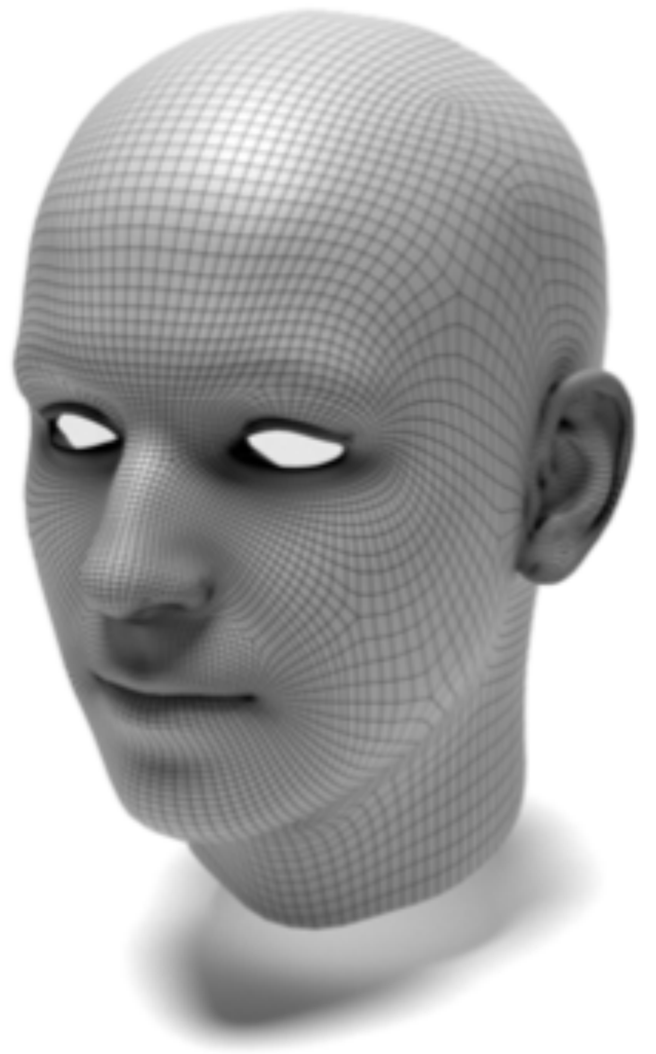
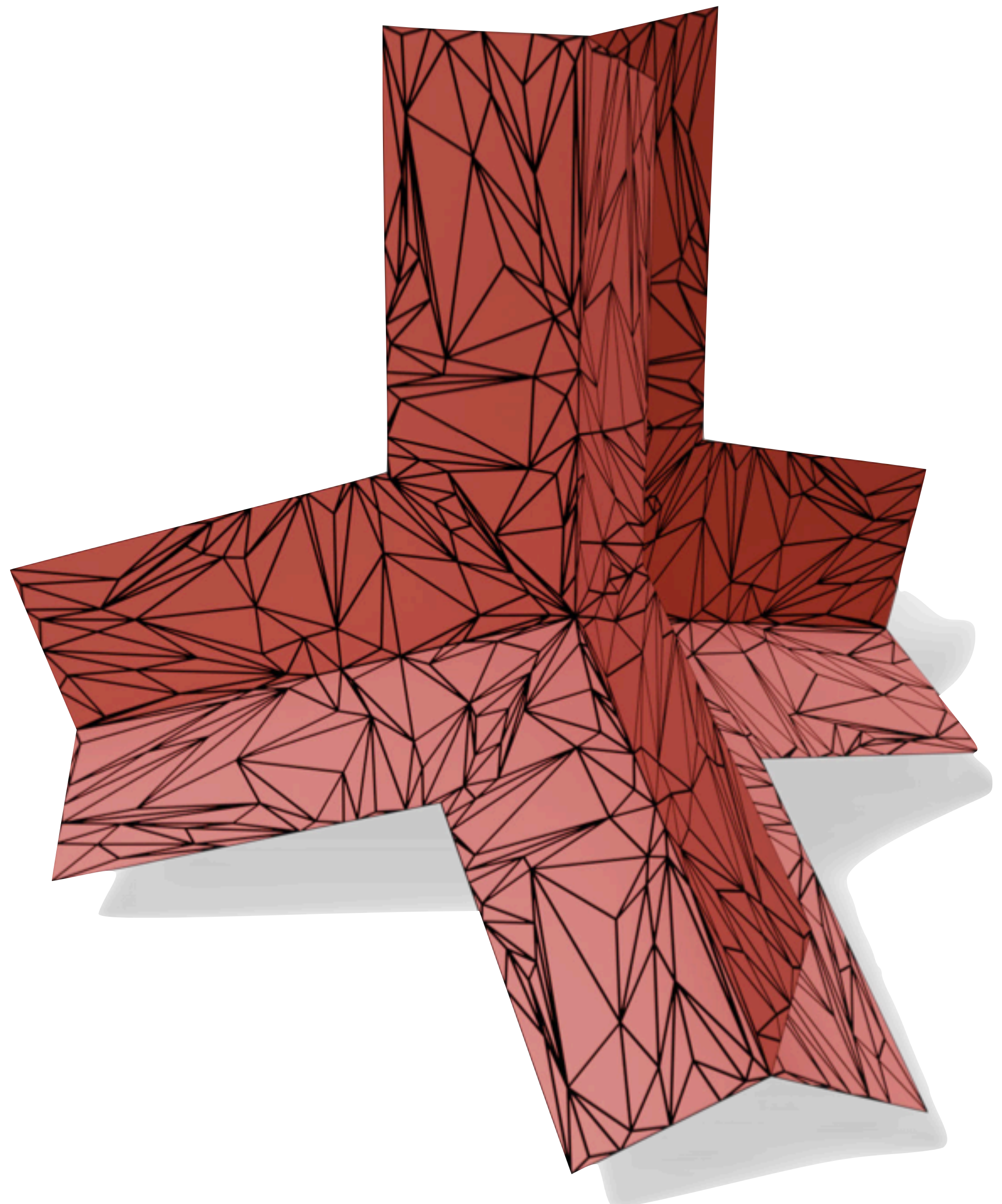
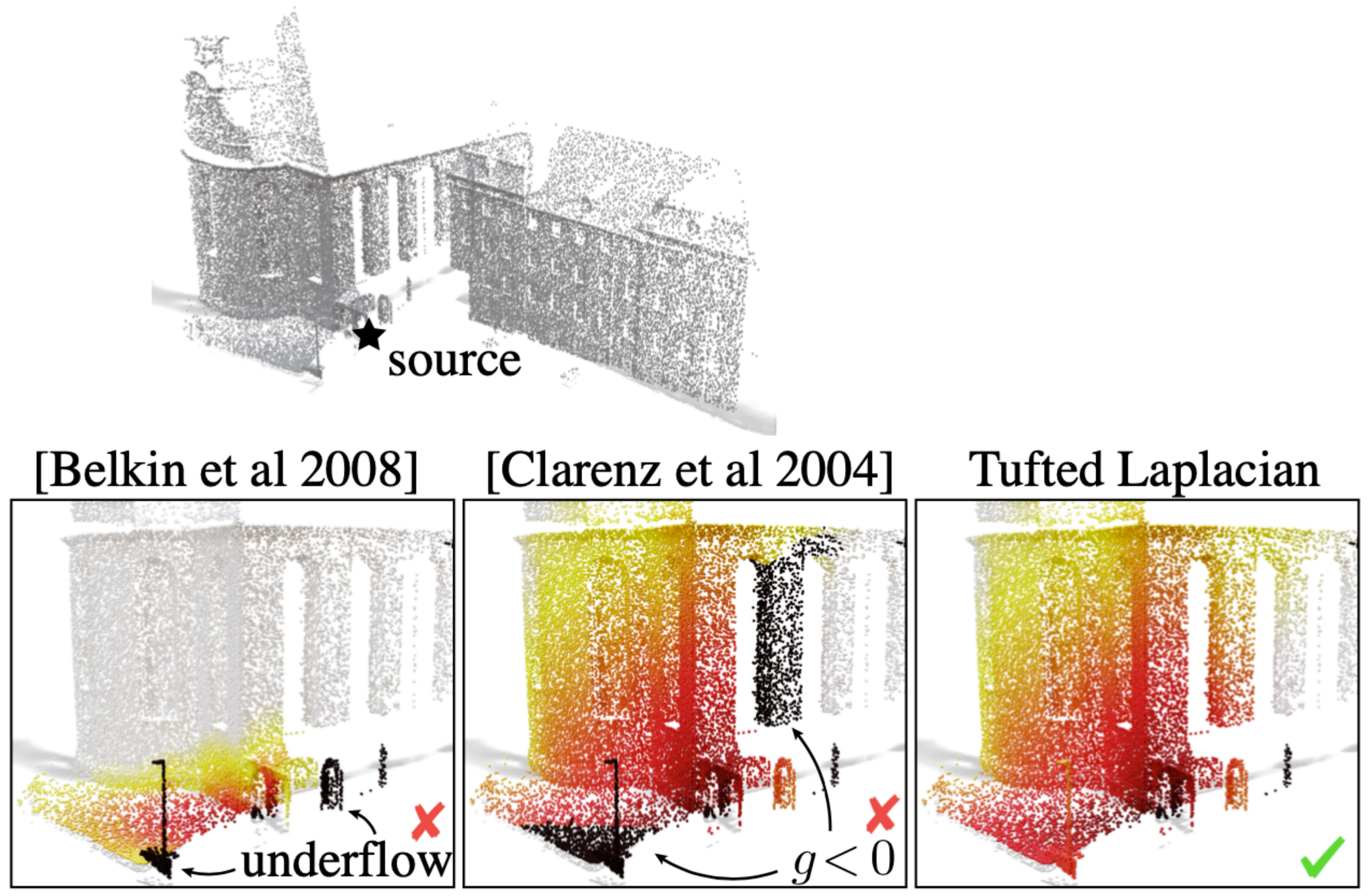
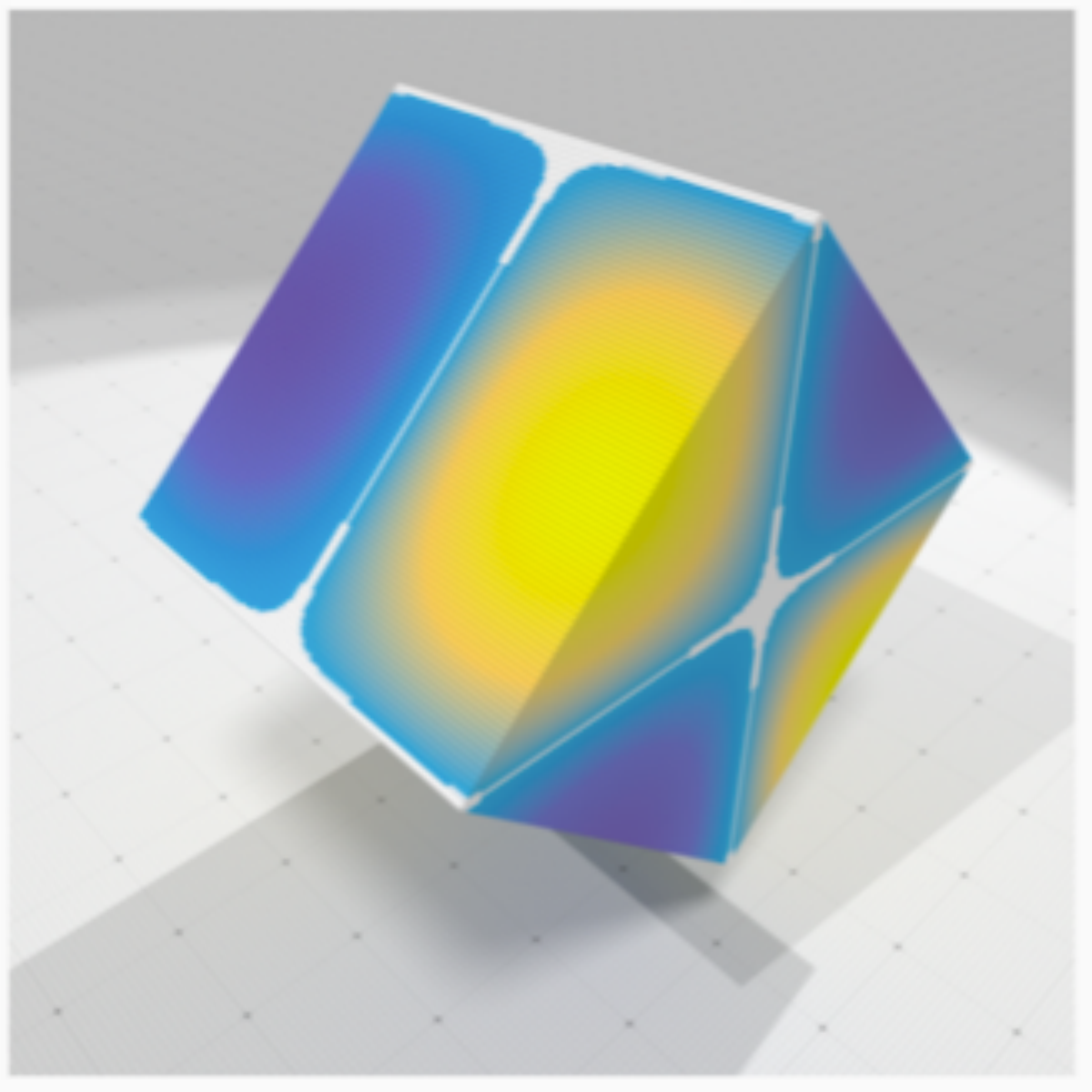
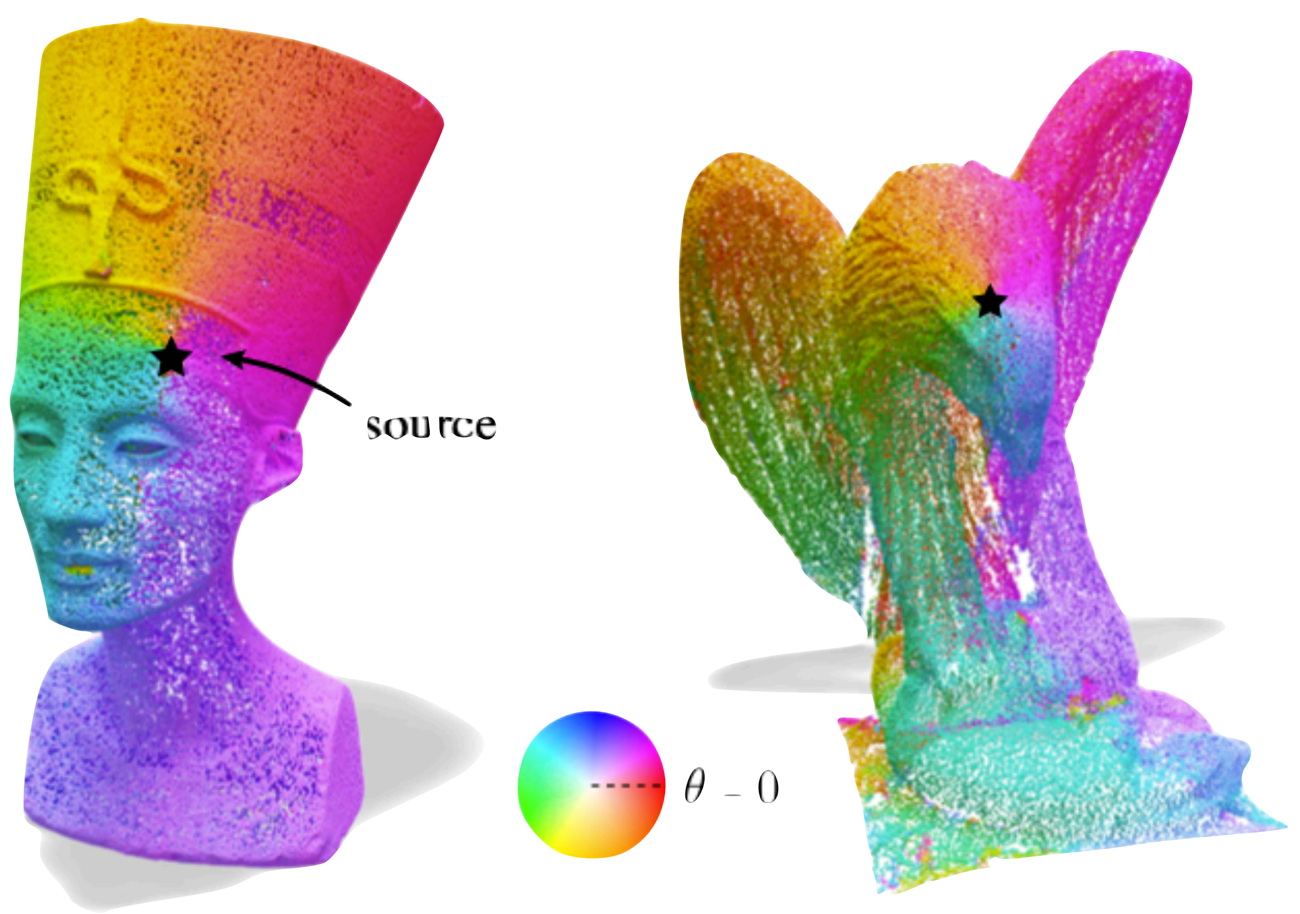
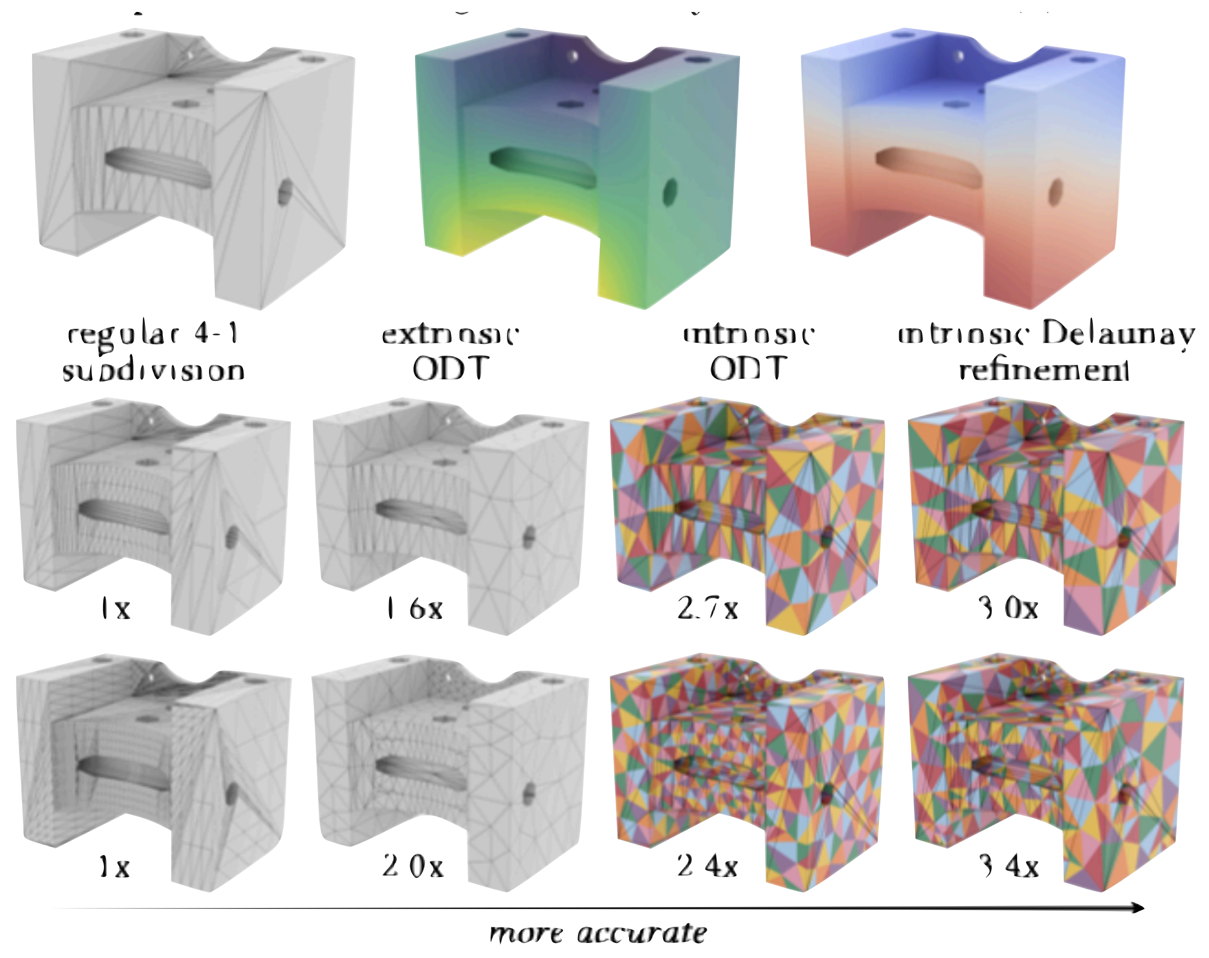


