Quasi-Affine Transformation in Higher Dimension

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1 Introduction

In many computer vision and image processing applications, we are facing new constraints due to the image sizes both in dimension with 3-D and 3-D+t medical acquisition devices, and in resolution with VHR (Very High Resolution) satellite images. This article deals with high performance image transformation using quasi-affine transforms (QATs for short), which can be viewed as a discrete version of general affine transformations. QAT can approximate rotations and scalings, and in some specific cases, QAT may also be one-to-one and onto mappings from $\mathbb{Z}^n$ to $\mathbb{Z}^n$, leading to exact computations. A similar approach has been proposed in [1] with the notion of *Generalized Affine Transform*. The author demonstrated that any isometric transform (equivolume affine transform) in dimension n can be decomposed into $(n^2 - 1)$ fundamental skew transformations which parameters can be optimized to obtain a reversible transformation. Note that the bound has been improved to $(3n - 3)$ in [2]. This approach is a generalization of the decomposition of rotations in dimension 2 into three shears (see for example [3]). In this paper, we focus on QATs since they provide a wider class of transformation (contracting and dilating transformations are allowed). Furthermore, we investigate direct transformation algorithms without the need of intermediate computations.

In dimension 2, the QAT appeared in several articles [4,5,6,7,8]. To summarize the main results, the authors have proved several arithmetical results on QAT in 2-D leading to efficient transformation algorithms. More precisely, thanks to periodic properties of pavings induced by the reciprocal map, the image transformation can be obtained using a set of precomputed canonical pavings. In this paper, we focus on a theoretical analysis of n-dimensional QAT. The idea is to investigate fundamental results in order to be able to design efficient transformation algorithms in dimension 2 or 3 as detailed in [9]. More precisely, we demonstrate the arithmetical and periodic structures embedded in $n-$dimensional QAT. Due to the space limitation, we only details the proofs of the main theorems (other proofs are available in the technical report [9]).

During the reviewing process of the paper, the article [10] was brought to our attention. Even if the framework differs, some results presented here can also be found in [10]. However, the closed formulas for the paving periodicity presented in Section 3.2 make our contribution complementary.

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In Section 2, we first detail preliminary notations and properties. Then, Section 3 contains the main theoretical results leading to a generic n-D transformation algorithm sketched in Section 4.

2 Preliminaries

2.1 Notations

Before we introduce arithmetical properties of QAT in higher dimension, we first detail the notations considered in this paper. Let n denotes the dimension of the considered space, V_i denotes the i^{th} coordinate of vector V, and $M_{i,j}$ denotes the $(i,j)^{th}$ coefficient of matrix M. We use the notation $\gcd(a, b, \ldots)$ for the greatest common divisor of an arbitrary number of arguments, and $\text{lcm}(a, b, \ldots)$ for their least common multiple.

Let $\left[\frac{a}{b}\right]$ denotes the quotient of the euclidean division of a by b, that is the integer $q \in \mathbb{Z}$ such that $a = bq + r$ satisfying $0 \leq r < |b|$ regardless of the sign of b[1]. We consider the following generalization to $n-$dimensional vectors:

$$\left[\frac{V}{b}\right] = \begin{pmatrix} \left[\frac{V_0}{b}\right] \\ \vdots \\ \left[\frac{V_{n-1}}{b}\right] \end{pmatrix} \text{ and } \left\{\frac{V}{b}\right\} = \begin{pmatrix} \left\{\frac{V_0}{b}\right\} \\ \vdots \\ \left\{\frac{V_{n-1}}{b}\right\} \end{pmatrix} . \tag{1}$$

2.2 Quasi-Affine Transformation Definitions

Defined in dimension 2 in [4,5,6,7,8], we consider a straightforward generalization to $\mathbb{Z}^n$ spaces.

Definition 1 (QAT). *A quasi-affine transformation is a triple* $(\omega, M, \boldsymbol{V}) \in \mathbb{Z} \times M_n(\mathbb{Z}) \times \mathbb{Z}^n$ *(we assume that* $\det(M) \neq 0$*). The associated application is:*

$$\begin{aligned} \mathbb{Z}^n &\longrightarrow \mathbb{Z}^n \\ X &\longmapsto \left[\frac{MX + V}{\omega}\right] . \end{aligned}$$

And the associated affine application is:

$$\begin{aligned} \mathbb{R}^n &\longrightarrow \mathbb{R}^n \\ X &\longmapsto \frac{MX + V}{\omega} . \end{aligned}$$

In other words, a QAT is the composition of the associated affine application and the integer part floor function.

