Codes identifiants dans des classes de graphes héréditaires

Aline Parreau

LIFL, Université de Lille 1

En collaboration avec :
Florent Foucaud, Sylvain Gravier, Adrian Kosowski, George Mertzios, Reza Naserasr et Petru Valicov

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Laboratoire d'Informatique Fondamentale de Lille

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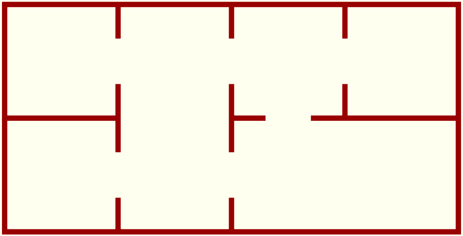
Fire detection in a museum?



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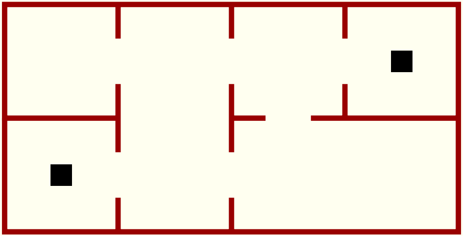

- Detector can detect fire in their room or in their neighborhood.

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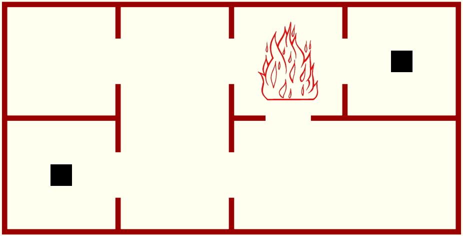

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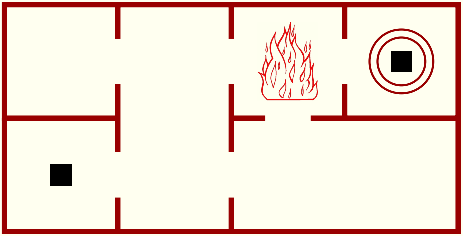

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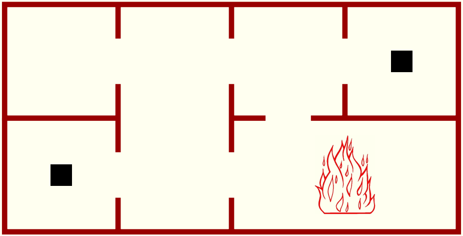

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- Each room must contain a detector or have a detector in a neighboring room.

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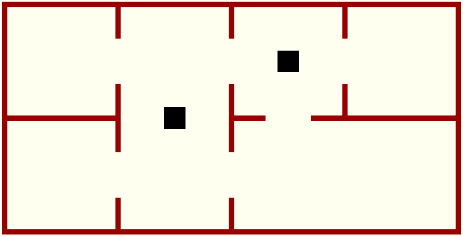

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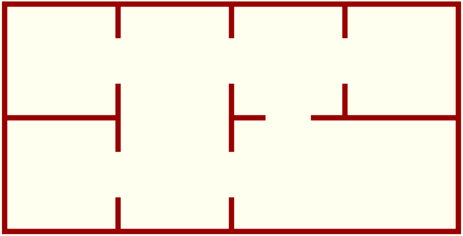
Modelization with a graph



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- Edges E: between two neighboring rooms

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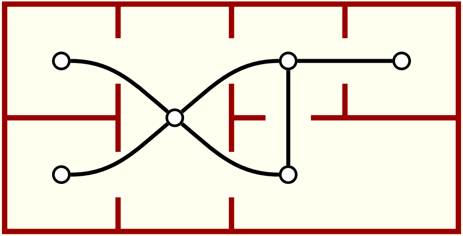

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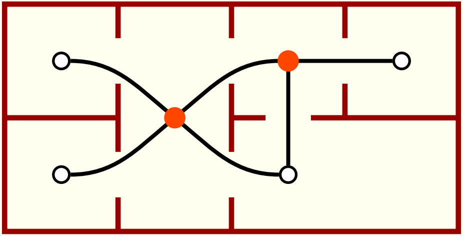

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$$\forall u \in V, N[u] \cap S \neq \emptyset$$

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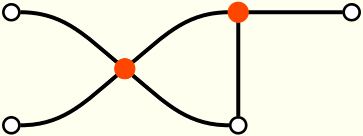

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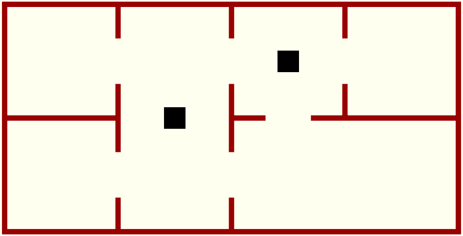