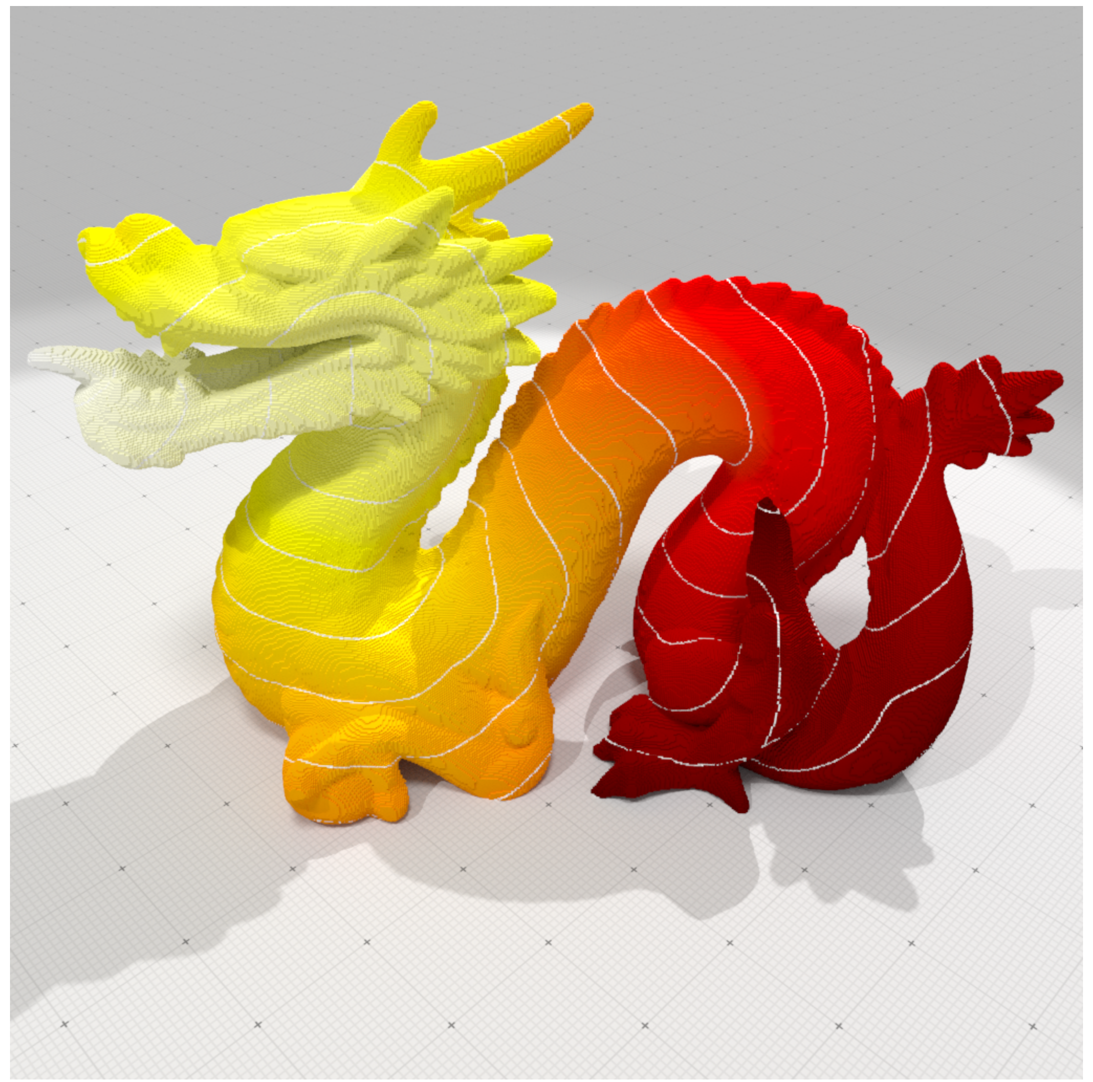








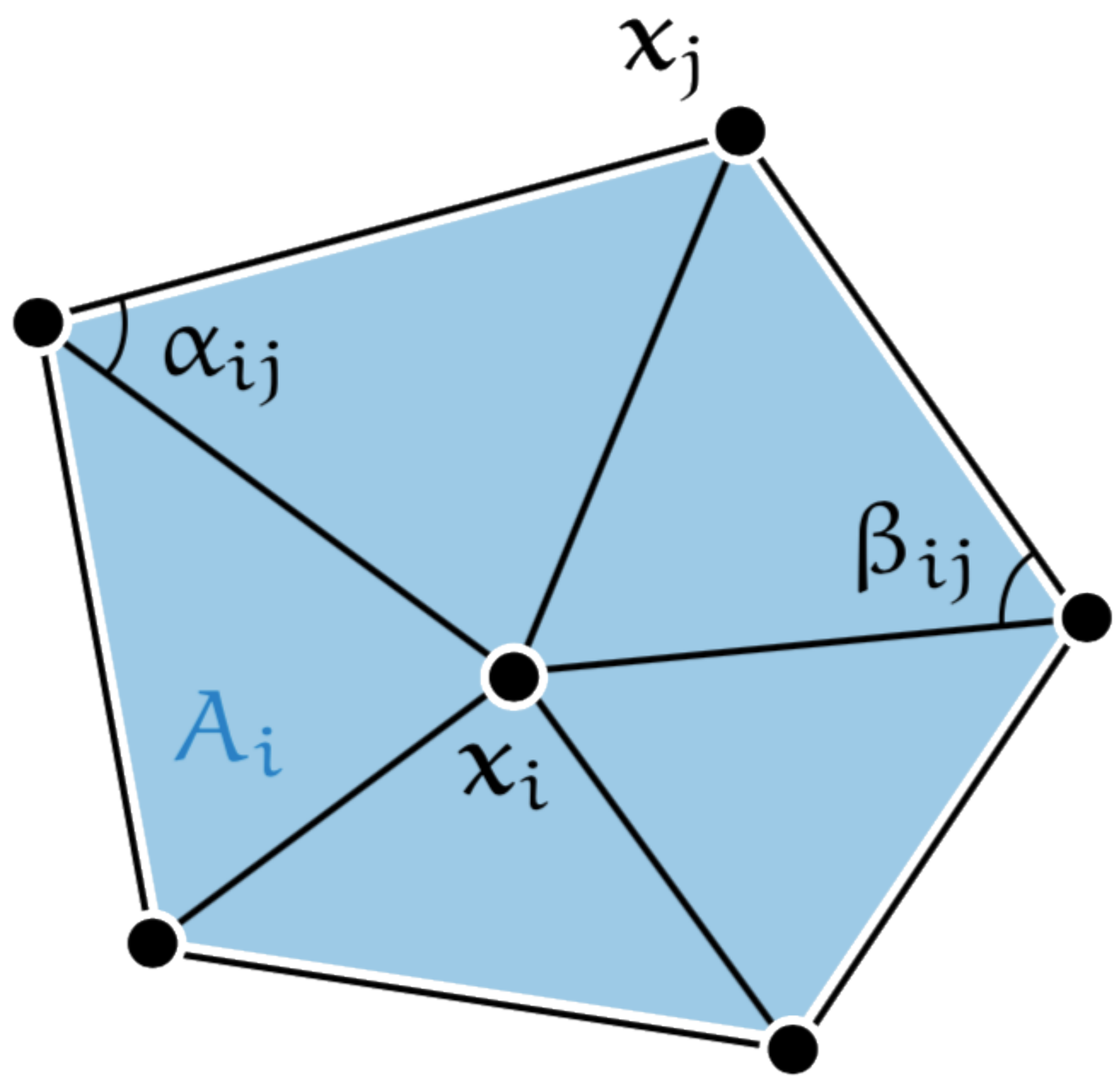




Laplace-Beltrami operators in the wild

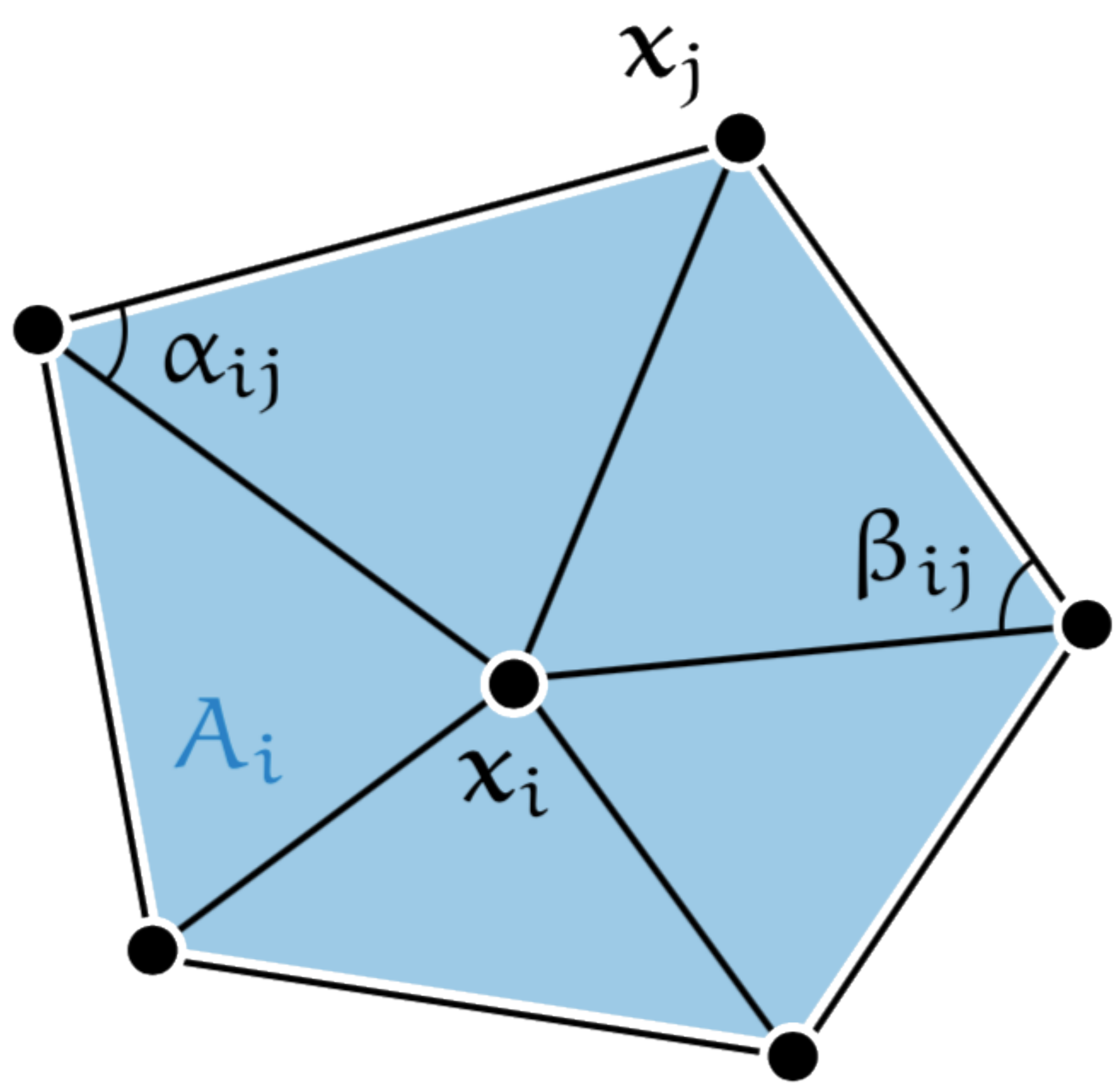
David Coeurjolly

Recap: Laplace-Beltrami on discrete (manifold) meshes



Recap: Laplace-Beltrami on discrete (manifold) meshes

$$(L_{COMBI}u)(x_i) := -deg(x_i)u(x_i) + \sum_{x_j \in \text{link}_0(x_i)} u(x_j)$$