Definition 2. *A QAT is said to be contracting if* $\omega^n > |\det(M)|$*, otherwise it is said to be dilating.*

[1] $\left\{\frac{a}{b}\right\}$ denotes the corresponding remainder $\left\{\frac{a}{b}\right\} = a - b\left[\frac{a}{b}\right]$.

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In other words, a QAT is contracting if and only if the associated affine application is contracting. Note that if $\omega^n = |\det(M)|$, the QAT is dilating, even if the associated affine application is an isometry.

Definition 3. *The inverse of a QAT* (ω, M, V) *is the QAT:*

$$(\det(M), \omega \operatorname{com}(M)^t, -\operatorname{com}(M)^t V)\,, \tag{2}$$

where ^{t} *denotes the transposed matrix and* $\operatorname{com}(M)$ *the co-factor matrix of* M^2.

The associated affine application of the inverse of a QAT is therefore the inverse of the affine application associated to the QAT. However, due to the nested floor function, the composition $f \cdot f^{-1}$ is not the identity function in the general case.

3 QAT Properties in Higher Dimensions

Without loss of generality, we suppose that the QAT is contracting.

3.1 Pavings of a QAT

A key feature of a QAT in dimension 2 is the paving induced by the reciprocal map of a discrete point. In the following, we adapt the definitions in higher dimensions and prove that a QAT in $\mathbb{Z}^n$ also carries a periodic paving.

Definition 4 (Paving). *Let* f *be a QAT. For* $Y \in \mathbb{Z}^n$*, we denote:*

$$P_Y = \{X \in \mathbb{Z}^n \mid f(X) = Y\} = f^{-1}(Y)\,, \tag{3}$$

P_Y *is called order 1 paving of index* Y *of* f.

P_Y can be interpreted as a subset of $\mathbb{Z}^n$ (maybe empty) that corresponds to the reciprocal map of Y by f. We easily show that the set of pavings of a QAT forms a paving of the considered space (see Fig. 1). In dimension 2, this definition exactly coincides with previous ones [4,6,7,8,5].

Definition 5. P_Y *is said arithmetically equivalent to* P_Z *(denoted* $P_Y \equiv P_Z$*) if:*

$$\forall X \in P_Y, \exists X' \in P_Z, \left\{\frac{MX + V}{\omega}\right\} = \left\{\frac{MX' + V}{\omega}\right\}. \tag{4}$$

Again, this definition is equivalent (as shown below) to those given in the literature.

Theorem 1. *The equivalence relationship is symmetric,* i.e.*:*

$$P_Y \equiv P_Z \Leftrightarrow P_Z \equiv P_Y\,. \tag{5}$$

Proof. The proof is given [9].

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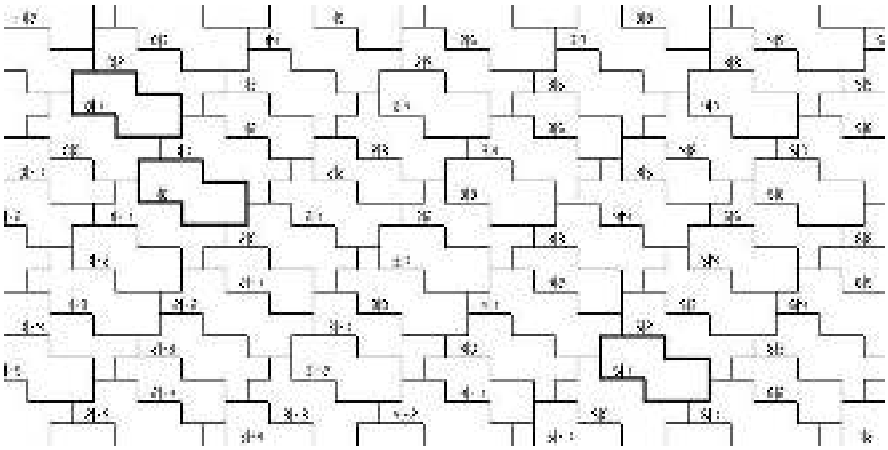

Fig. 1. Pavings of the QAT $\left(84, \begin{pmatrix} 12 & -11 \\ 18 & 36 \end{pmatrix}, \begin{pmatrix} 150 \\ -500 \end{pmatrix}\right)$ with their indexes (in 2D, these are the couples: $Y = (x, y)$).

