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# Laplace-Beltrami operator on Digital Curves

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## Abstract

Many problems in image analysis, digital processing and shape optimization are expressed as variational problems and involve the discretization of laplacians. Indeed, PDEs containing Laplace-Beltrami operator arise in surface fairing, mesh smoothing, mesh parametrization, remeshing, feature extraction, shape matching, etc. The discretization of the laplace-Beltrami operator has been widely studied, but essentially in the plane or on triangulated meshes. In this paper, we propose a digital Laplace-Beltrami operator, which is based on the heat equation described by [BSW08] and adapted to 2D digital curves. We give elements for proving its theoretical convergence and present an experimental evaluation that confirms its convergence property.

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**Keywords:** JFIG 2016, laplacians, heat equation, convolution, digital surfaces

## 1. Introduction

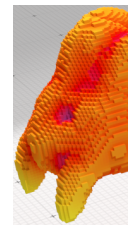
Many problems in image analysis, digital processing and shape optimization are expressed as variational problems and involves the discretization of laplacians. Indeed, PDEs containing Laplace-Beltrami operator arise in surface fairing, mesh smoothing, mesh parametrization, remeshing, feature extraction, shape matching, etc (see for example [LZ08]). For instance, computing geodesics on triangulated surfaces can be formulated from the heat diffusion equation [CWW13] or by a limit probability distribution of Dirac measures in optimal transportation problems [SRGB14].

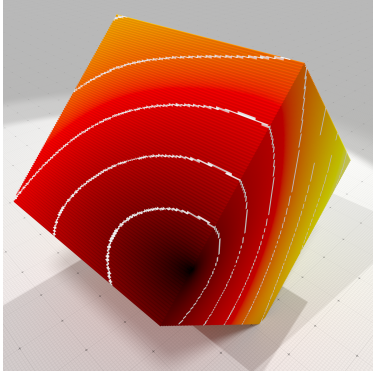
The subject of the laplace-Beltrami operator and its discretization has been widely studied. Some approaches rely on the theory of exterior calculus which has been developed in the computational mathematics and geometry processing community, with focus on triangular meshes. Purely combinatorial approaches can be found in [GP10]. Another approach developed in [Hir03] relies more on the relation between the discrete structure and the continuous one. On triangular meshes, those formulations are closely related to the famous cotangent formula [PP93], and is equivalent to specific cases of finite element method [LZ08].

An important objective when proposing discretizations of the laplace-Beltrami operator is to give convergence results: as meshes refine and tend toward the underlying manifold under certain properties, the approximated laplace-Beltrami operator should tends toward the usual one on the manifold. On arbitrary triangular meshes, it is shown that the computed laplace-Beltrami operator cannot recover all the properties of the smooth manifold one [WMKG07]. Regard-

ing discrete exterior calculus, Hildebrandt *et al.* [HPW06] provided convergence results when the triangulated meshes tend toward the continuous manifold with those properties: Hausdorff distance tends to zero, mesh normals tend to surface normals and the mesh is projected one-to-one on surface. Similar proofs exist in the context of finite element method [AFW06a, AFW06b] and for chainlet discrete calculus [Har06].

In the case of triangular meshes, as the structure becomes thinner, estimated intrinsic geometric informations converge toward the real ones and can be easily injected inside the calculus. In our case, we want to define the Laplace-Beltrami operator on a different geometric structure: it is called a digital surface and is the border of a subset of the digital space  $\mathbb{Z}^d$ . One can see them geometrically as unions of faces of  $d$ -dimensional cubes. Digital surfaces can be defined as digitizations of manifolds but such digitizations present several difficulties: points are spaced evenly, they do not interpolate the smooth surface and elementary normals are not informative. Therefore, in opposition to triangular meshes, one needs to be more clever to recover geometric informations such as normals, areas or curvature for example. Fortunately, this topic was studied in depth in the last ten years and several progresses were made during this period. Parameter-free tangent and normal estimation along 2D digital curves were established [LVdV07, dVLF07] using properties of maximal digital straight segments. This approach was extended to 3D curves in [PJKL12]. Further works using digital version of integral invariants [CLL14, PWHY09] induced convergent estimation of the normal vector field along digital surfaces in arbitrary dimension as well as the whole curvature tensor [CLL14, LCL14].





**Figure 1:** Implementation of geodesic distance computation described in [CWW13] using a discrete exterior calculus on digital surfaces. Isocontour in white on the left staired face of the cube shows metrics failure.