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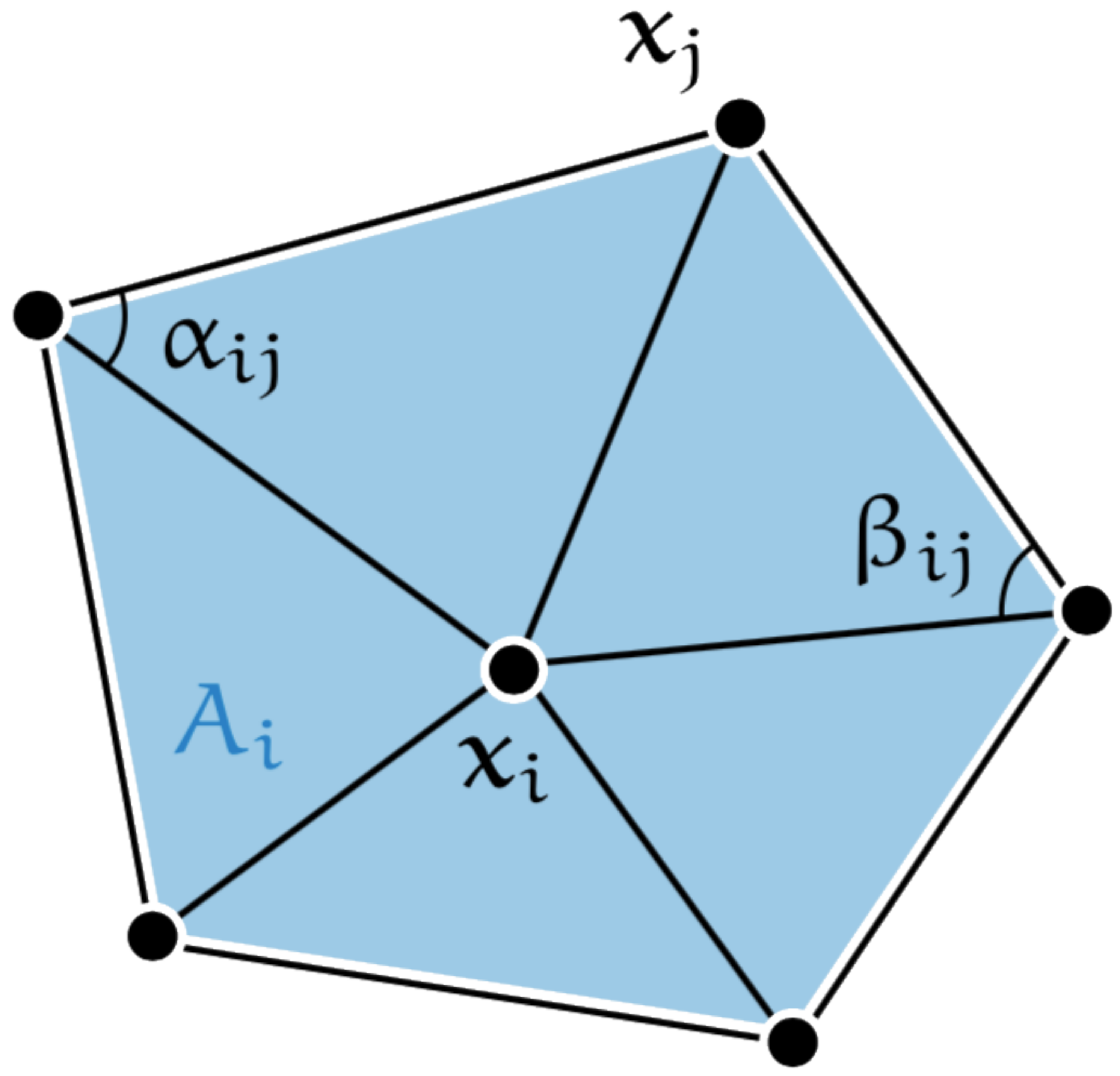

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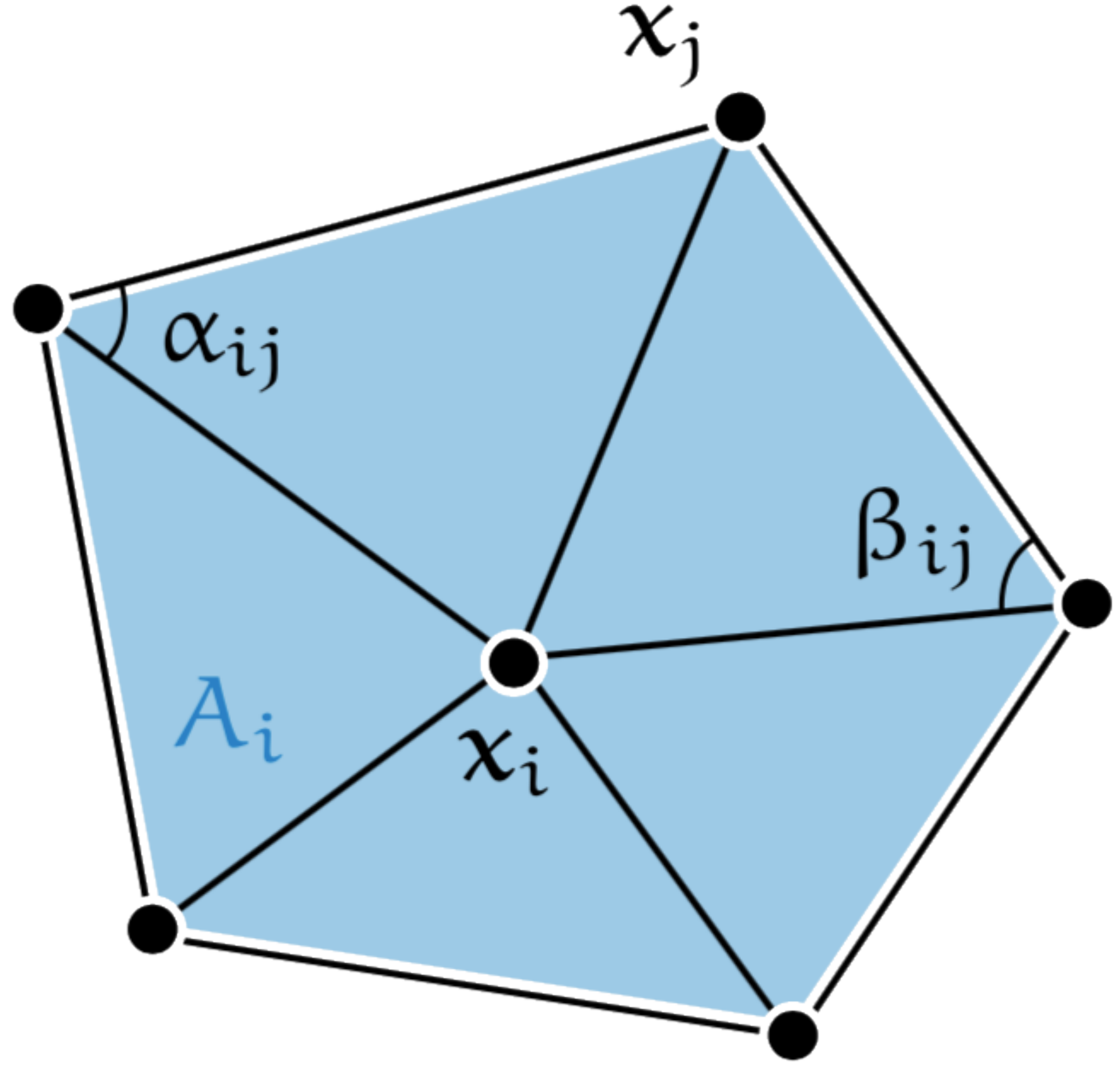
$$(L_{COT}u)(x_i) := \frac{1}{2\mathscr{A}_{x_i}} \sum_{x_j \in \text{link}_0(x_i)} (\cot(\alpha_{ij}) + \cot(\beta_{ij}))(u(x_i) - u(x_j))$$



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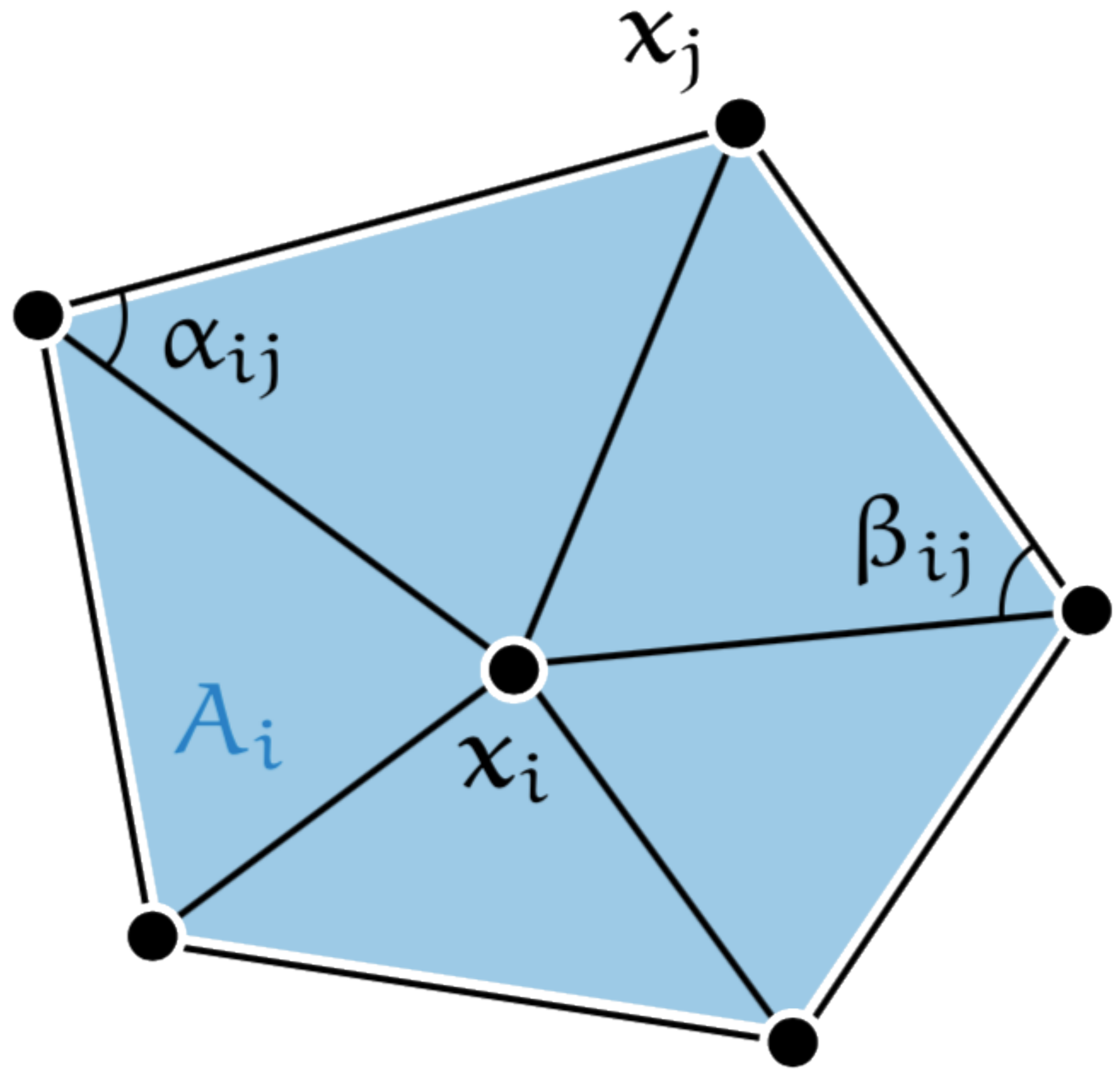
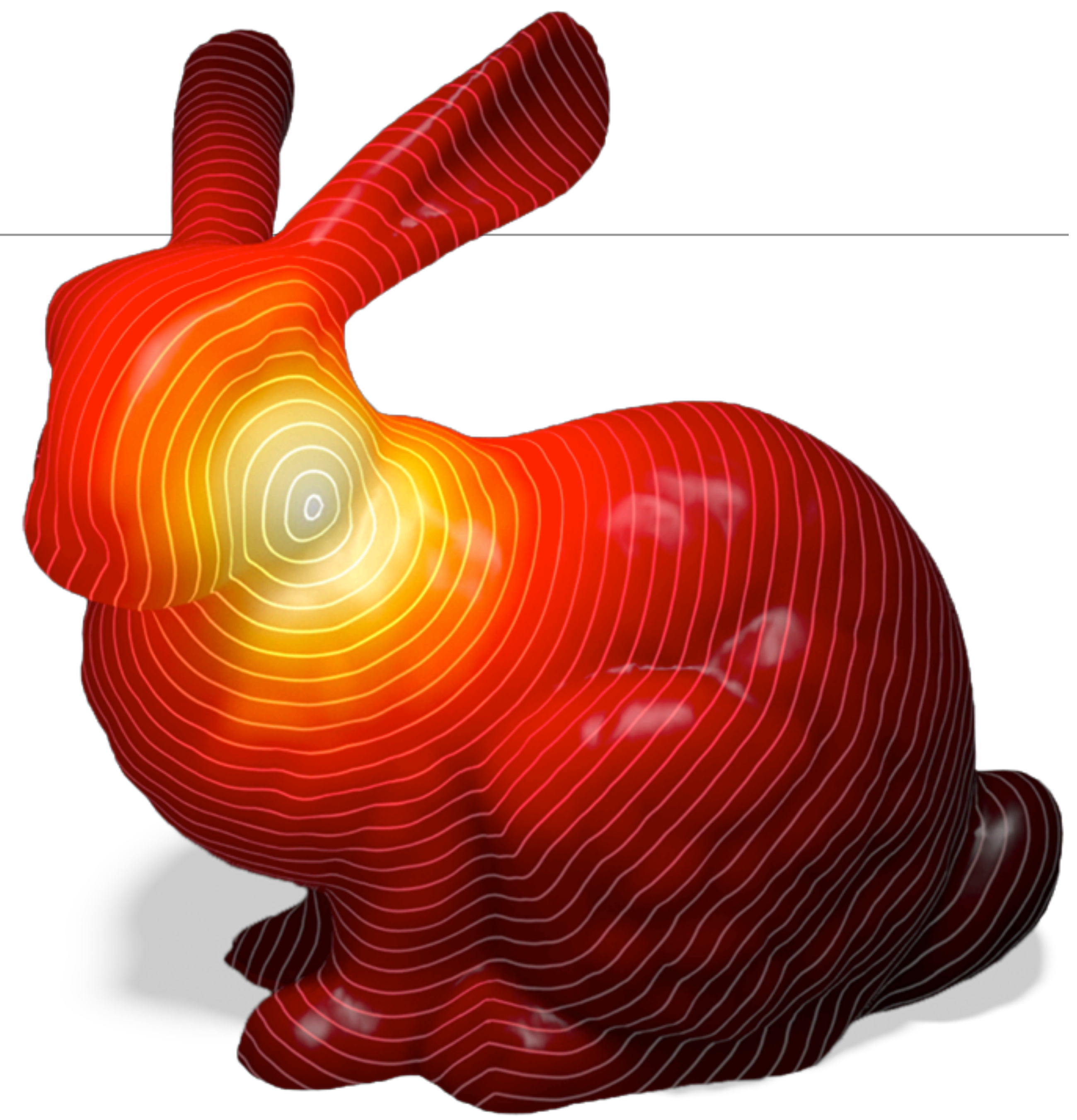
$$(L_{Mesh}u)(x_i) := \frac{1}{4\pi t^2} \sum_f \frac{A_f}{3} \sum_{x_j \in V(f)} e^{-\frac{\|x_i - x_j\|^2}{4t}}(u(x_j) - u(x_i))$$