$P_{0,1}$		$P_{5,1}$		$P_{1,0}$	
X	$\left\{\frac{MX+V}{\omega}\right\}$	X	$\left\{\frac{MX+V}{\omega}\right\}$	X	$\left\{\frac{MX+V}{\omega}\right\}$
$\begin{pmatrix} 3 \\ 15 \end{pmatrix}$	$\begin{pmatrix} 21 \\ 10 \end{pmatrix}$	$\begin{pmatrix} 27 \\ 3 \end{pmatrix}$	$\begin{pmatrix} 21 \\ 10 \end{pmatrix}$	$\begin{pmatrix} 6 \\ 11 \end{pmatrix}$	$\begin{pmatrix} 17 \\ 4 \end{pmatrix}$
$\begin{pmatrix} 5 \\ 14 \end{pmatrix}$	$\begin{pmatrix} 56 \\ 10 \end{pmatrix}$	$\begin{pmatrix} 29 \\ 2 \end{pmatrix}$	$\begin{pmatrix} 56 \\ 10 \end{pmatrix}$	$\begin{pmatrix} 8 \\ 10 \end{pmatrix}$	$\begin{pmatrix} 52 \\ 4 \end{pmatrix}$
$\begin{pmatrix} 4 \\ 15 \end{pmatrix}$	$\begin{pmatrix} 33 \\ 28 \end{pmatrix}$	$\begin{pmatrix} 28 \\ 3 \end{pmatrix}$	$\begin{pmatrix} 33 \\ 28 \end{pmatrix}$	$\begin{pmatrix} 7 \\ 11 \end{pmatrix}$	$\begin{pmatrix} 29 \\ 22 \end{pmatrix}$
$\begin{pmatrix} 6 \\ 14 \end{pmatrix}$	$\begin{pmatrix} 68 \\ 28 \end{pmatrix}$	$\begin{pmatrix} 30 \\ 2 \end{pmatrix}$	$\begin{pmatrix} 68 \\ 28 \end{pmatrix}$	$\begin{pmatrix} 9 \\ 10 \end{pmatrix}$	$\begin{pmatrix} 64 \\ 22 \end{pmatrix}$
$\begin{pmatrix} 3 \\ 16 \end{pmatrix}$	$\begin{pmatrix} 10 \\ 46 \end{pmatrix}$	$\begin{pmatrix} 27 \\ 4 \end{pmatrix}$	$\begin{pmatrix} 10 \\ 46 \end{pmatrix}$	$\begin{pmatrix} 6 \\ 12 \end{pmatrix}$	$\begin{pmatrix} 6 \\ 40 \end{pmatrix}$
$\begin{pmatrix} 5 \\ 15 \end{pmatrix}$	$\begin{pmatrix} 45 \\ 46 \end{pmatrix}$	$\begin{pmatrix} 29 \\ 3 \end{pmatrix}$	$\begin{pmatrix} 45 \\ 46 \end{pmatrix}$	$\begin{pmatrix} 8 \\ 11 \end{pmatrix}$	$\begin{pmatrix} 41 \\ 40 \end{pmatrix}$
$\begin{pmatrix} 7 \\ 14 \end{pmatrix}$	$\begin{pmatrix} 80 \\ 46 \end{pmatrix}$	$\begin{pmatrix} 31 \\ 2 \end{pmatrix}$	$\begin{pmatrix} 80 \\ 46 \end{pmatrix}$	$\begin{pmatrix} 10 \\ 10 \end{pmatrix}$	$\begin{pmatrix} 76 \\ 40 \end{pmatrix}$
$\begin{pmatrix} 4 \\ 16 \end{pmatrix}$	$\begin{pmatrix} 22 \\ 64 \end{pmatrix}$	$\begin{pmatrix} 28 \\ 4 \end{pmatrix}$	$\begin{pmatrix} 22 \\ 64 \end{pmatrix}$	$\begin{pmatrix} 7 \\ 12 \end{pmatrix}$	$\begin{pmatrix} 18 \\ 58 \end{pmatrix}$
$\begin{pmatrix} 6 \\ 15 \end{pmatrix}$	$\begin{pmatrix} 57 \\ 64 \end{pmatrix}$	$\begin{pmatrix} 30 \\ 3 \end{pmatrix}$	$\begin{pmatrix} 57 \\ 64 \end{pmatrix}$	$\begin{pmatrix} 9 \\ 11 \end{pmatrix}$	$\begin{pmatrix} 53 \\ 58 \end{pmatrix}$
$\begin{pmatrix} 5 \\ 16 \end{pmatrix}$	$\begin{pmatrix} 34 \\ 82 \end{pmatrix}$	$\begin{pmatrix} 29 \\ 4 \end{pmatrix}$	$\begin{pmatrix} 34 \\ 82 \end{pmatrix}$	$\begin{pmatrix} 8 \\ 12 \end{pmatrix}$	$\begin{pmatrix} 30 \\ 76 \end{pmatrix}$
$\begin{pmatrix} 7 \\ 15 \end{pmatrix}$	$\begin{pmatrix} 69 \\ 82 \end{pmatrix}$	$\begin{pmatrix} 31 \\ 3 \end{pmatrix}$	$\begin{pmatrix} 69 \\ 82 \end{pmatrix}$	$\begin{pmatrix} 10 \\ 11 \end{pmatrix}$	$\begin{pmatrix} 65 \\ 76 \end{pmatrix}$

