)	Approx
All	$\log n$	log APX-h
Split	$\log n$	log APX-h
Interval	$n^{1/2}$	open
Unit Interval	n	2
Bipartite	$\log n$	log APX-h
Line graphs	$n^{1/2}$	4
Chordal	$\log n$	log APX-h
Planar	n	7
Cograph	n	1

Back to identifying codes

Graph class	Lower bound (order)	Approx	VC dim
All	$\log n$	log APX-h	∞
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Interval	$n^{1/2}$	open	2
Unit Interval	n	2	2
Bipartite	$\log n$	log APX-h	∞
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A dichotomy result

Theorem

VC-dim of $\mathcal{C}$?

Infinite	Finite d
$\Downarrow$	$\Downarrow$
There are infinetly many G,	For all G,
$\gamma^{ID}(G) \approx \log \lvert V \rvert$	$\gamma^{ID}(G) \geq c \lvert V \rvert^{1/d}$

Proof - Case with infinite VC dimension

Proposition

If $\mathcal{C}$ has infinite VC-dimension, for any integer k, $\mathcal{C}$ contains a graph G with at least 2^k vertices and an identifying code $\leq 2k$.

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Proposition

If $\mathcal{C}$ has infinite VC-dimension, for any integer k, $\mathcal{C}$ contains a graph G with at least 2^k vertices and an identifying code $\leq 2k$.

Proof:

- Let $G \in \mathcal{C}$ of VC-dim k and X be a shattered set of size k.
- Let Y be a set shattering X.
- $C = X$ identifies all vertices of Y.
- Add to C at most k vertices of Y to identify vertices of X.

$\rightarrow$ $G' = G[X \cup Y]$ satisfies the claim.

Proof - Case with finite VC dimension

Proposition

If $\mathcal{C}$ has finite VC-dimension d, $\forall G \in \mathcal{C}$, $\gamma^{ID}(G) \geq c|V|^{1/d}$.

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Proof: direct consequence of:

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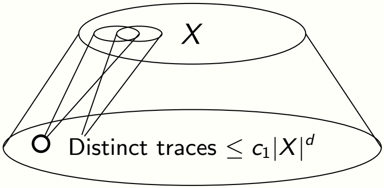
Let X be a subset of vertices of graph G of VC-dimension d. The number of distinct traces on X is at most $\sum_{i=1}^{d} \binom{|X|}{i} \leq c_1|X|^d$.



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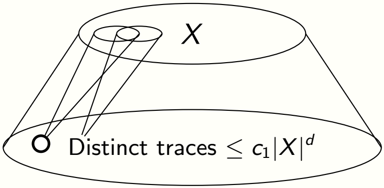

Identifying code $\gamma^{ID}(G)$

All vertices $|V|$ $\leq c_1\gamma^{ID}(G)^d$

Back to the table

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- Lower bound not optimal (ex: Line graphs)

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- What about approximation ?

Inapproximability in infinite VC dimension

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If $\mathcal{C}$ has ∞ VC-dimension, $\textsc{Min-Id-Code}$ is log-APX-hard on $\mathcal{C}$.

Inapproximability in infinite VC dimension

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Consequence of:

Proposition

If $\mathcal{C}$ has infinite VC-dimension, $\mathcal{C}$ contains:

- all bipartite graphs, or
- all split graphs, or
- all cobipartite graphs.

and

Theorem Foucaud, 2013

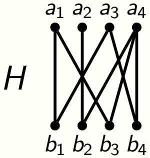
$\textsc{Min-Id-Code}$ is log-APX-hard on bipartite, split and cobipartite graphs.

Sketch of the proof

Proposition

If $\mathcal{C}$ has infinite VC-dimension, $\mathcal{C}$ contains all bipartite graphs, or all split graphs, or all cobipartite graphs.

1. $\mathcal{C}$ is full-crossing bipartite:
   For any bipartite graph $H = (A \cup B, E)$, by adding edges on A or on B to H, we can get an element H' of $\mathcal{C}$.



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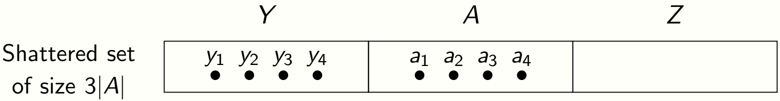

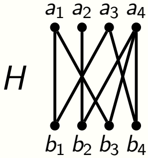

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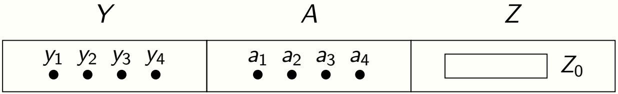
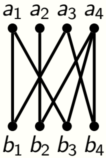