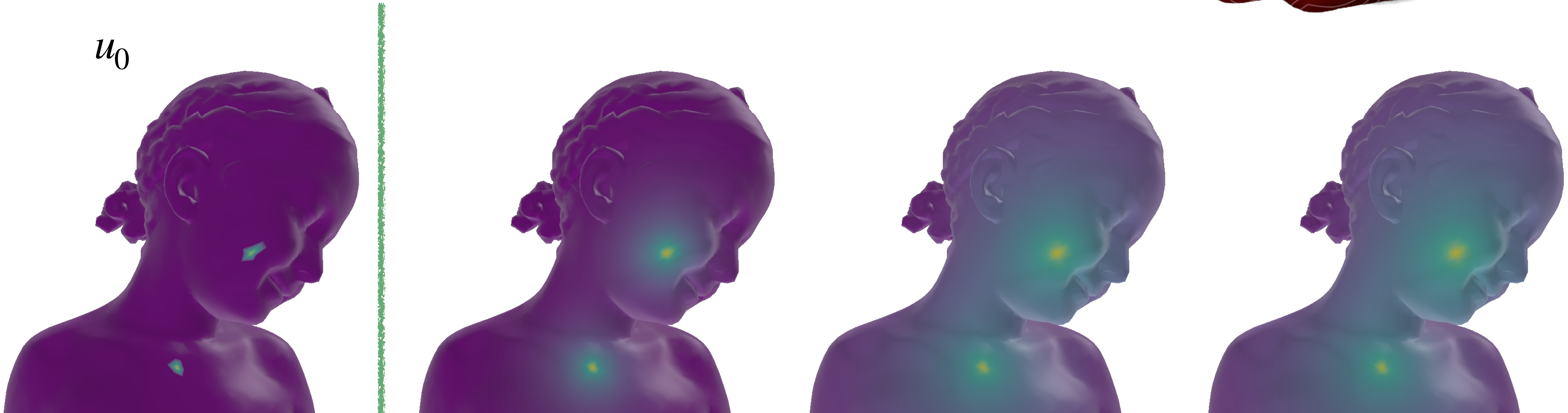
Recap: Laplace-Beltrami on manifold meshes



$$u(x, t) : \Omega \times \mathbb{R}^+ \to \mathbb{R}$$

$$\frac{\partial u}{\partial t} = \Delta u \qquad \text{subject to} \qquad u(x,0) = u_0(x)$$

u_0



Recap: What is the best discretization? TL;DR: There is no free lunch

- SYM: $L_{ij} = L_{ji}$
- LOC: $L_{ij} = 0$ if e_{ij} is not an edge of M
- LIN: $(Lu)(x_i) = 0$ for linear functions
- POS: $L_{ij} \geq 0$ (aka suff. condition for maximal principle)
- PSD: L is PSD (aka Dirichlet energy = $\sum L_{ij}(u(x_i) - u(x_j))^2$)
- CON: $L_n \to \Delta$ (solutions of Dirichlet problems converge to smooth solutions)

	Ref.	*SYM*	*LOC*	*LIN*	*POS*	*PSD*	*CON*	PCON
MEAN VALUE	[9]	○	●	●	●	○	○	?
INTRINSINC DEL	[3]	●	○	●	●	●	?	?
$\mathcal{L}_{COMBI}$	[26]	●	●	○	●	●	○	○
$\mathcal{L}_{COT}$	[8, 10]	●	●	●	○	●	●	○
$\mathcal{L}_{MESH}$	[2]	○	○	?	●	●	●	●

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$\mathcal{L}_{MESH}$	[2]	○	○	?	●	●	●	●

$$\lim_{\epsilon \to 0} \| L_\epsilon u - \Delta u \|_{L^\infty} = \lim_{\epsilon \to 0} \sup_{x \in \mathcal{M}} | L_\epsilon u(x) - \Delta u(x) | = 0 , \quad \forall u \in C^2(\mathcal{M})$$

Some theoretical details

- CON: $L_n \to \Delta$ (solutions of Dirichlet problems converge to smooth solutions)
- Given a sequence of meshes M_n that converges to a smooth manifold $\mathscr{M}$ in Hausdorff distance with a one-to-one and onto mapping $M_n \to \mathscr{M}$, then:

« On the convergence of metric and geometric properties of polyhedral surfaces », Klaus Hildebrandt, Konrad Polthier & Max Wardetzky Geometriae Dedicata, (2006)

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$\Leftrightarrow$ *convergence of* L_n

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But…

- What about ***bad*** meshes ?
- What about polygonal meshes ?
- What about nonmanifold meshes ?
- What about point clouds ?
- What about digital surfaces ?

Laplace-Beltrami on ***bad*** meshes

Recap: What is the best discretization? TL;DR: There is no free lunch

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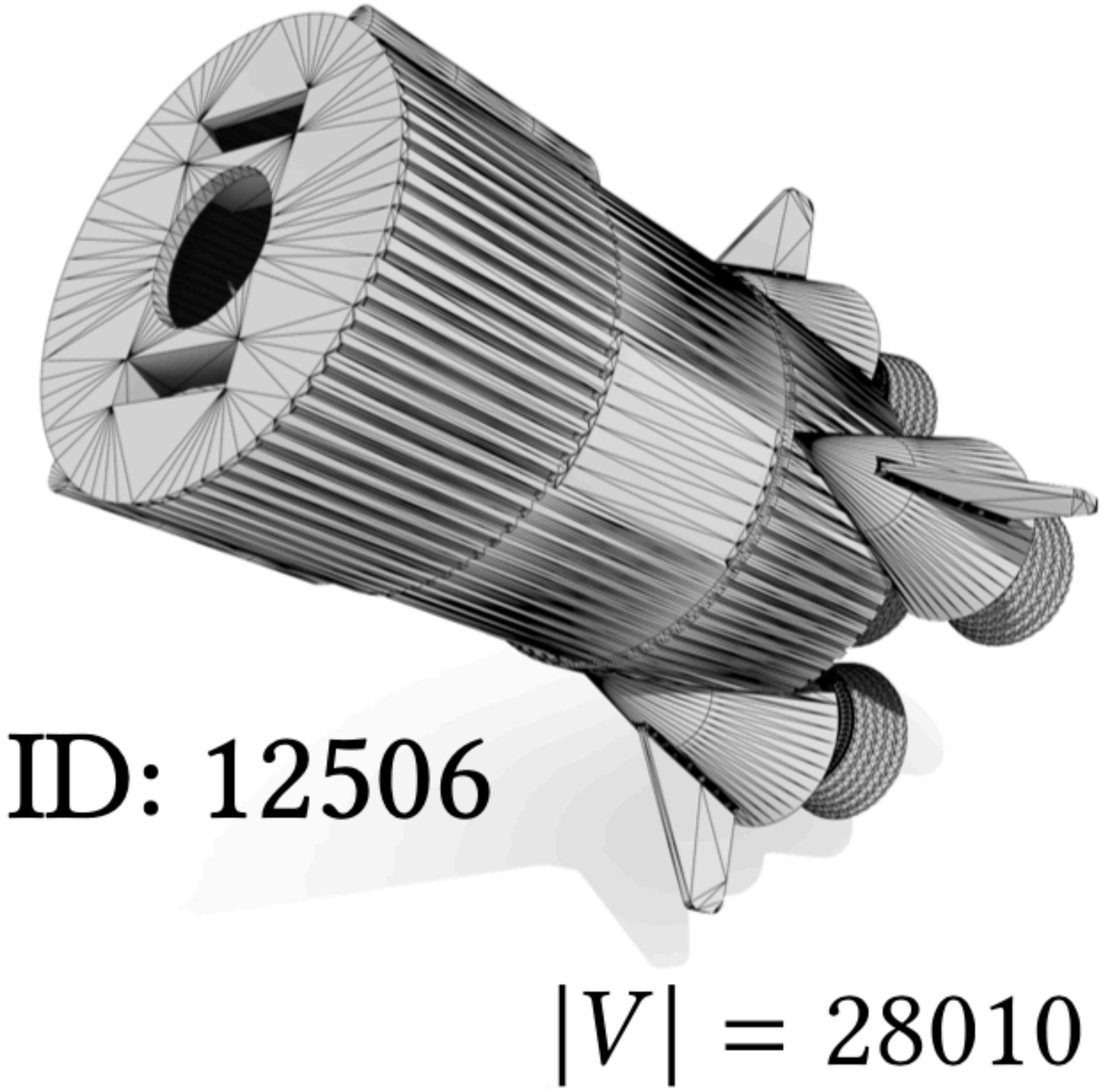
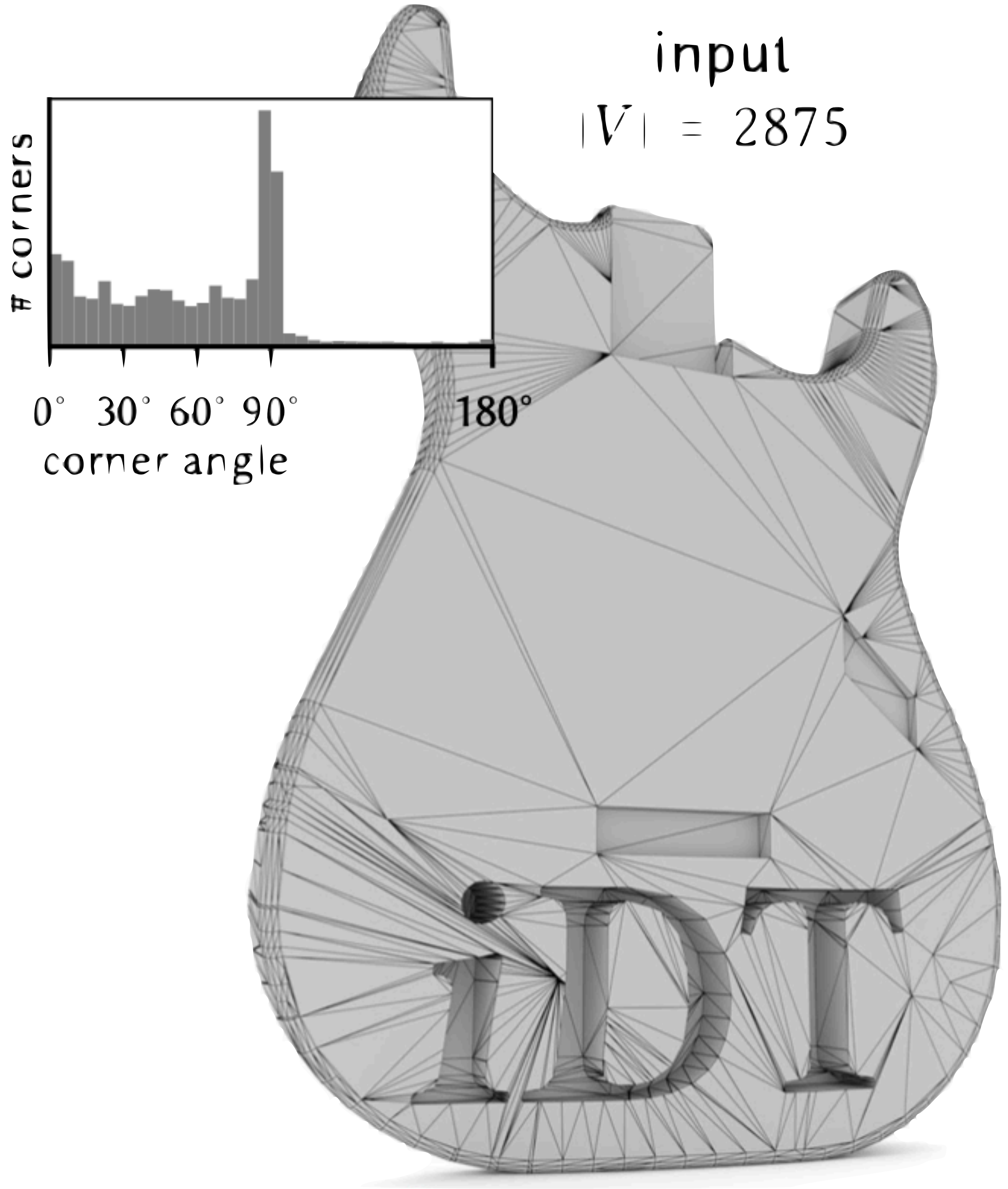
MEAN VALUE	[9]	○	●	●	●	○	○	?
INTRINSINC DEL	[3]	●	○	●	●	●	?	?
$\mathcal{L}_{COMBI}$	[26]	●	●	○	●	●	○	○
$\mathcal{L}_{COT}$	[8, 10]	●	●	●	○	●	●	○
$\mathcal{L}_{MESH}$	[2]	○	○	?	●	●	●	●