As shown on Fig.1 and Fig.5, using classical DEC operator, one cannot construct a convergent Laplace-Beltrami operator in 1D even by using convergent geometric estimators. Indeed, theoretical analysis on 2D curves shows that one cannot expect convergence of the discretized Laplacian using exterior calculus toward the real Laplace-Beltrami on Riemannian Manifold. Recent work [LT14] studied the closeness between a digital surface in a space of dimension  $d$  and its underlying smooth manifold structure. They particularly proved convergence of discretized integrals using convergent estimated normals on digital surface.

In this paper, we address the problem of computing a digital Laplace-Beltrami operator on 2D digital curves by using the work of [BSW08] based on the heat equation. We focus our study on 1D curves embedded in 2D as even in this simple case, the discretization of the operator is not trivial. We give elements for proving its theoretical convergence and present an experimental evaluation that confirms its convergence property.

## 2. Digital Surfaces

In all the paper, we assume that there is some ideal shape  $\mathcal{M}$  in the space with a smooth topological boundary. We recall the definition of the Gauss Digitization process, which makes the link between the continuous shape and its digital approximation:

**Definition 1** Let  $h > 0$  be the sampling grid step. The *Gauss Digitization* of a shape  $\mathcal{M} \subset \mathbb{R}^d$  is defined as  $\mathcal{D}_h(\mathcal{M}) := \mathcal{M} \cap (h\mathbb{Z})^d$ , where  $d$  is the dimension of the space.

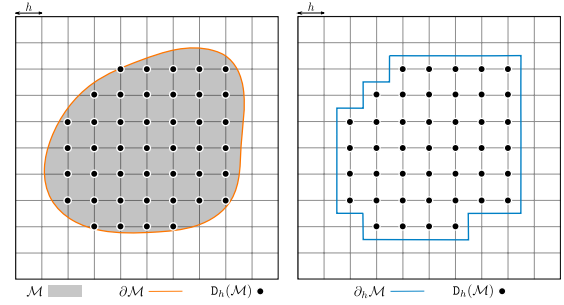
The digitization process has therefore a very simple scheme: it considers the discrete points of the infinite regular grid with sample rate  $h$  and keeps only the ones inside the shape (see Fig.2). We need also the notion of  $h$ -cube of a discrete point: for some  $z \in (h\mathbb{Z})^d$  it is the closed  $d$ -dimensional axis-aligned cube of  $\mathbb{R}^d$  centered on  $z$  with edge length  $h$ . We denote it by  $\mathcal{Q}_z^h$ . The collection of all  $h$ -cubes of the discrete points of a digitized shape is called its *h-cube embedding*. Finally, we arrive at the structure on which we will do computations:

**Definition 2** The digitized  $h$ -boundary of  $\mathcal{M}$ , denoted by  $\partial_h \mathcal{M}$ , is the topological boundary of the  $h$ -cube embedding of the Gauss digitization of  $\mathcal{M}$ :

$$\partial_h \mathcal{M} := \partial \left( \bigcup_{z \in \mathcal{D}_h(\mathcal{M})} \mathcal{Q}_z^h \right).$$

Geometrically it is a “staircased” surface that approximates the boundary of the continuous shape  $\mathcal{M}$ , and is also called a digital surface.

We wish to prove that our digital laplacian on the digitized boundary converge to the standard laplacian on the continuous shape boundary. The convergence setup will be the following: given an estimated operator on the digital surface and a real operator on the smooth structure, we look at pointwise convergence between those two quantities as the grid step  $h$  tends toward zero (that is when  $\partial_h \mathcal{M} \xrightarrow{h \rightarrow 0} \partial \mathcal{M}$  in the Hausdorff sense). More details about digital surfaces and their relationship with the underlying manifold can be found in [LT14].



**Figure 2:** Illustration of the notations used in this paper. A smooth shape  $\mathcal{M}$ , its Gauss digitization  $\mathcal{D}_h(\mathcal{M})$ , its  $h$ -cube embedding  $\mathcal{Q}_h \mathcal{D}_h(\mathcal{M})$  and its  $h$ -boundary  $\partial_h \mathcal{M}$