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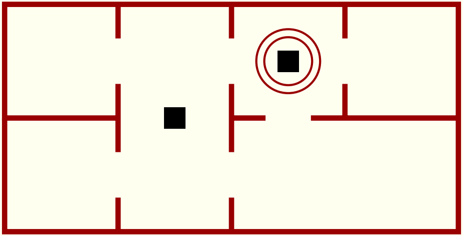

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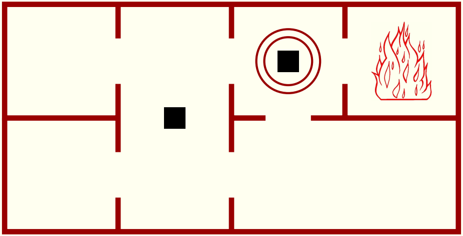

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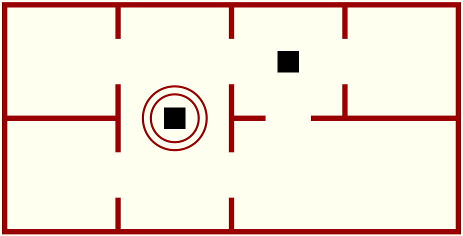

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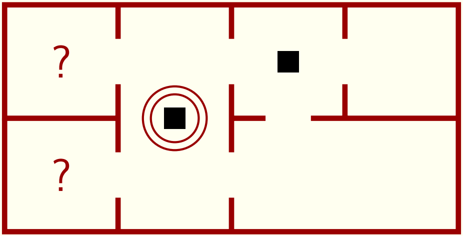
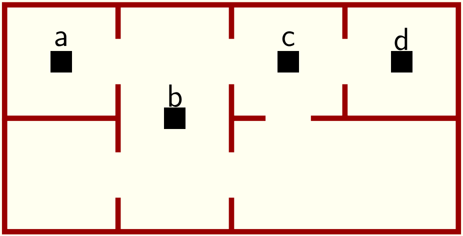


Where is the fire ?

To locate the fire, we need more detectors.

Identifying where is the fire



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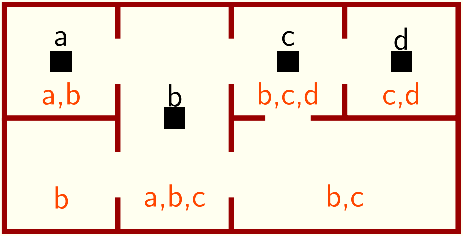
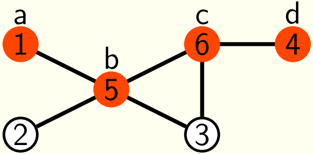


In each room, the set of detectors in the neighborhood is unique.

Modelization with a graph

Identifying code C = subset of vertices which is

- dominating : $\forall u \in V, N[u] \cap C \neq \emptyset$,
- separating : $\forall u, v \in V, N[u] \cap C \neq N[v] \cap C$.



V \ C	a	b	c	d
1	•	•	-	-
2	-	•	-	-
3	-	•	•	-
4	-	-	•	•
5	•	•	•	-
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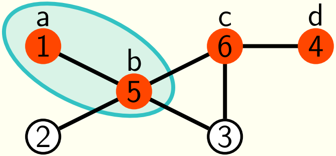

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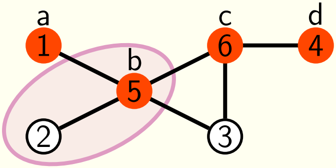

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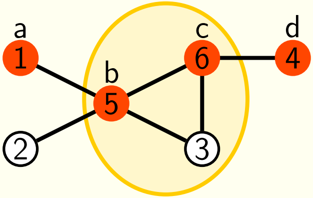

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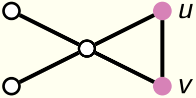
Facts about identifying codes

- Introduced in 1998 by Karpvosky, Chakrabarty and Levitin
- Existence $\Leftrightarrow$ no twins in the graph:



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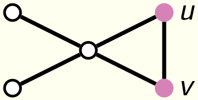

Twins: $N[u] = N[v]$

Given a twin-free graph G, what is the size $\gamma^{ID}(G)$ of minimum identifying code ?

A difficult question...

IDENTIFYING CODE : Given a twin-free graph G and an integer k, is there an identifying code of size k in G?

Proposition Charon, Hudry, Lobstein, 2001

IDENTIFYING CODE is NP-complete.