Bad meshes





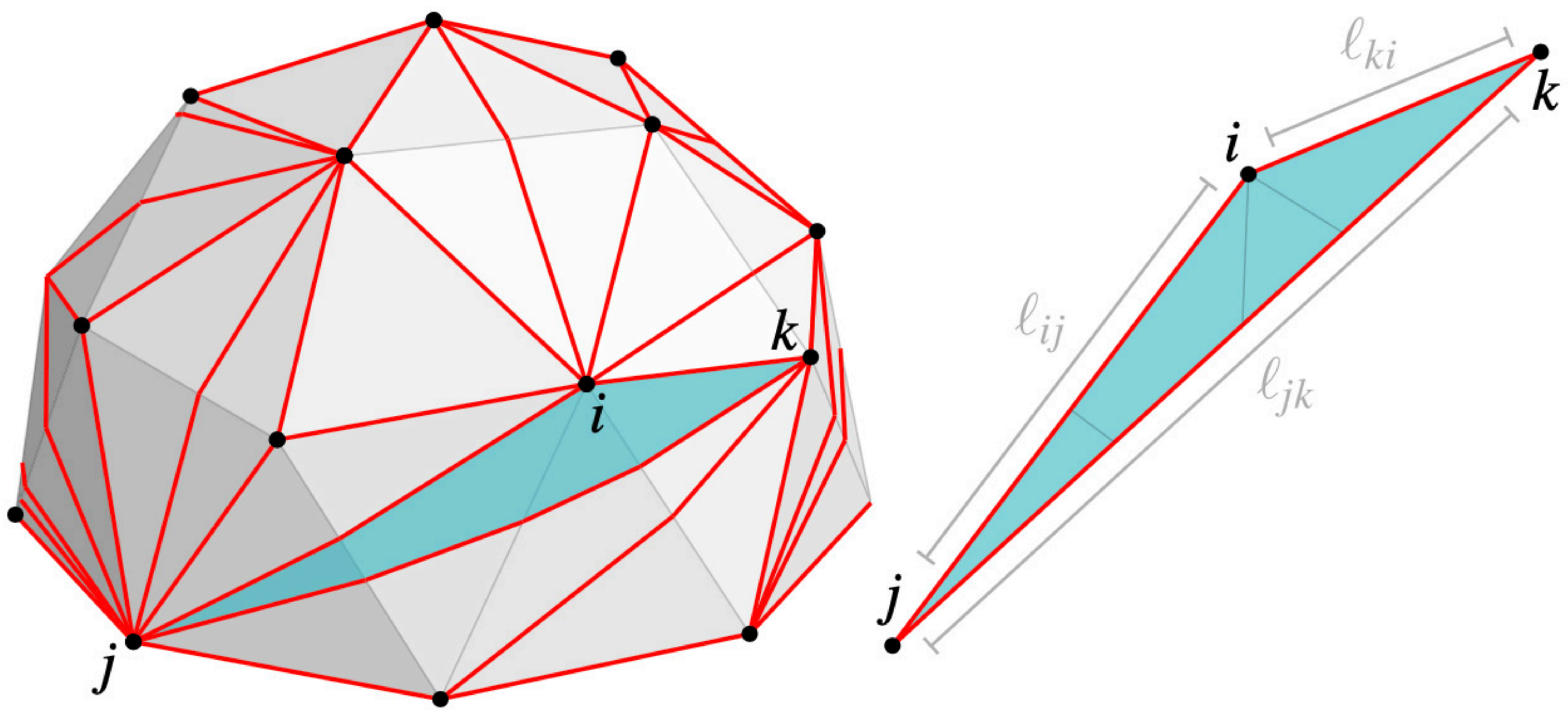
⇒ *negative cotan weights*

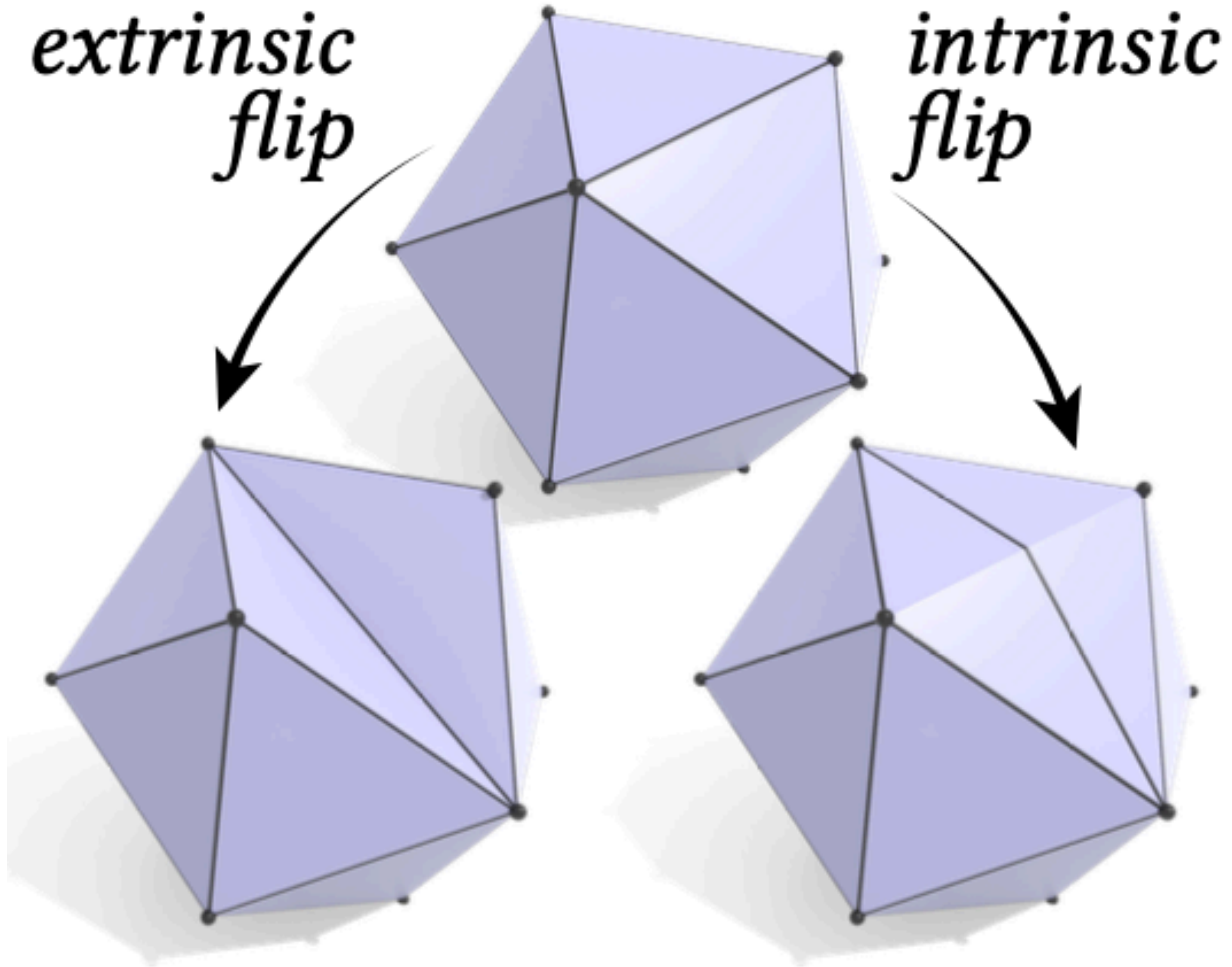
⇒ *numerical instabilities, non-real eigenvalues….*

Options: *remeshing or intrinsic implicit operators*

Intrinsic triangulation and intrinsic Delaunay triangulation

- Edge flipping: update intrinsic representation of the triangulation



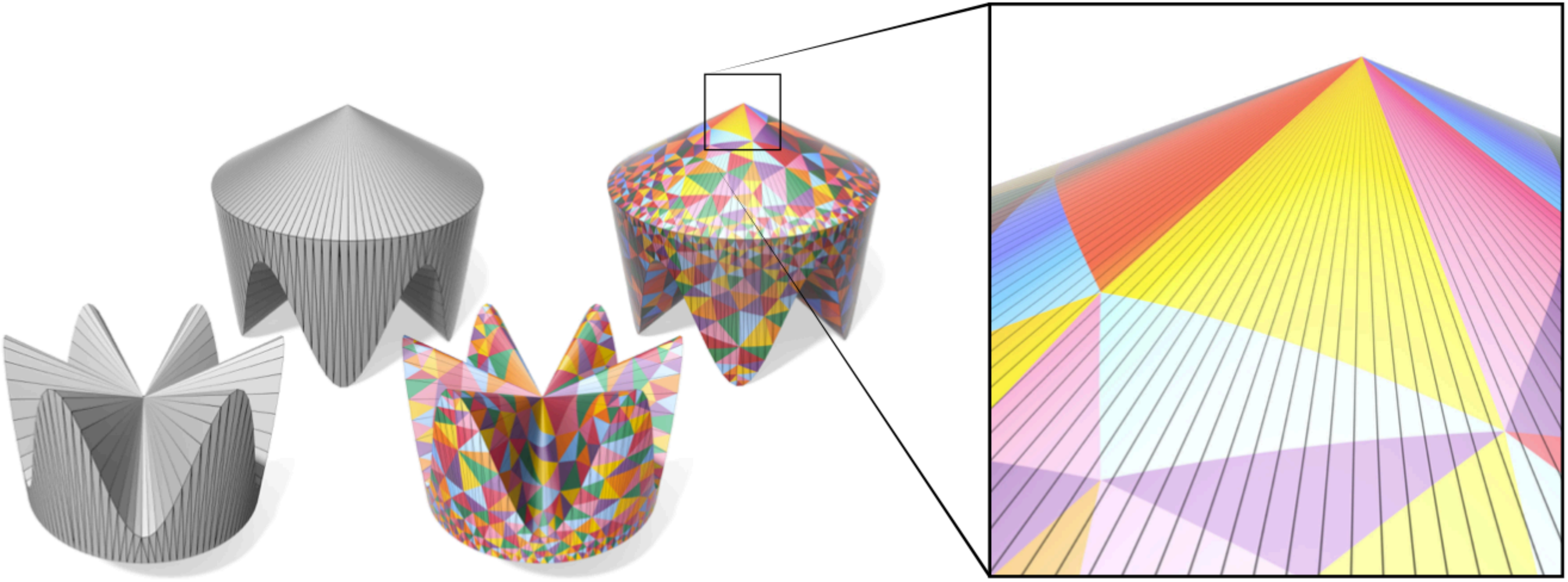


⇒ enforce positivity of cotangent weights

⇒ allows refinement

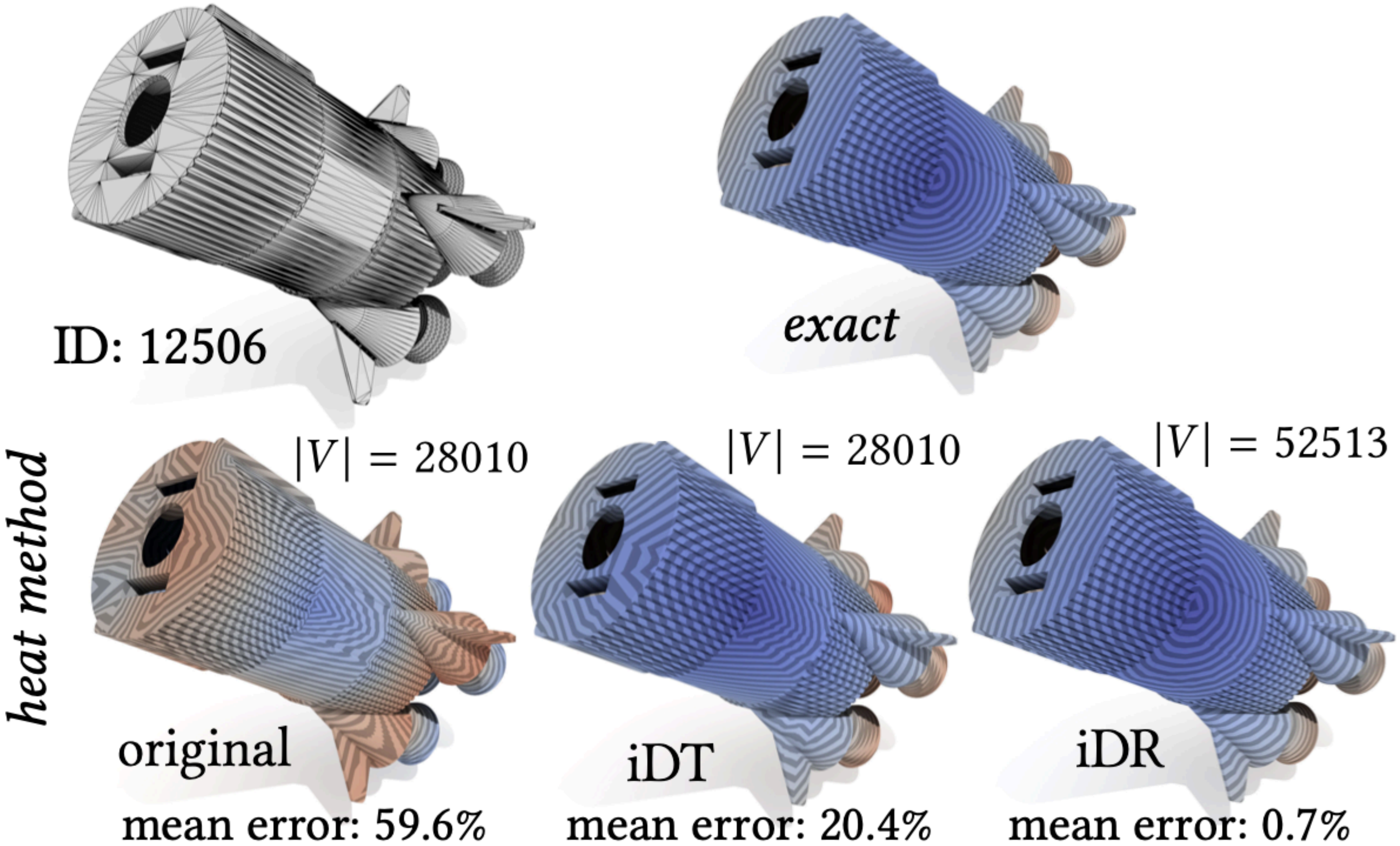
« Navigating Intrinsic Triangulations », Sharp, Soliman, Crane, ACM TOG (2019)

Intrinsic triangulation and intrinsic Delaunay triangulation



⇒ *Efficient datastructure based on vertex embedding and edge lengths (geodesic arcs)*

Intrinsic Delaunay triangulation and refinement



Laplace-Beltrami on polygonal meshes

Polygonal meshes

- Non planar, non convex faces

« Discrete Laplacians on general polygonal meshes ». Alexa, Wardetzky, ACM Transactions on Graphics 30, 4 (2011), 102:1– 102:10.
« Polygon Laplacian Made Simple », Bunge, Herholz, Kashdan, Botsch, CGF 2020
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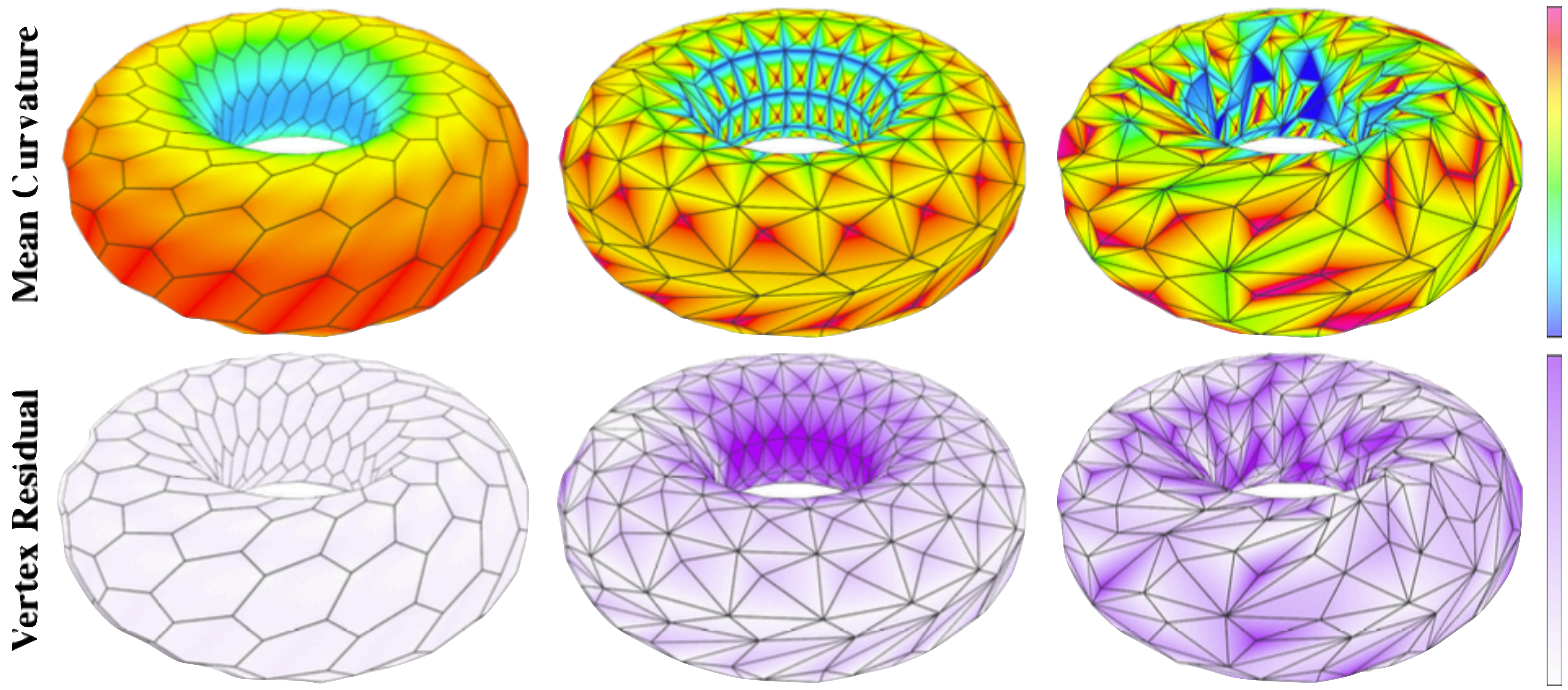

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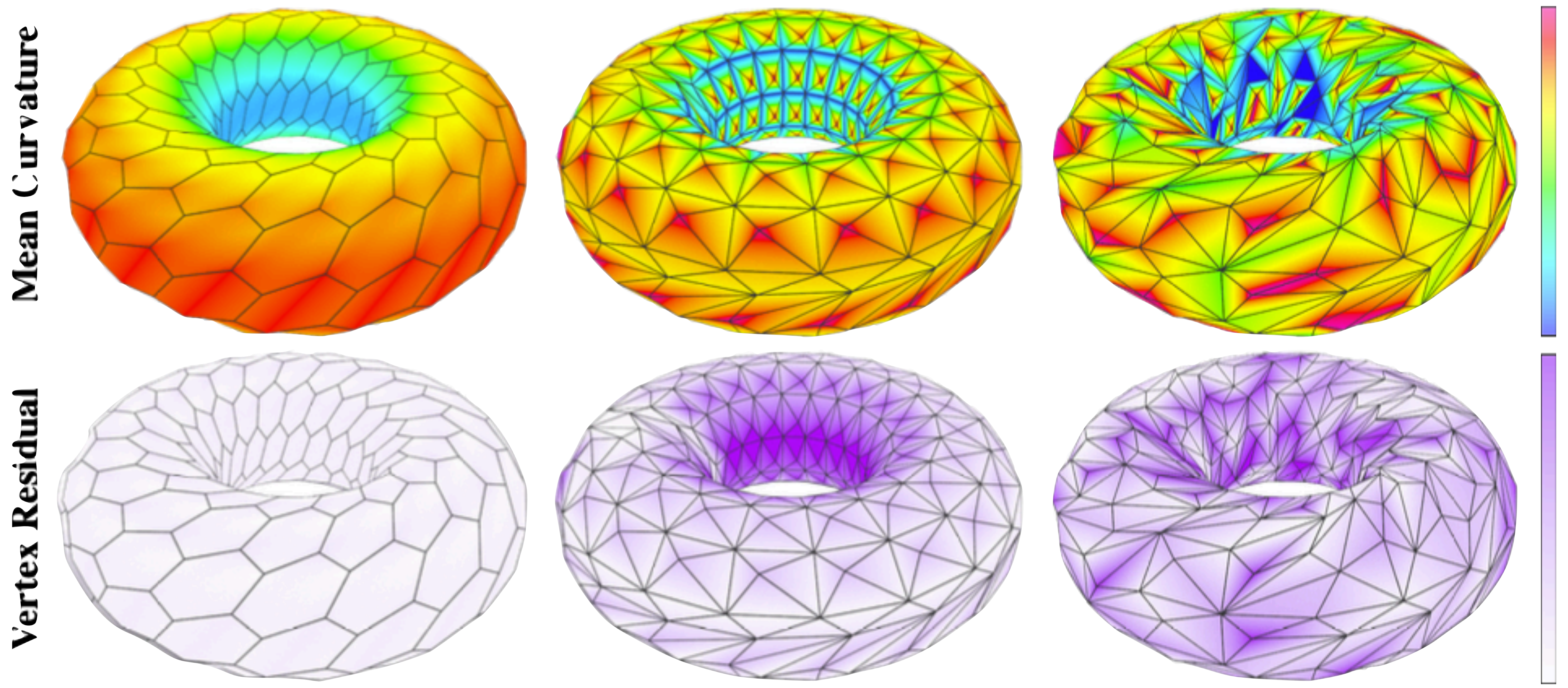
Polygonal meshes