Table 1. Pavings of index $(0, 1)$, $(5, 1)$ and $(1, 0)$ of the QAT $\left(84, \begin{pmatrix} 12 & -11 \\ 18 & 36 \end{pmatrix}, \begin{pmatrix} 150 \\ -500 \end{pmatrix}\right)$

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Figure 1 illustrates arithmetically equivalent pavings: the pavings of index $(0, 1)$ and $(5, 1)$ are arithmetically equivalent (see Table 1).

Definition 6. *P_Y and P_Z are said geometrically equivalent if:*

$$\exists \boldsymbol{v} \in \mathbb{Z}^n, P_Y = T_{\boldsymbol{v}} P_Z \,, \tag{6}$$

where $T_{\boldsymbol{v}}$ denotes the translation of vector $\boldsymbol{v}$.

In Figure 1, the pavings of indexes $(0, 1)$ and $(1, 0)$ are geometrically equivalent. In image processing purposes, when we want to transform a $n-$dimensional image by a QAT, geometrically equivalent pavings will allow us to design fast transformation algorithms.

Theorem 2. *If $P_Y \equiv P_Z$, then P_Y and P_Z are geometrically equivalent. Since $P_Y \equiv P_Z$, there exists $X \in P_Y$ and $X' \in P_Z$ such that:*

$$\left\{\frac{MX + V}{\omega}\right\} = \left\{\frac{MX' + V}{\omega}\right\} .$$

Then $\boldsymbol{v} = X - X'$ is the translation vector:

$$P_Y = T_{\boldsymbol{v}} P_Z \,.$$

In dimension 2, this theorem is also proved in [7].

Proof. The proof is given in [9].

In a computational point of view, if a paving P_Y has been already computed, and if we know that $P_Y \equiv P_Z$, then P_Z can be obtained by translation of P_Y. In Figure 1, the pavings of index $(0, 1)$ and $(5, 1)$ are arithmetically equivalent (see Table 1), therefore they are geometrically equivalent (as we can check on the figure). Note that the inverse implication is false: in Figure 1, the pavings of index $(0, 1)$ and $(1, 0)$ are geometrically equivalent but they are not arithmetically equivalent (see Table 1).

3.2 Paving Periodicity

Definition 7. $\forall 0 \leq i < n$, *We define the set $\mathcal{A}_i$ as follows:*

$$\mathcal{A}_i = \{\alpha \in \mathbb{N}^* \mid \exists (\beta_j)_{0 \leq j < i} \in \mathbb{Z}^i, \forall (y_0, \ldots, y_{n-1}) \in \mathbb{Z}^n, \\ P_{y_0,\ldots,y_i+\alpha,\ldots,y_{n-1}} \equiv P_{y_0+\beta_0,\ldots,y_{i-1}+\beta_{i-1},y_i,\ldots,y_{n-1}}\}$$

Theorem 3 (Perdiodicity). *The set of QAT pavings is $n-$periodic, in other words*

$$\forall 0 \leq i < n, \mathcal{A}_i \neq \emptyset$$

[2] Remind that $M \operatorname{com}(M)^t = \operatorname{com}(M)^t M = \det(M) I_n$.

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Proof. The proof is given in Sect. A.1.

If we consider $\alpha = |det(M)|$ as in Sect. A.1, we have demonstrated the periodic structure of QAT pavings since $P_Y \equiv P_{Y+\alpha e_i}$ for each i. We investigate now the quantities α_i which are minimal for each dimension i,

Definition 8. *$\forall 0 \leq i < n$, let us consider $\alpha_i = \min(\mathcal{A}_i)$. We define $\{\beta_j^i\}_{0\leq j<i} \in \mathbb{Z}^i$ and $U_i \in \mathbb{Z}^n$ such that*

$$\forall (y_0, \ldots, y_{n-1}) \in \mathbb{Z}^n, P_{y_0,\ldots,y_i+\alpha_i,\ldots,y_{n-1}} = T_{\boldsymbol{U_i}} P_{y_0+\beta_0^i,\ldots,y_{i-1}+\beta_{i-1}^i,y_i,\ldots,y_{n-1}} .$$

Thanks to Theorem 2 and using notations of Def. 8, let $X \in P_{y_0,\ldots,y_i+\alpha_i,\ldots,y_{n-1}}$ and $X' \in P_{y_0+\beta_0^i,\ldots,y_{i-1}+\beta_{i-1}^i,y_i,\ldots,y_{n-1}}$, such that $\left\{\frac{MX+V}{\omega}\right\} = \left\{\frac{MX'+V}{\omega}\right\}$. Then, we have $U_i = X - X'$.