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Optimization problem hard to approximate:

Proposition Laifenbled, Trachtenberg, 2006 & Suomela, 2007

Best polynomial approximation: factor $\log(|V|)$

Bounds on $\gamma^{ID}(G)$ using the number of vertices

- n : number of vertices
- Upper bound: $\gamma^{ID}(G) \leq n - 1$ (Bertrand and Gravier, Moncel 2001)

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$$\gamma^{ID}(G) \geq \log(n+1)$$

Tight example:

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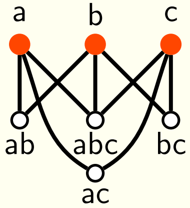

What about restricted classes of graphs?

Coding theory classes:

- path, cycles
- rectangular , hexagonal, triangular grids
- hypercubes
- ...

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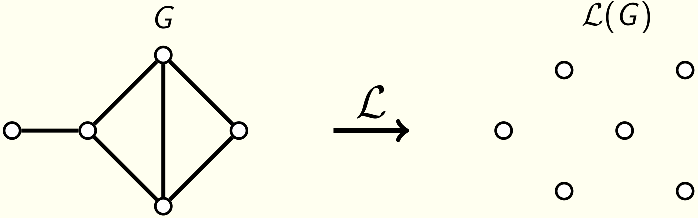
Heriditary classes ?

$\rightarrow$ line graphs

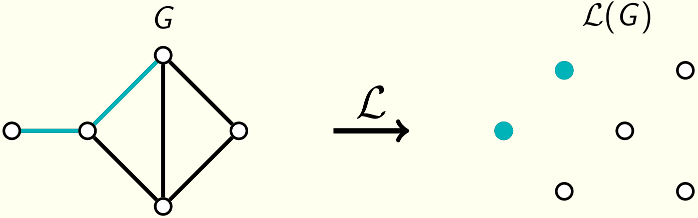
$\rightarrow$ interval graphs

Identifying codes in line graphs

Identifying code in line graphs



Identifying code in line graphs



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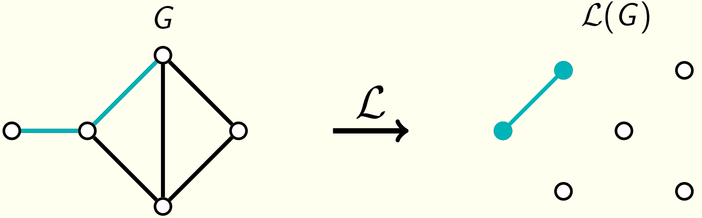

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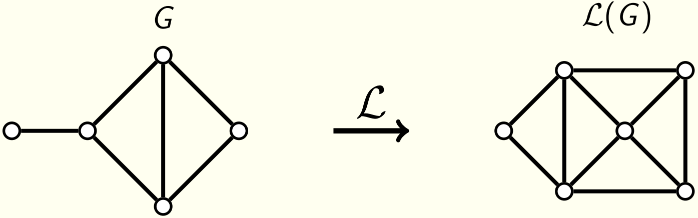

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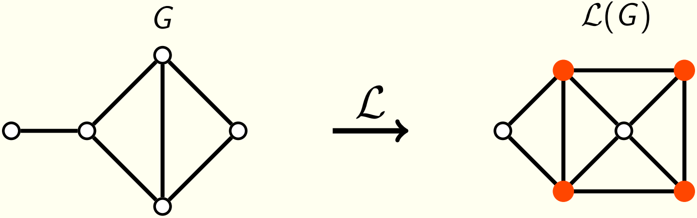

Identifying code

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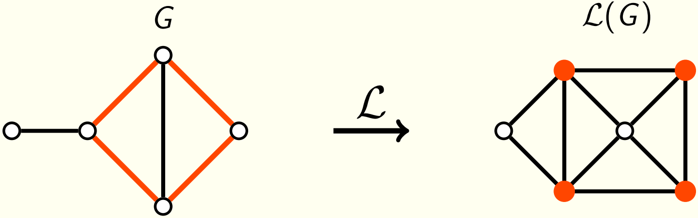
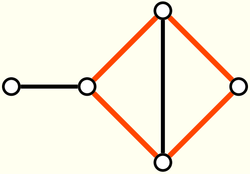