- Non planar, non convex faces

- ***Options***:



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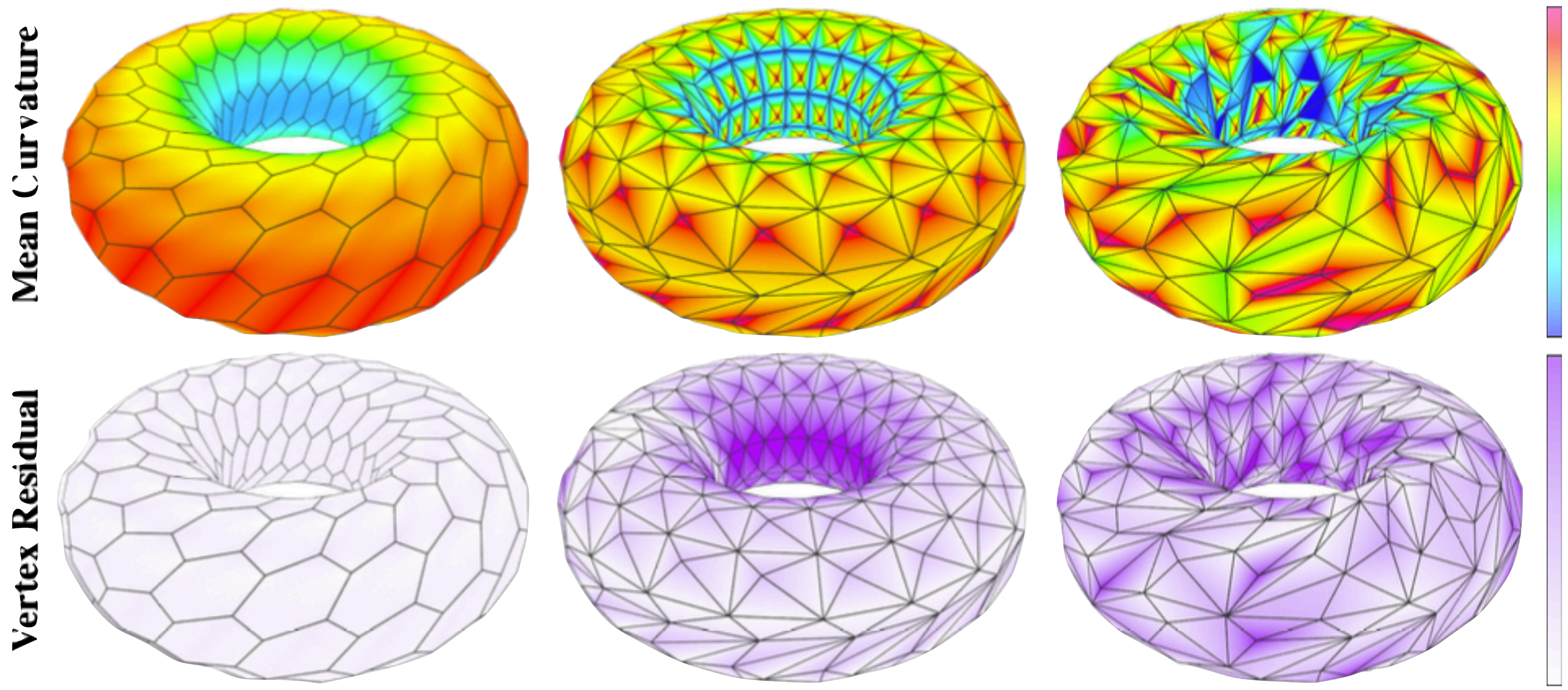
- Non planar, non convex faces

- ***Options***:

  - triangulate and use L_{COT}



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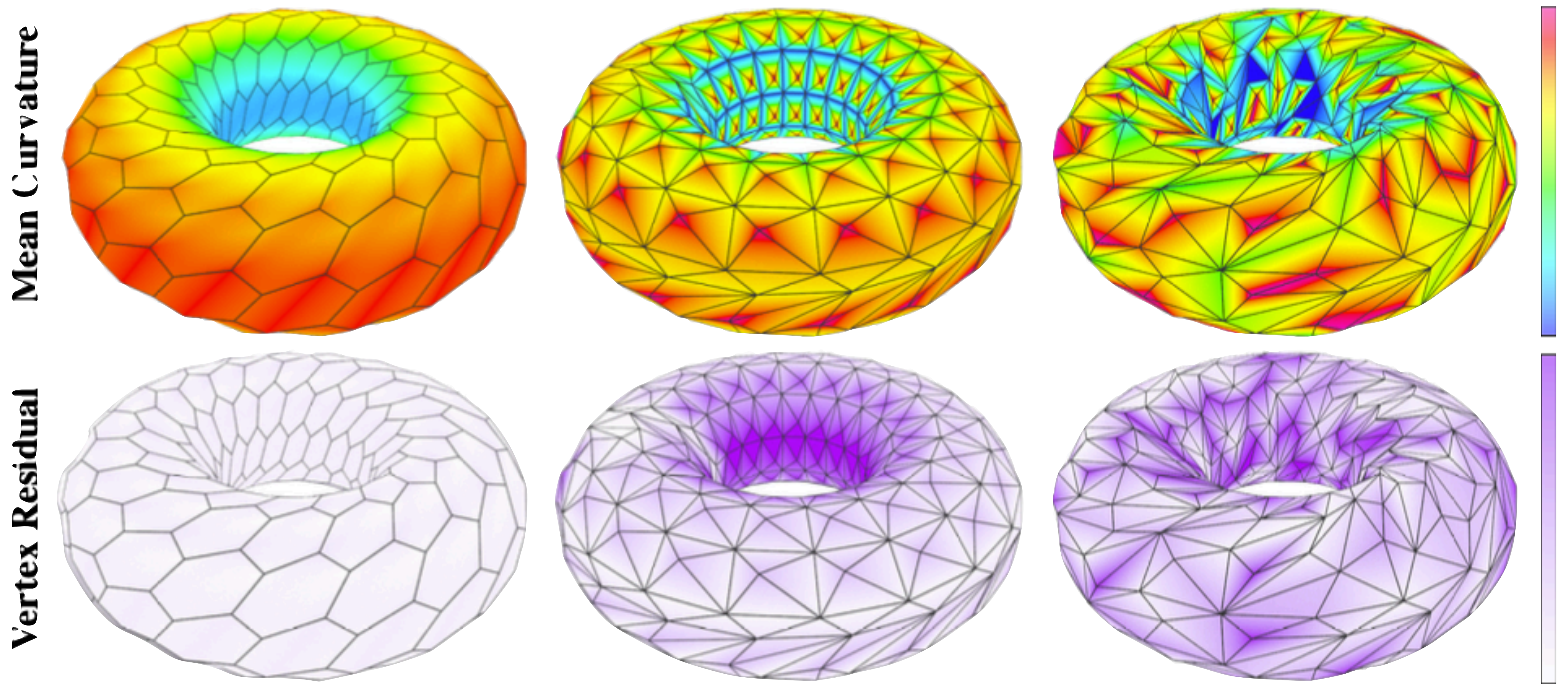
Polygonal meshes

- Non planar, non convex faces
- ***Options***:
  - triangulate and use L_{COT}
  - insert virtual vertices but how?



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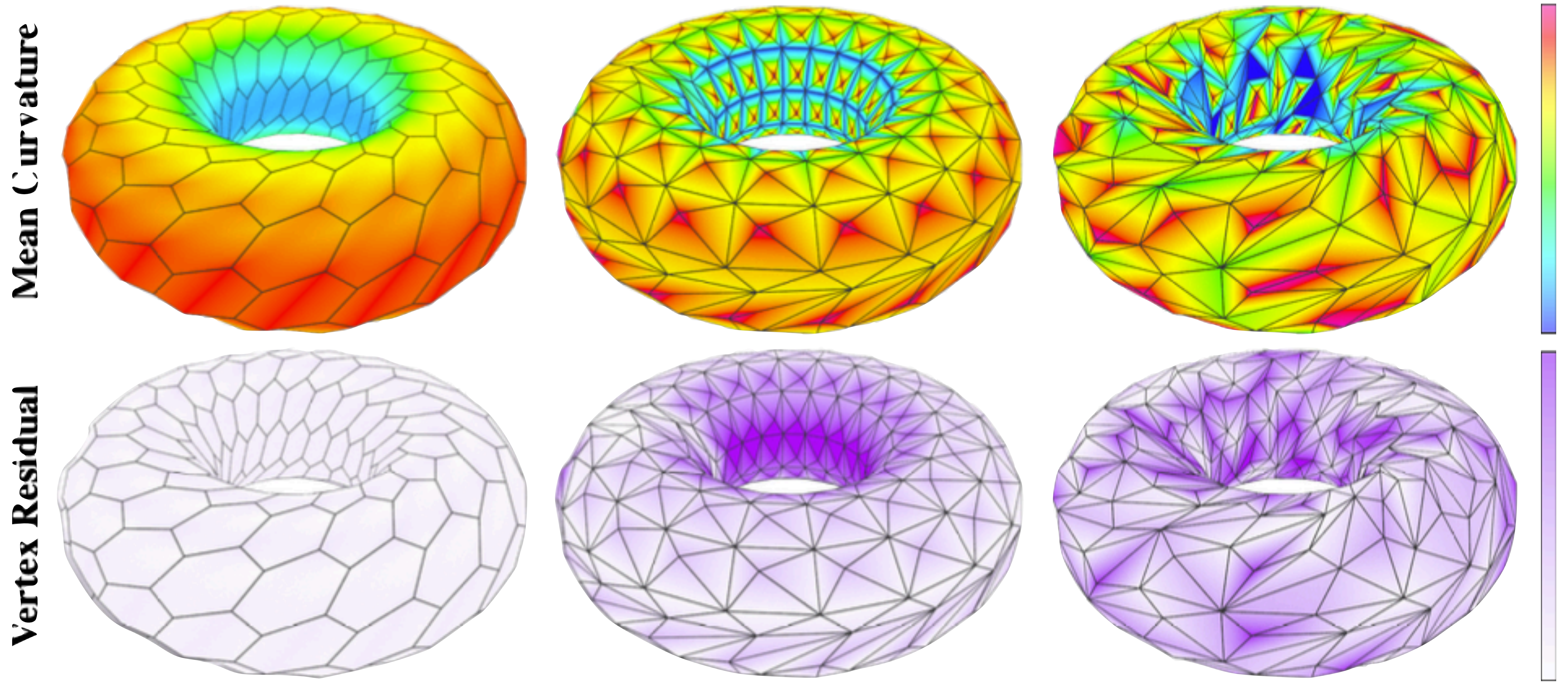
Polygonal meshes

- Non planar, non convex faces
- ***Options***:
  - triangulate and use L_{COT}
  - insert virtual vertices but how?
  - define *polygonal differential operators*



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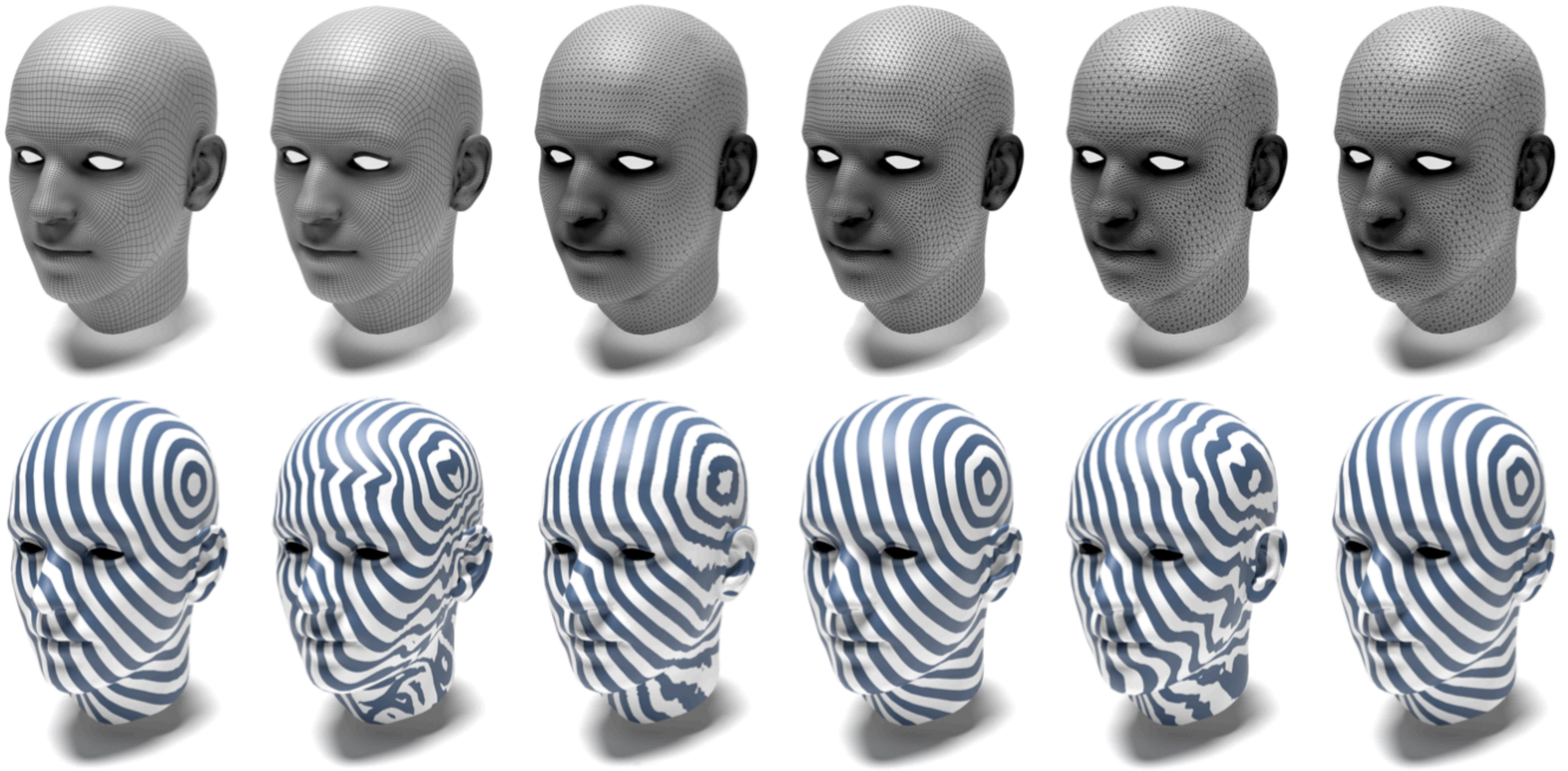
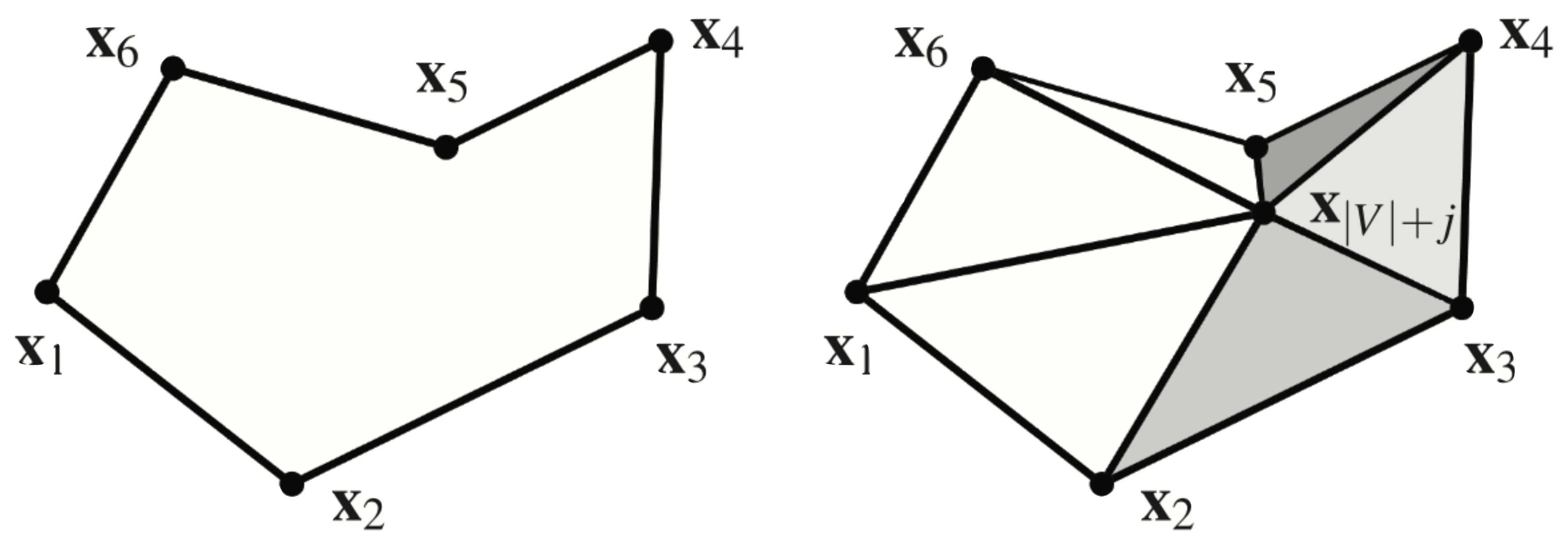