The quantities α_i, β_j^i and U_i can be computed in dimension 2 and 3 (see [9]) by using greatest common divisors and Euclide's algorithm. The computation of α_2 in 3D already involves a consequent number of intermediate variables, that is why these computations seem to be hard to generalize in arbitrary dimension. Therefore, we will suppose here that quantities α_i, β_j^i and U_i are given.

To paraphrase above results, α_i and its associated U_i and $\{\beta_j^i\}$ allows us to *reduce* the i^{th} component of Y while preserving the geometrically equivalence relationship. If we repeat this *reduction* process to each component from $n-1$ down-to 0, we construct a point Y^0 such that P_Y and P_{Y^0} are geometrically equivalent. The following theorem formalizes this principle and defines the initial period paving P_{Y^0}.

Theorem 4. *$\forall (y_0, \ldots, y_{n-1}) \in \mathbb{Z}^n$, we have $P_{y_0,\ldots,y_{n-1}} = T_W P_{y_0^0,\ldots,y_{n-1}^0}$ with*

$$W = \sum_{i=0}^{n-1} w_i U_i$$
$$\textit{and } \forall n > i \geq 0, \begin{cases} w_i = \left\lfloor \frac{y_i + \sum_{j=i+1}^{n-1} w_j \beta_i^j}{\alpha_i} \right\rfloor \\ y_i^0 = \left\{ \frac{y_i + \sum_{j=i+1}^{n-1} w_j \beta_i^j}{\alpha_i} \right\} \end{cases} .$$

Proof. The proof is given in Sect. A.2.

Therefore, if we already computed the pavings $P_{y_0^0,\ldots,y_{n-1}^0}$ for $0 \leq y_i^0 < \alpha_i$, we can obtain any paving by translation of one of these pavings.

3.3 Super-paving of a QAT

We now describe how to compute these initial period pavings based on the notion of super-paving (see Fig. 2).

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Definition 9. *A super-paving of a QAT is the set $\mathcal{P}$ such that*

$$\mathcal{P} = \bigcup_{0 \leq Y^0 < (\alpha_0, \ldots, \alpha_{n-1})} P_{Y^0}$$

In other words, the super-paving is the union of all pavings of the initial period. In dimension 2, this definition coincides with definitions given in [7,6,8].

Theorem 5. *$\mathcal{P}$ is the paving $P_{(0,\ldots,0)}$ of the QAT defined by:*

$$\left(\omega \, \mathrm{lcm}_{0 \leq i < n}(\alpha_i), \begin{pmatrix} \theta_0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \theta_{n-1} \end{pmatrix} M, \begin{pmatrix} \theta_0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \theta_{n-1} \end{pmatrix} V \right) , \tag{7}$$

with $\forall 0 \leq i < n$,

$$\theta_i = \frac{\mathrm{lcm}_{0 \leq j < n}(\alpha_j)}{\alpha_i} .$$

Proof. The proof is given in [9].

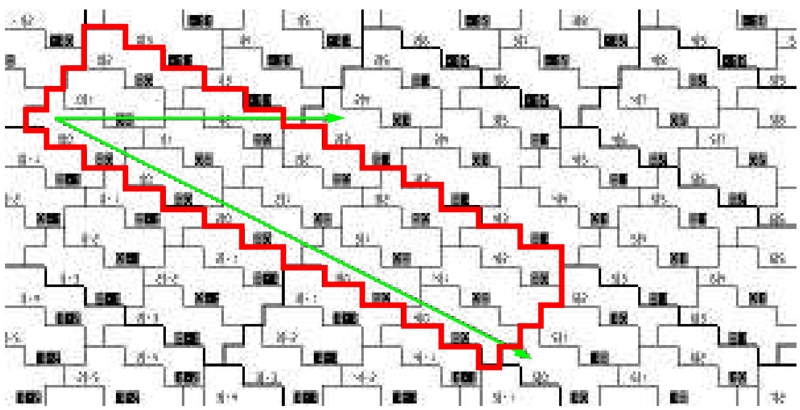

Fig. 2. Super-paving decomposition of the QAT defined in Fig. 1. Arrows illustrate a basis of the periodic structure, numbers on white background are the y_i, and numbers on black background are the w_i.