Edge identifying code $\longleftrightarrow$ Identifying code

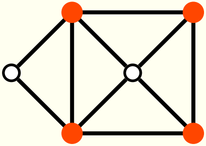
Identifying code in line graphs

G



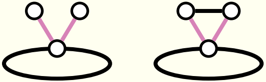
$\xrightarrow{\mathcal{L}}$

$\mathcal{L}(G)$

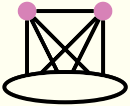


Edge identifying code $\longleftrightarrow$ Identifying code

$\gamma^{EID}(G) = \gamma^{ID}(\mathcal{L}(G))$



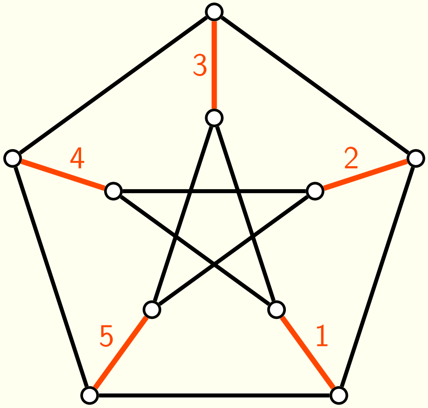
$\longleftrightarrow$



Pendant edges

Twins

Example with Petersen graph



$$\gamma^{EID}(\mathcal{P}) \leq 5$$

Lower bound of γ^{EID} using n

Proposition Foucaud, Gravier, Naserasr, P., Valicov, 2012

$$\gamma^{EID}(G) \geq \frac{n}{2}$$

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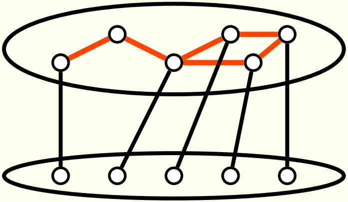
C_E, k edges on n' vertices

$X = V(G) \setminus V(C_E)$



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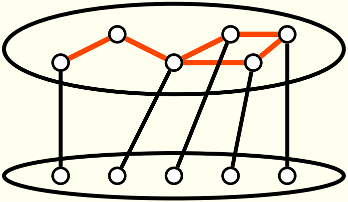

- Assume C_E is connected
- If C_E is a cycle, $|X| \leq n' \leq k$,

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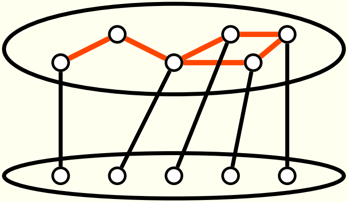

- Assume C_E is connected
- If C_E is a cycle, $|X| \leq n' \leq k$,
- If C_E is a tree, $n' - 1 = k$ and $|X| \leq n' - 2$
- In both cases, $n = |X| + n' \leq 2k$

Consequence for line graphs

- $\gamma^{EID}(G) \geq \frac{n}{2}$

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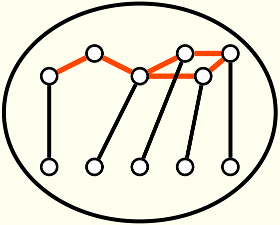
Corollary

If H is a line graph,

$$\gamma^{ID}(H) \geq \frac{\sqrt{2n}}{2}.$$

$\rightarrow$ Larger than the $\log(n)$ bound!

Alternative proof



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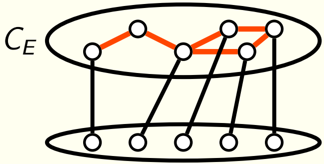

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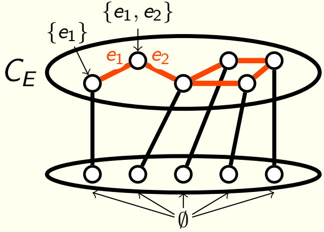

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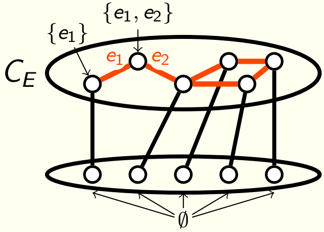

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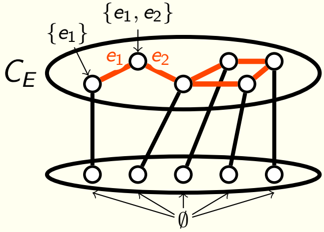

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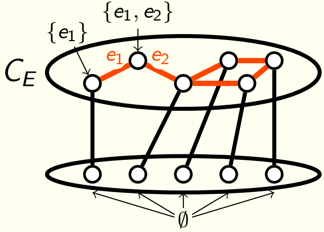

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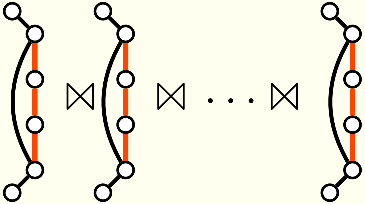
Leads to the following tight bound:

$$\gamma^{EID}(G) \geq \frac{3\sqrt{2}}{2}\sqrt{|E(G)|}$$



Still difficult

EDGE-IDCODE : Given G pendant-free and k, $\gamma^{EID}(G) \leq k$?