a) ours b) [AW11] c) refinement d) max-angle e) min-area f) intrinsic Delaunay

Option 1: virtual vertices

- ***Implicit virtual*** vertex per non-triangular face

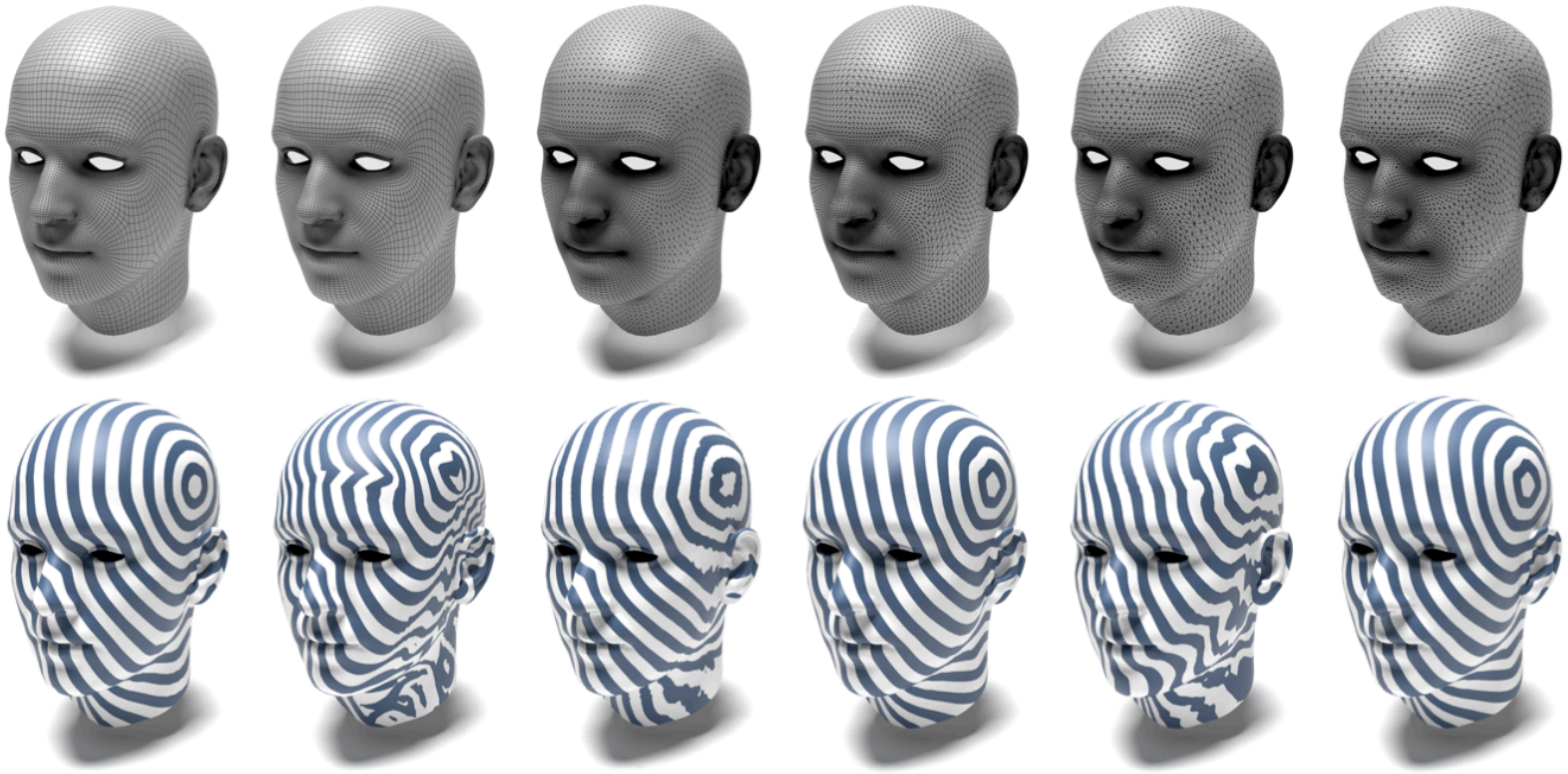
$$\mathbf{x}_f = \arg\min_{\mathbf{x}} \sum_{i=1}^{n} \text{area}\left(\mathbf{x}_i, \mathbf{x}_{i+1}, \mathbf{x}_f\right)^2 .$$



- ***Galerkin*** trick (coarse-fine mappings for operators) to have implicit construction of L (still $V \times V$ matrix)

⇒ Numerically ok, parameter free, experimental convergence, easy to compute

⇒ PSD issues may still occur



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Option 2: Complete DEC calculus for polygonal meshes

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- General idea: define **gradient**, **covariant derivatives**, **inner product** operators to construct a proper Laplace-Beltrami operator.

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$$\mathbf{G}_f = -\frac{1}{a_f}[\mathbf{n}_f]\mathbf{E}_f^t\mathbf{A}_f.$$

gradient

Symbol	Meaning	Definition
n_f	Number of vertices	$v_1, \ldots, v_{n_f} \in f$
$\mathbf{X}_f$	Vertex positions	$\mathbf{X}_f = [\mathbf{x}_{v_1} \ldots \mathbf{x}_{v_{n_f}}]^t \in \mathbb{R}^{n_f \times 3}$
$\mathbf{D}_f$	Difference operator	$\mathbf{D}_f^{i,i+1} = 1,\ \mathbf{D}_f^{i,i} = -1$
$\mathbf{A}_f$	Average operator	$\mathbf{A}_f^{i,i+1} = \mathbf{A}_f^{i,i} = 1/2$
$\mathbf{E}_f$	Edge vectors	$\mathbf{E}_f = \mathbf{D}_f\mathbf{X}_f$
$\mathbf{B}_f$	Edge midpoints	$\mathbf{B}_f = \mathbf{A}_f\mathbf{X}_f$
$\mathbf{c}_f$	Face center	$\mathbf{c}_f = \mathbf{X}_f^t\mathbf{1}_f/n_f$
$\mathbf{a}_f$	Polygonal vector area	$\mathbf{a}_f = 1/2\sum_{v_i \in f}\mathbf{x}_{v_i} \times \mathbf{x}_{v_{i+1}}$
a_f	Area of polygonal face	$a_f = \lvert\mathbf{a}_f\rvert$
$\mathbf{n}_f$	Normal of polygonal face	$\mathbf{n}_f = \mathbf{a}_f/a_f$
$\mathbf{h}_f$	Vertex heights for polygonal face	$\mathbf{h}_f = (\mathbf{X}_f - \mathbf{1}_f\mathbf{c}_f^t)\mathbf{n}_f$

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$$\mathbf{G}_f = -\frac{1}{a_f}[\mathbf{n}_f]\mathbf{E}_f^t\mathbf{A}_f.$$

gradient

$$\mathbf{V}_f = \mathbf{E}_f\left(\mathbf{I} - \mathbf{n}_f\mathbf{n}_f^t\right).$$

flat

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sharp

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inner prod. 1-form

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Laplace-Beltrami

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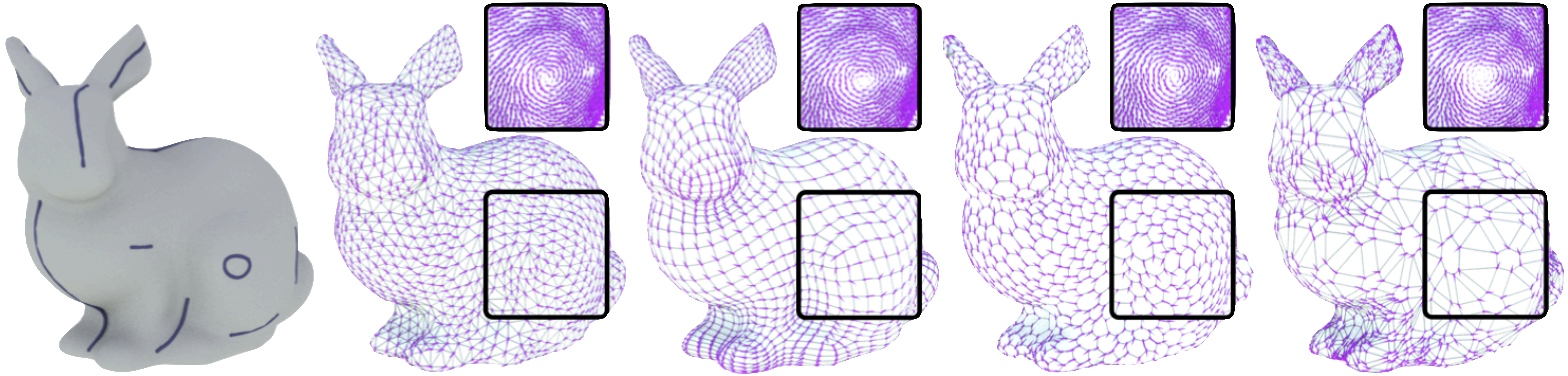
Laplace-Beltrami

- + operators on **direction fields**



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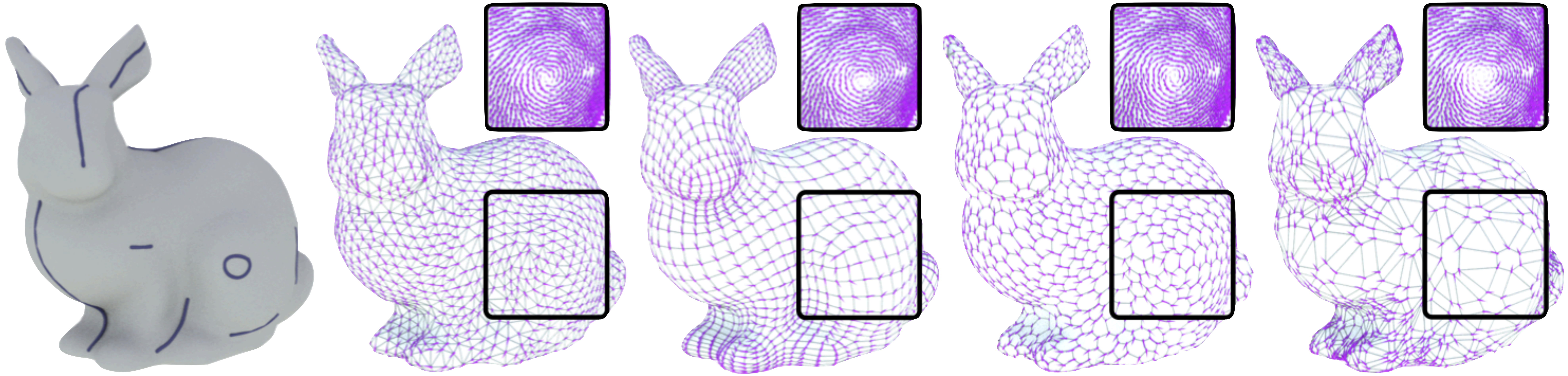
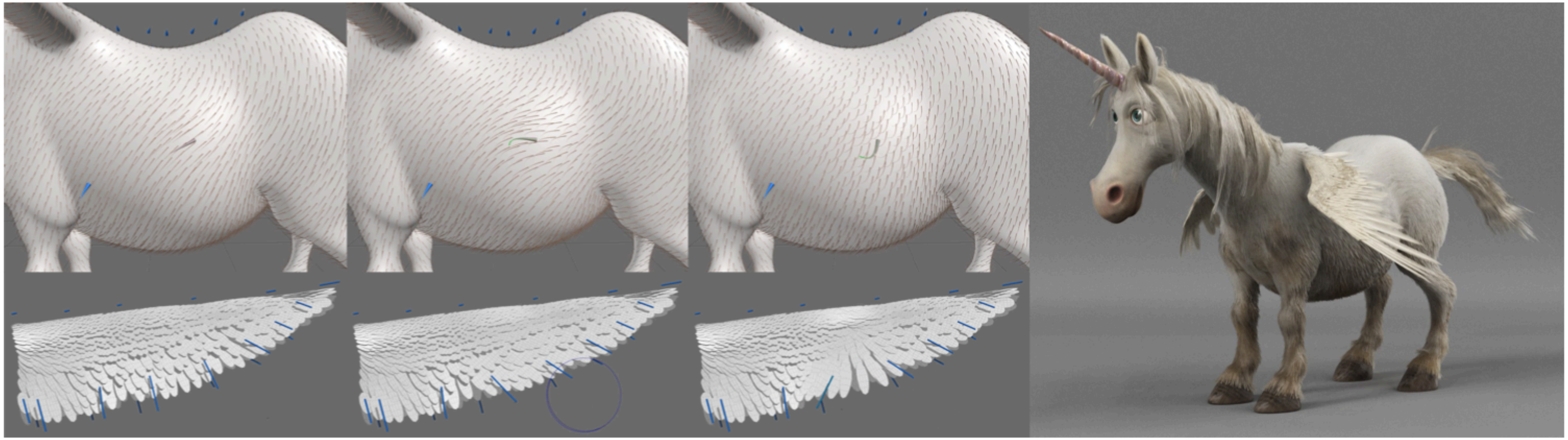
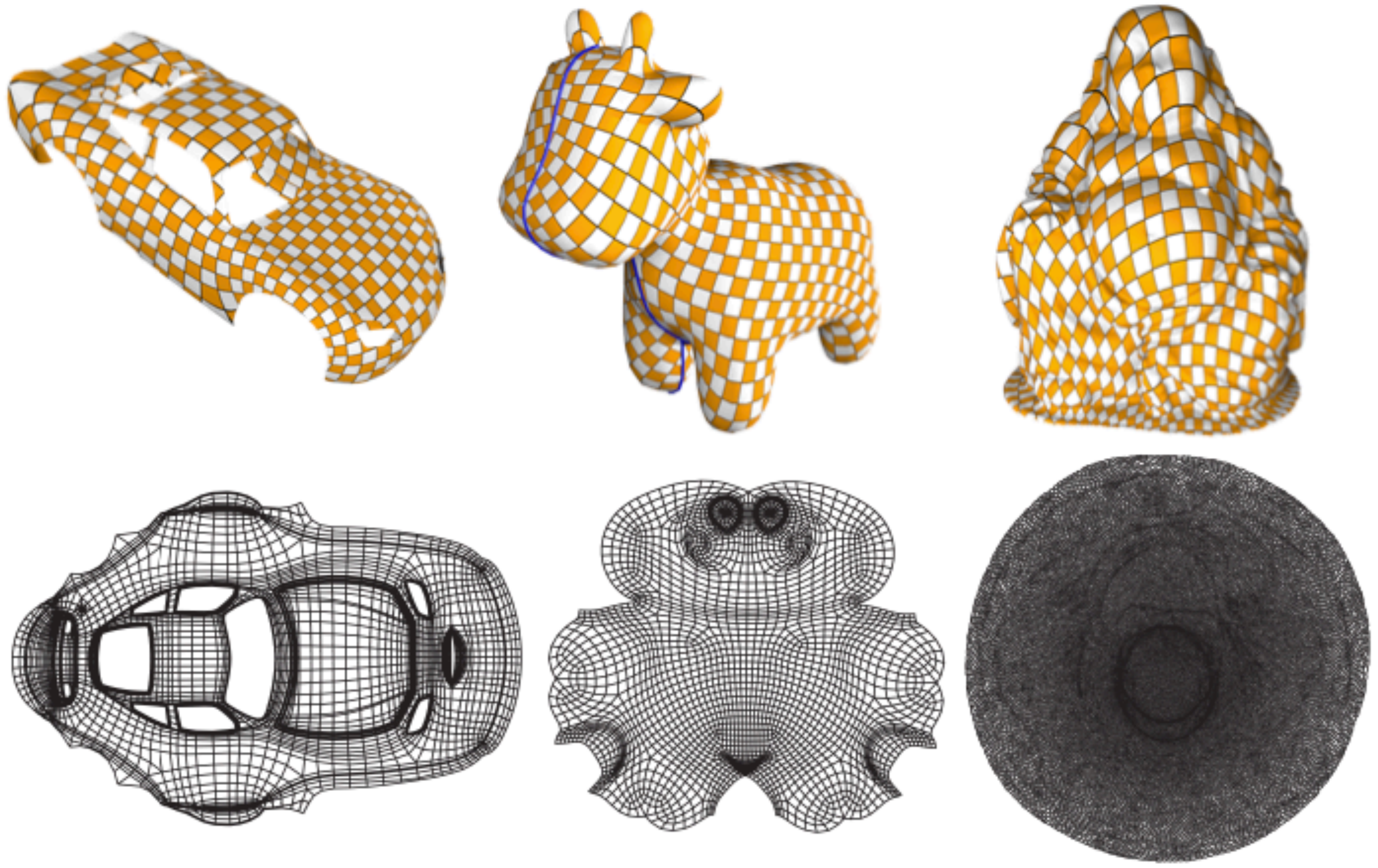
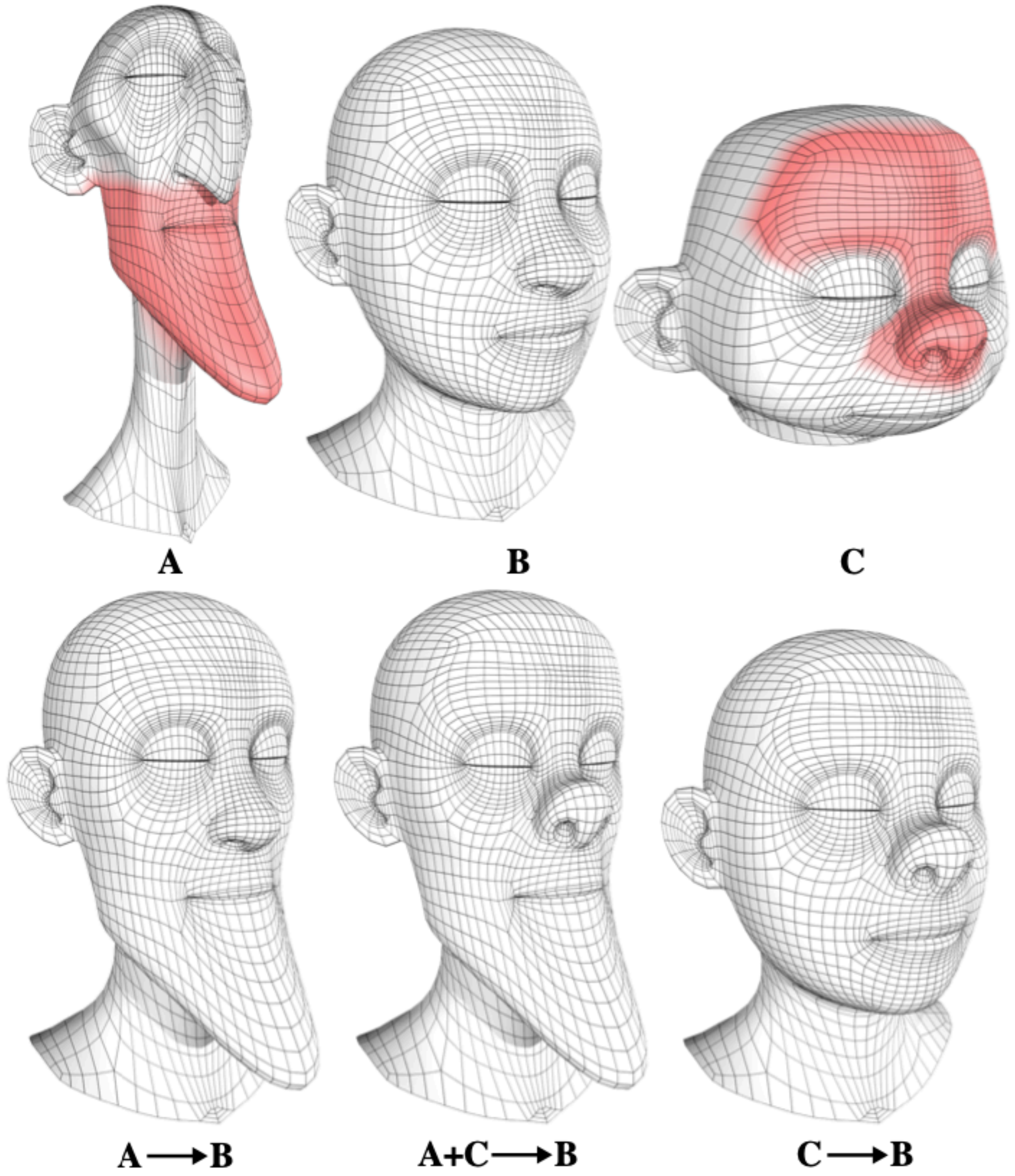