## 3. Discretization on triangular meshes

We first describe the work of [BSW08], which proposes a discrete laplacian for triangulated meshes. Let  $g_0 : \partial \mathcal{M} \mapsto \mathbb{R}$  be a smooth function (ie the initial temperature distribution on  $\partial \mathcal{M}$ ). Let  $g : \partial \mathcal{M} \times (0, T) \mapsto \mathbb{R}$  a time-dependent function which solves the partial differential equation called the Heat Equation:

$$\Delta g = \frac{\partial}{\partial t} g, \quad (1)$$

with initial condition  $g(0) := g(\cdot, 0) = g_0 : \partial \mathcal{M} \mapsto \mathbb{R}$ . The function  $g(t) := g(\cdot, t)$  is therefore the temperature distribution on  $\mathcal{M}$  at time  $t$ . In  $\mathbb{R}^d$ , one can find (see [Ros97] for example) an exact solution:

$$g(x, t) = \frac{1}{(4\pi t)^{\frac{d}{2}}} \int_{y \in \mathbb{R}^d} e^{-\frac{\|x-y\|^2}{4t}} g_0(y) dy, \quad (2)$$

where  $H_{\mathbb{R}^d}^t := \frac{1}{(4\pi t)^{\frac{d}{2}}} e^{-\frac{\|x-y\|^2}{4t}}$  is called the heat kernel. On an arbitrary manifold, the heat kernel involves a more complex construction (see [Ros97]). Fortunately, we know from [Mol75] that on a compact subset of  $\partial \mathcal{M}$  the real heat kernel is not too far from the  $\mathbb{R}^d$  one.

The very idea of [BSW08] is to make the heat equation not a property of the laplacian but a definition by injecting Eq.2 into Eq.1:

$$\Delta g(x, 0) = \lim_{t \rightarrow 0} \frac{1}{t} \left( \int_{y \in \partial \mathcal{M}} H_{\partial \mathcal{M}}^t(x, y) g(y) dy - f(x) \right) \quad (3a)$$

$$= \lim_{t \rightarrow 0} \frac{1}{t} \int_{y \in \partial \mathcal{M}} H_{\partial \mathcal{M}}^t(x, y) (g(y) - g(x)) dy. \quad (3b)$$

Then they give the following discretization of the laplacian at a vertex  $w$  of a triangular mesh  $K$ :

$$L_K^t g(w) := \frac{1}{4\pi t^2} \sum_{p \in V} e^{-\frac{\|p-w\|^2}{4t}} (g(p) - g(w)) \mathcal{A}(p), \quad (4)$$

where  $V$  is the set of vertices of  $K$  and  $\mathcal{A}$  represents the mean area of triangles that surround  $p$ . They prove a convergence theorem under the assumption that vertices of  $K$  lie on the underlying manifold, and some additional properties on  $K$  (see [BSW08]).

On digital structures, we must show how to compute both  $\|\cdot\|$  and  $\mathcal{A}$ . The former relies on computing convergent distances in 2D, and areas in 3D which is straightforward. The later becomes non-trivial in 3D, as one must know how to compute distance between two points on a surface. Therefore, as stated in the introduction, we present the discretization of the operator on 1D curves embedded in 2D.

#### 4. Discretization on digital curves

We describe how to discretize  $\|\cdot\|$  and  $\mathcal{A}$  from Eq.4. First, we adapt Eq.4 onto our digital curve:

$$\tilde{\Delta}_h g(\hat{w}) := \frac{1}{\sqrt{4\pi t}} \sum_{\hat{c}_i \in \partial_h \mathcal{M}} e^{-\frac{\tilde{d}(\hat{c}_i, \hat{w})}{4t}} (g(\hat{w}) - g(\hat{c}_i)) \mu_h(\hat{c}_i), \quad (5)$$

where  $\hat{w}$  and  $\hat{c}$  are vertices (element of dimension 0 of  $\partial_h \mathcal{M}$  as in Fig.4) on our digital curve,  $\tilde{d}$  is the approximated distance between two vertices and  $\mu_h$  is the elementary measure associated to a vertex. To measure such lengths we use the Integral Invariant normal estimator [LCL14], whose convergence speed is in  $\mathcal{O}(h^{\frac{2}{3}})$ . For each linel  $l$  of the digitized boundary (a linel is an edge joining two vertices), we define its measure  $\mu_h(l)$  as:

$$\mu_h(l) := \langle \hat{\mathbf{n}}(l), \mathbf{n}_e(l) \rangle, \quad (6)$$

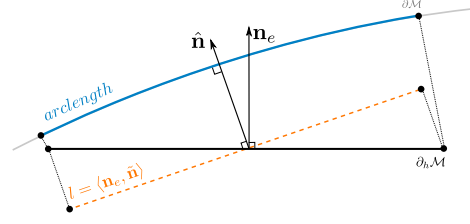
where  $\langle \cdot, \cdot \rangle$  is the dot product,  $\hat{\mathbf{n}}$  the estimated normal and  $\mathbf{n}_e$  the trivial normal to the linel (see Fig.3). It is known (see [CLR12]) that this measure gives a good approximation of the real projected arclength on the curve. Then we have:

$$\mu_h(\hat{c}_i) = \frac{\mu_h(l_i) + \mu_h(l_{i+1})}{2} \quad \text{and} \quad \tilde{d}(\hat{c}_i, \hat{w}) = \sum_j \mu_h(l_j), \quad (7)$$

where  $j$  lies in the range of indices corresponding to the linels between  $\hat{c}_i$  and  $\hat{w}$ . The  $\mu_h$  function is the digital analog of  $\mathcal{A}$  in Eq.4 on triangular meshes.

We state the following conjecture:

**Proposition 1** As  $h$  tends toward 0, the maximum pointwise error between the estimated laplacian described in Eq.5 and the real Laplace-Beltrami operator on the manifold tends toward 0 with a convergence speed in  $\mathcal{O}(h^{\frac{2}{3}})$ , if one chooses parameter  $t$  in  $\Theta(h^{\frac{2}{3}})$ .

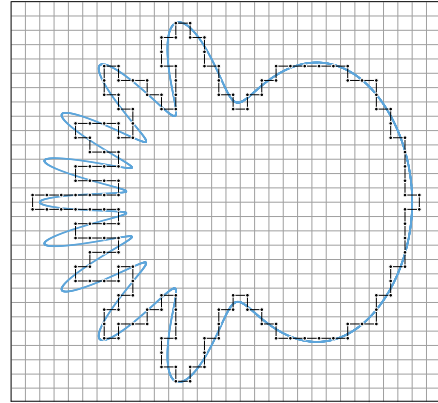


**Figure 3:** The length of a linel  $l$  is drawn in orange as the result of the dot product between the estimated normal  $\hat{\mathbf{n}}$  and the elementary normal  $\mathbf{n}_e$ . It is an approximation of the projected arclength, drawn in blue.

#### 5. Convergence results

We have implemented this digital Laplace-Beltrami operator with the DGtal library [DGt]. Input continuous smooth shapes are given as parameterized 2D closed curves , which are digitized using Gauss process (see an example in Fig.4). Normal vectors along the digitized boundary are then estimated. These vectors are used to compute both the length between any two points and the measure of each point. Finally, we compare the real laplacian defined on the input curve with our estimated laplacian (from Eq.5) on the digitized curve.

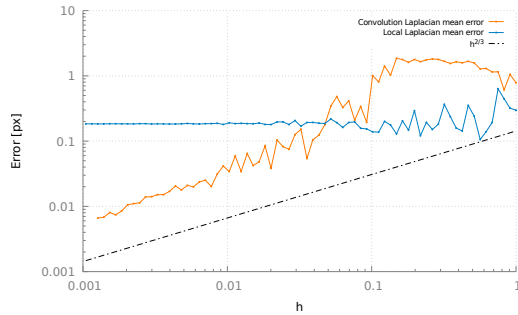
We wish to confirm Prop.1. The choice of  $t$  is given by the following conditions: it must be as close as possible to 0, but not too close, otherwise we would fall back into a local laplacian, and convergence would be impossible. Therefore we choose  $t$  to be some  $h^{\frac{2}{3}}$  for  $h$  the grid step. This choice of  $t$  leads to a convergence in  $\mathcal{O}(h^{\frac{2}{3}})$  as shown in Fig.5.



**Figure 4:** An example of a 2D curve in blue and its digitization in black with  $h = 0.1$ . Pointels are black dots, linels are black line segments.

#### 6. Conclusion and discussion

We derived from [BSW08] a convolutional laplacian based on the heat equation. We showed its digitization and implementation on 2D digitized curves using Gauss process. We have checked Prop.1 by experimentation. We did not provide theoretical analysis, as it is still an ongoing work: the idea is to reformulate the theorem stated in [LT14] into a local one



**Figure 5:** Convergence results using  $t = \frac{2}{3}$  on accelerated flower shape. Graphs in logarithm scale are to be read from right to left: as  $h$  tends toward 0, the error on laplacian estimation decreases with speed  $h^{\frac{2}{3}}$ .

and combine it with gaussian integration properties. Further work will include the implementation of this laplacian on 2D surfaces embedded in 3D as well as theoretical convergence proofs.

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