$\forall i, N[a_i] \cap Z \neq Z_0$

H



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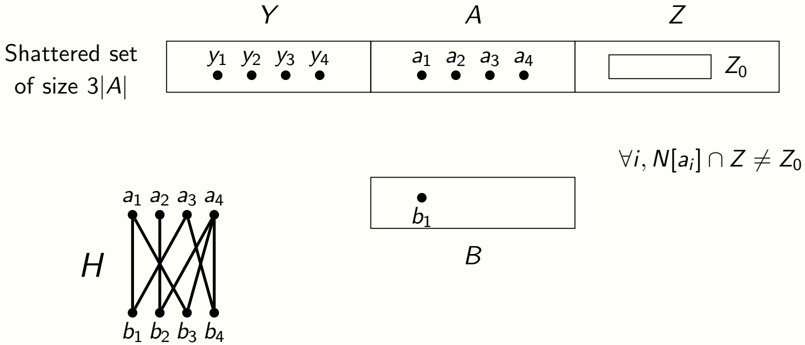

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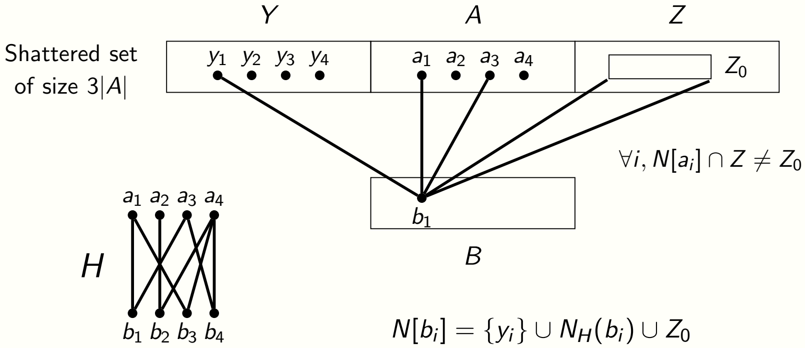

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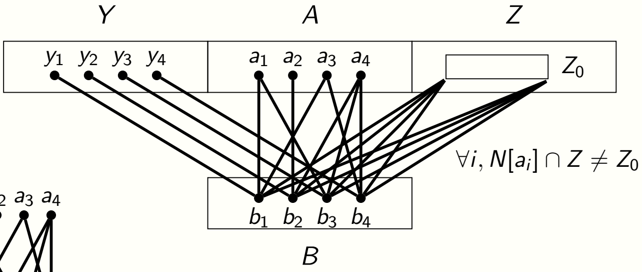

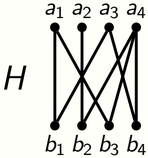

$$N[b_i] = \{y_i\} \cup N_H(b_i) \cup Z_0$$

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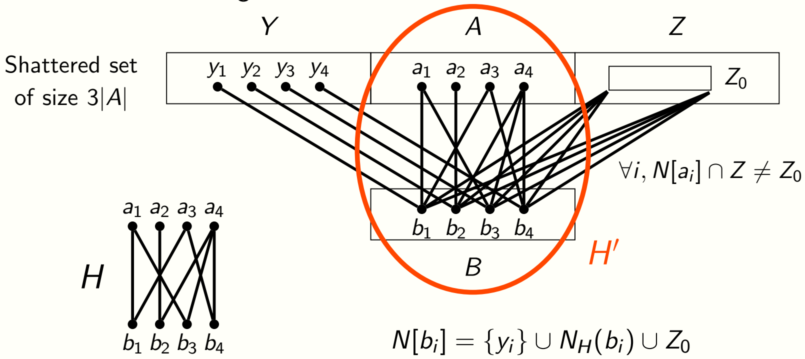

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2. H' induces cliques or stable sets on A and B.

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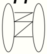
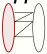
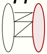
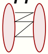
3. Conclusion :

For any bipartite H, H^0  or H^1  or H^2  or H^3  $\in \mathcal{C}$

(H_n) : sequence of universal bipartite graphs.
$\Rightarrow \exists i \in \{0, 1, 2, 3\}, H_n^i \in \mathcal{C}$ for infinitely many n.
$\Rightarrow$ All bipartites ($i = 0$) or all splits ($i = 1, 2$) or all cobipartites ($i = 3$) are in $\mathcal{C}$.

In the finite case ?

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Is there a constant approximation in finite VC-dimension?

A class of finite VC-dimension with no good approximation

Theorem

MIN-ID-CODE cannot be approximated within a $o(\log |V|)$ factor in polynomial time for the class of bipartite C_4-free graphs.

- Class of VC-dimension 2
- Reduction from SET COVERING WITH INTERSECTION 1.

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Gracias !