Hence, we can associate a canonical paving to each point of the super-paving. More precisely, the super-paving allows us to compute the equivalence classes for the arithmetical equivalence relationship between two pavings.

3.4 Paving Construction

In this section, we focus on an arithmetic paving construction algorithm. Hence, using the results of the previous section, such a construction algorithm will be used to compute canonical pavings in the super-paving.

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Definition 10. *The matrix T is the Hermite Normal Form of the QAT matrix M if:*

- *T is upper triangular, with coefficients $\{T_{ij}\}$ such that $T_{ii} > 0$;*
- *$\exists H \in GL_n(\mathbb{Z}), MH = T$.*

If M is nonsingular integer matrix, the Hermite Normal Form exists. Note also that if $H \in M_n(\mathbb{Z})$, then $H \in GL_n(\mathbb{Z}) \Leftrightarrow |\det(H)| = 1$.

For example, given $\begin{pmatrix} 12 & -11 \\ 18 & 36 \end{pmatrix}$, we have: $\begin{pmatrix} 12 & -11 \\ 18 & 36 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 35 & 12 \\ 0 & 18 \end{pmatrix}$.

Using the Hermite Normal Form, we can design a fast paving computation algorithm formalized in the following theorem:

Theorem 6. *$\forall Y \in \mathbb{Z}^n$, let $MH = T$ be the Hermite Normal Form of the QAT matrix M, then*

$$P_Y = \{HX \mid \forall n > i \geq 0, A_i(X_{i+1}, \ldots, X_{n-1}) \leq X_i < B_i(X_{i+1}, \ldots, X_{n-1})\} \tag{8}$$

With

$$A_i(X_{i+1}, \ldots, X_{n-1}) = -\left\lfloor \frac{-\omega Y_i + \sum_{j=i+1}^{n-1} T_{i,j} X_j + V_i}{T_{i,i}} \right\rfloor ,$$

$$B_i(X_{i+1}, \ldots, X_{n-1}) = -\left\lfloor \frac{-\omega (Y_i + 1) + \sum_{j=i+1}^{n-1} T_{i,j} X_j + V_i}{T_{i,i}} \right\rfloor .$$

In [7,8], a similar result can be obtained in dimension 2. However, the Hermite Normal Form formalization allows us to prove the result in higher dimension. To prove Theorem 6, let us first consider the following technical lemma:

Lemma 1. *Let $a, b, q, x \in \mathbb{Z}$ with $q > 0$, then*

$$a \leq qx < b \Leftrightarrow -\left\lfloor \frac{-a}{q} \right\rfloor \leq x < -\left\lfloor \frac{-b}{q} \right\rfloor .$$

Proof. The proof is detailed in [9].

We can now prove the Theorem 6 (cf Sect. A.3). The implementation of the construction algorithm is straightforward: we just have to consider n nested loop such that the loop with level i goes from A_i to B_i quantities. See [9] for details in dimension 2 and 3.

4 A Generic QAT Algorithm

In Algorithm 1, we give the generic algorithm applying a contracting QAT f to an image $\mathcal{A}$ (see Fig. 3). The principle is that we give to each pixel Y of image $\mathcal{B}$ the average color of the paving P_Y in image $\mathcal{A}$. If f is a dilating QAT, we obtain the very similar Algorithm 2 which principle is that firstly we replace f

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Algorithm 1: Generic QAT algorithm for a contracting QAT

```
Input: a contracting QAT f := (ω, M, V), an image A : Z^n → Z
Output: a transformed image B : Z^n → Z
Compute the Hermite Normal Form of the matrix M;
Determine the minimal periodicities {α_i} and vectors {U_i};
Use Theorems 5 and 6 to compute the canonical pavings in the super-paving P;
foreach Y ∈ B do
    Find Y^0 and W such that P_Y = T_W P_{Y^0} by using Theorem 4;
    sum ← 0;
    foreach Z ∈ P_{Y^0} do
        c ← A(T_W Z);            // we read the color in the initial image
        sum ← sum + c;
    B(Y) ← sum/|P_{Y^0}|;                          // we set the color
```

Algorithm 2: Generic QAT algorithm for a dilating QAT

```
Input: a dilating QAT f := (ω, M, V), an image A : Z^n → Z
Output: a transformed image B : Z^n → Z
Replace f with f^{-1};
Compute the Hermite Normal Form of the matrix M;
Determine the minimal periodicities {α_i} and vectors {U_i};
Use Theorems 5 and 6 to compute the canonical pavings in the super-paving P;
foreach Y ∈ A do
    Find Y^0 and W such that P_Y = T_W P_{Y^0} by using Theorem 4;
    c ← A(Y);                    // we read the color in the initial image
    foreach Z ∈ P_{Y^0} do
        B(T_W Z) ← c;                              // we set the color
```

with f^{-1}, and then we give the color of each pixel Y of image $\mathcal{A}$ to each pixel of P_Y in image $\mathcal{B}$. In both algorithms, some elements cannot be computed in arbitrary dimension n. Indeed, even if there exist algorithms to compute the Hermite Normal Form of an arbitrary square integer matrix [11], there is no generic algorithm to obtain the minimal periodicities $\{\alpha_i\}$.