Theorem Foucaud, Gravier, Naserasr, P., Valicov, 2012

EDGE-IDCODE is NP-complete even for planar subcubic bipartite graphs with large girth.

Reduction from PLANAR ($\leq 3, 3$)-SAT.

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Reduction from PLANAR ($\leq 3, 3$)-SAT.

Corollary

IDENTIFYING CODE is NP-complete even for perfect planar 3-colorable line graphs with maximum degree 4.

But approximable

Proposition Foucaud, Gravier, Naserasr, P., Valicov, 2012

EDGE-IDCODE (resp. IDCODE in line graphs) has a polynomial 4-approximation.

Due to:

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Proposition Foucaud, Gravier, Naserasr, P., Valicov, 2012

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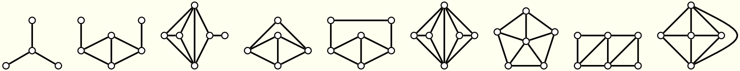
Open question

Better approximation?

Conclusion for line graph

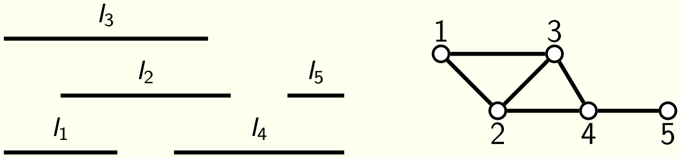
- Class of graph for which $\gamma^{ID}(G) \geq \Theta(\sqrt{n})$.
- Defined by forbidden induced subgraphs:



- Is the lower bound still true with less restrictions?
  $\rightarrow$ Not true for claw-free graphs.

Identifying codes in interval graphs

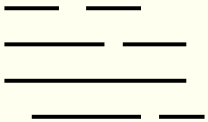
Interval graphs



Lower bound

Proposition Foucaud, Naserasr, P., Valicov, 2012+

If G is an interval graph, $\gamma^{ID}(G) > \sqrt{2n}$



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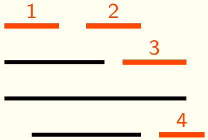

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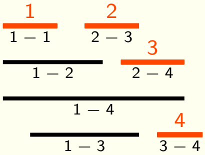

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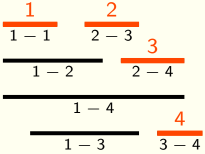

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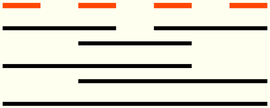
Tight



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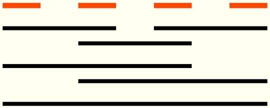

Link with line graphs ?

$\rightarrow$ each vertex is *defined* with two cliques

Complexity

Proposition Foucaud, Kosowski, Mertzios, Naserasr, P., Valicov

IDENTIFYING CODE is NP-Complete for interval graphs.

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Reduction from 3-DIMENSIONAL MATCHING:

- Instance: A,B, C sets and $\mathcal{T} \subset A \times B \times C$ some triples
- Question: is there a perfect 3-dimensional matching $M \subset T$, i.e., each element of $A \cup B \cup C$ appears exactly once in M?

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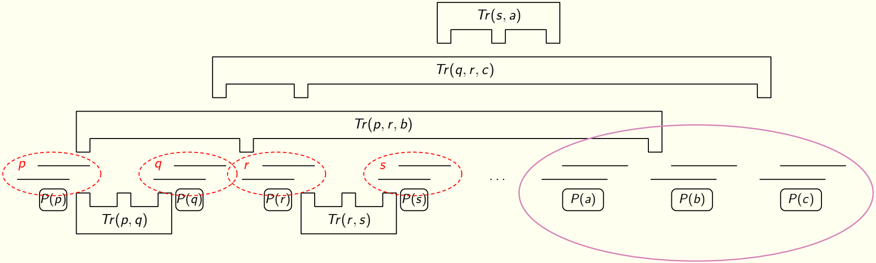

Unit interval graphs

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Proposition

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Open question

What is the complexity of $\textsc{IDCode}$ in unit interval graphs?

Conclusion

	Lower Bound	Complexity	Approximability
Line	$\sqrt{n}$	NP-c	4
Interval	$\sqrt{n}$	NP-c	OPEN
Unit interval	$\frac{n}{2}$	OPEN	2
Claw-free	$\log n$	NP-c	$\log n$
Tree	$\frac{3n}{7}$	Linear	-
Cograph	$\frac{n}{2}$	Linear	-
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Merci !