In [9], we detail the computation of the minimal periodicities in dimension 2 and 3. We also demonstrate with a complete experimental analysis that algorithms 1 and 2 outperform classical techniques to transform an image by affine functions.

5 Conclusion and Future Works

In this paper, we have demonstrated that in higher dimension, Quasi-Affine Transformations contain arithmetical properties leading to the fact that the induced pavings are $n-$periodic. Furthermore, thanks to the Hermite Normal Form of the QAT matrix, we have presented efficient algorithms to construct a

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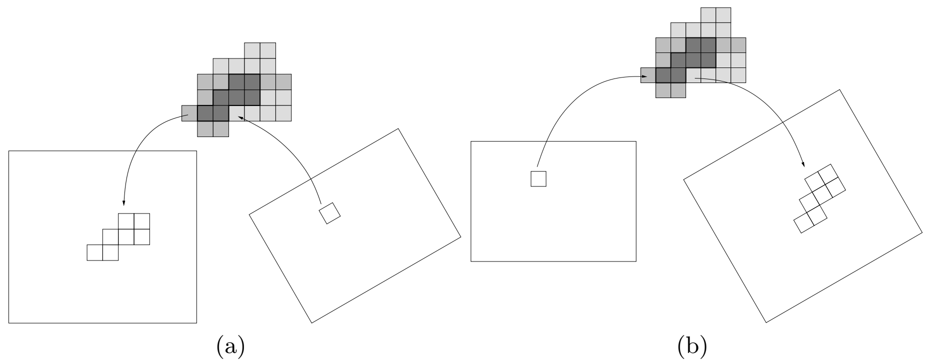

Fig. 3. Illustration in dimension 2 of the QAT algorithm when f is contracting (a) and dilating (b). In both cases, we use the canonical pavings contained in the super-paving to speed-up the transformation.

given paving and to compute a set of canonical pavings. From all these theoretical results, fast transformation algorithms have been designed in [9]. However, several future works exist. First, as detailed in Sections 3.1 and 3.3, the super-paving of a QAT contains a set of arithmetically distinct pavings. However, two arithmetical distinct pavings may be geometrically equivalent. Hence, a subset of the super-paving may be enough to design a fast algorithm. In dimension 2, in [4,6,7,8], the authors have investigated another structure, so-called generative strip, which removes some arithmetical distinct pavings whose geometry are identical. Even if the generalization in higher dimension of this object is not trivial, it may be interesting to investigate theoretical techniques to reduce the canonical paving set. Finally, a generic algorithm to compute the minimal periodicities is challenging.

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A Appendix: Proofs

A.1 Theorem 3

Proof. Given $0 \leq i < n$, let us suppose that $\forall 0 \leq j < i, \beta_j = 0$ and $\alpha = |\det(M)|$. Let $Y \in \mathbb{Z}^n$, $X \in P_Y$, and

$$X' = X + \frac{\det(M)}{|\det(M)|}\omega \operatorname{com}(M)^t e_i$$

with e_i being the $i-$th vector of the canonical basis of $\mathbb{R}^n$. We prove that $\mathcal{A}_i \neq \emptyset$ since $P_Y \equiv P_{Y+\alpha e_i}$:

$$\begin{aligned} MX' + V = MX + V + M\frac{\det(M)}{|\det(M)|}\omega \operatorname{com}(M)^t e_i &= \omega Y + \left\{\frac{MX+V}{\omega}\right\} + \omega \frac{\det(M)}{|\det(M)|} M \operatorname{com}(M)^t e_i \\ &= \omega Y + \left\{\frac{MX+V}{\omega}\right\} + \omega \frac{\det(M)}{|\det(M)|} det(M) e_i \\ &= \omega Y + \left\{\frac{MX+V}{\omega}\right\} + \omega |\det(M)| e_i = \omega(Y + \alpha e_i) + \left\{\frac{MX+V}{\omega}\right\} \end{aligned}$$

Hence, $\left\{\frac{MX'+V}{\omega}\right\} = \left\{\frac{MX+V}{\omega}\right\}$ and thus $X' \in P_{Y+\alpha e_i}$. Finally, $P_{Y+\alpha e_i} \equiv P_Y$ which proves that $\alpha \in \mathcal{A}_i$.□

A.2 Theorem 4

Proof. Let us denote $\mathcal{T}(j)$ the proposition

$$P_{y_0,\ldots,y_{n-1}} = T_{\sum_{i=j}^{n-1} w_i U_i} P_{y_0 + \sum_{i=j}^{n-1} w_i \beta_0^i, \ldots, y_{j-1} + \sum_{i=j}^{n-1} w_i \beta_{j-1}^i, y_j^0, \ldots, y_{n-1}^0} .$$

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We consider the following induction: given $n > p \geq 0$, we suppose $\mathcal{T}(p+1)$ and prove $\mathcal{T}(p)$. As a consequence of Def. 8,

$$\forall (z_1, \ldots, z_{n-1}) \in \mathbb{Z}^n, P_{z_0,\ldots,z_p+\alpha_p,\ldots,z_{n-1}} = T_{U_p} P_{z_0+\beta_0^p,\ldots,z_{p-1}+\beta_{p-1}^p,z_p,\ldots,z_{n-1}} .$$

Hence, $\forall k \in \mathbb{Z}$, we have

$$P_{z_0,\ldots,z_p+k\alpha_p,\ldots,z_{n-1}} = T_{kU_p} P_{z_0+k\beta_0^p,\ldots,z_{p-1}+k\beta_{p-1}^p,z_p,\ldots,z_{n-1}} .$$

With $\begin{cases} k = w_p \\ \forall 0 \leq j < p, z_j = y_j + \sum_{i=p+1}^{n-1} w_i \beta_j^i \\ \forall p \leq j < n, z_j = y_j^0 \end{cases}$, we obtain

$$P_{y_0+\sum_{i=p+1}^{n-1} w_i\beta_0^i,\ldots,y_{p-1}+\sum_{i=p+1}^{n-1} w_i\beta_{p-1}^i,y_p^0+w_p\alpha_p,y_{p+1}^0,\ldots,y_{n-1}^0} = T_{w_pU_p} P_{y_0+\sum_{i=p}^{n-1} w_i\beta_0^i,\ldots,y_{p-1}+\sum_{i=p}^{n-1} w_i\beta_{p-1}^i,y_p^0,\ldots,y_{n-1}^0} . \tag{9}$$

Since $\mathcal{T}(p+1)$ is true, and since $y_p + \sum_{j=p+1}^{n-1} w_j \beta_p^j = y_p^0 + \alpha_p w_p$, we have

$$P_{y_0,\ldots,y_{n-1}} = T_{\sum_{i=p+1}^{n-1} w_i U_i} P_{y_0+\sum_{i=p+1}^{n-1} w_i\beta_0^i,\ldots,y_{p-1}+\sum_{i=p+1}^{n-1} w_i\beta_{p-1}^i,y_p^0+\alpha_p w_p,y_{p+1}^0,\ldots,y_{n-1}^0} \tag{10}$$

We can identify the left side of Eq. (9) to the right part of the right side of (10), summing up the translation vectors leads to $\mathcal{T}(p)$. Since $\mathcal{T}(n) : P_{y_0,\ldots,y_{n-1}} = T_0 P_{y_0,\ldots,y_{n-1}}$ is trivial, we prove $\mathcal{T}(0)$ and thus the theorem.□

A.3 Theorem 6

Proof. Let $X, Y, Z \in \mathbb{Z}^n$ such that $X = H^{-1}Z$,

$$\begin{aligned} Z \in P_Y &\Leftrightarrow \left\lfloor \frac{MZ+V}{\omega} \right\rfloor = Y \\ &\Leftrightarrow \left\lfloor \frac{TX+V}{\omega} \right\rfloor = Y \\ &\Leftrightarrow \forall 0 \leq i < n, \omega y_i \leq \sum_{j=i}^{n-1} T_{i,j} X_j + V_i < \omega(y_i+1) \\ &\Leftrightarrow \forall 0 \leq i < n, \omega y_i - \sum_{j=i+1}^{n-1} T_{i,j} X_j - V_i \leq T_{i,i} X_i < \omega(y_i+1) - \sum_{j=i+1}^{n-1} T_{i,j} X_j - V_i . \end{aligned}$$

Thanks to Lemma 1, $Z \in P_Y$ is equivalent to

$$\forall 0 \leq i < n, A_i(X_{i+1}, \ldots, X_{n-1}) \leq X_i < B_i(X_{i+1}, \ldots, X_{n-1}) .\square$$

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