⇒ very generic framework



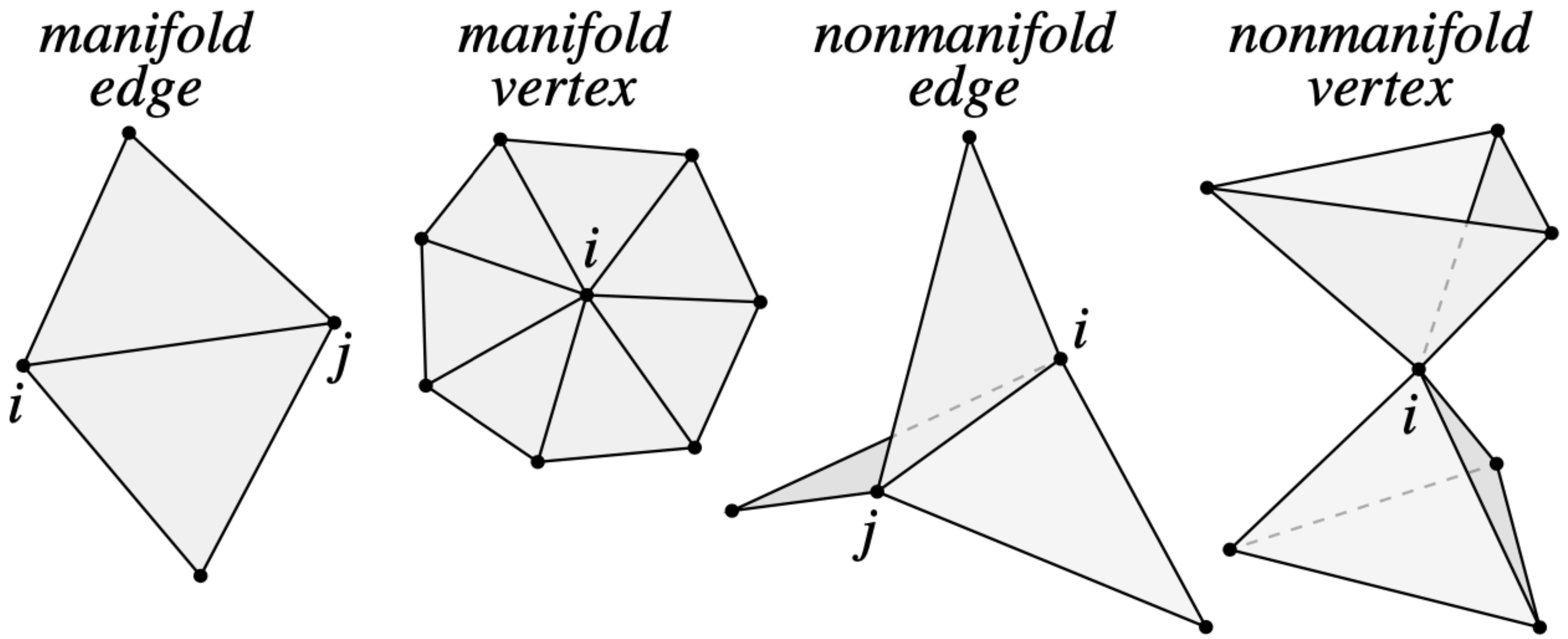


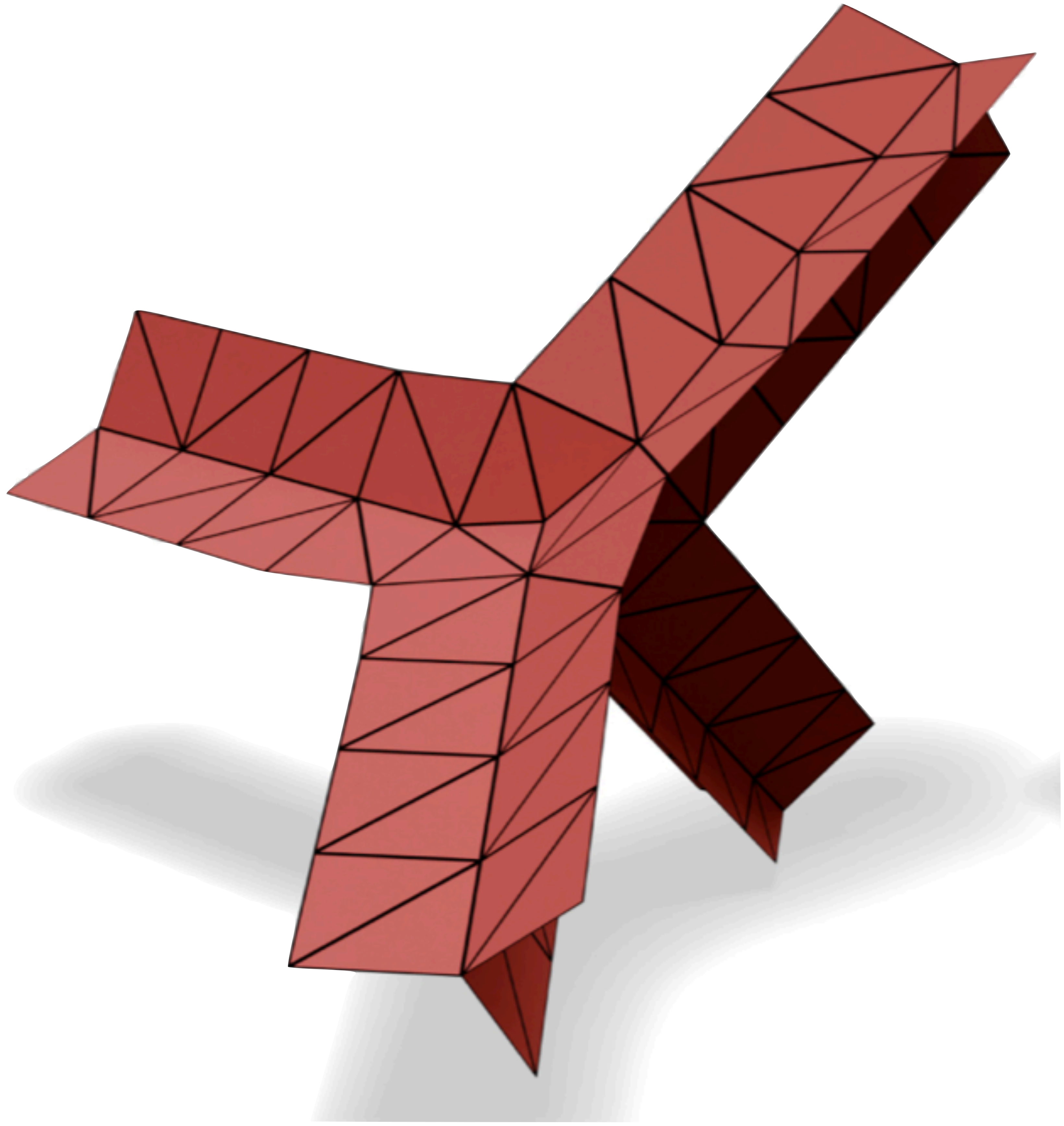

A
B
C
A⟶B
A+C⟶B
C⟶B

Laplace-Beltrami on nonmanifold meshes

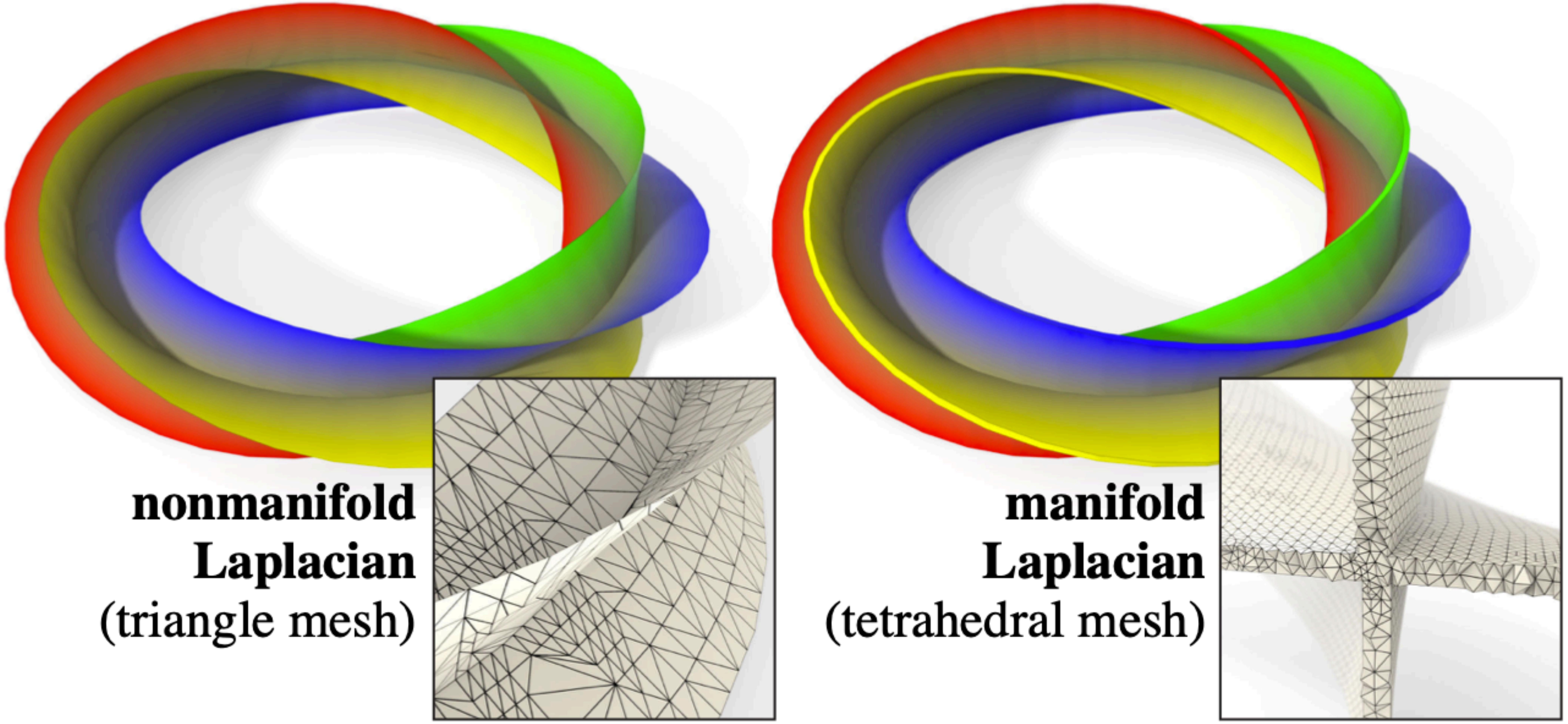
« A Laplacian for Nonmanifold Triangle Meshes »,
Sharp, Crane, SGP CGF 2020

Nonmanifold meshes



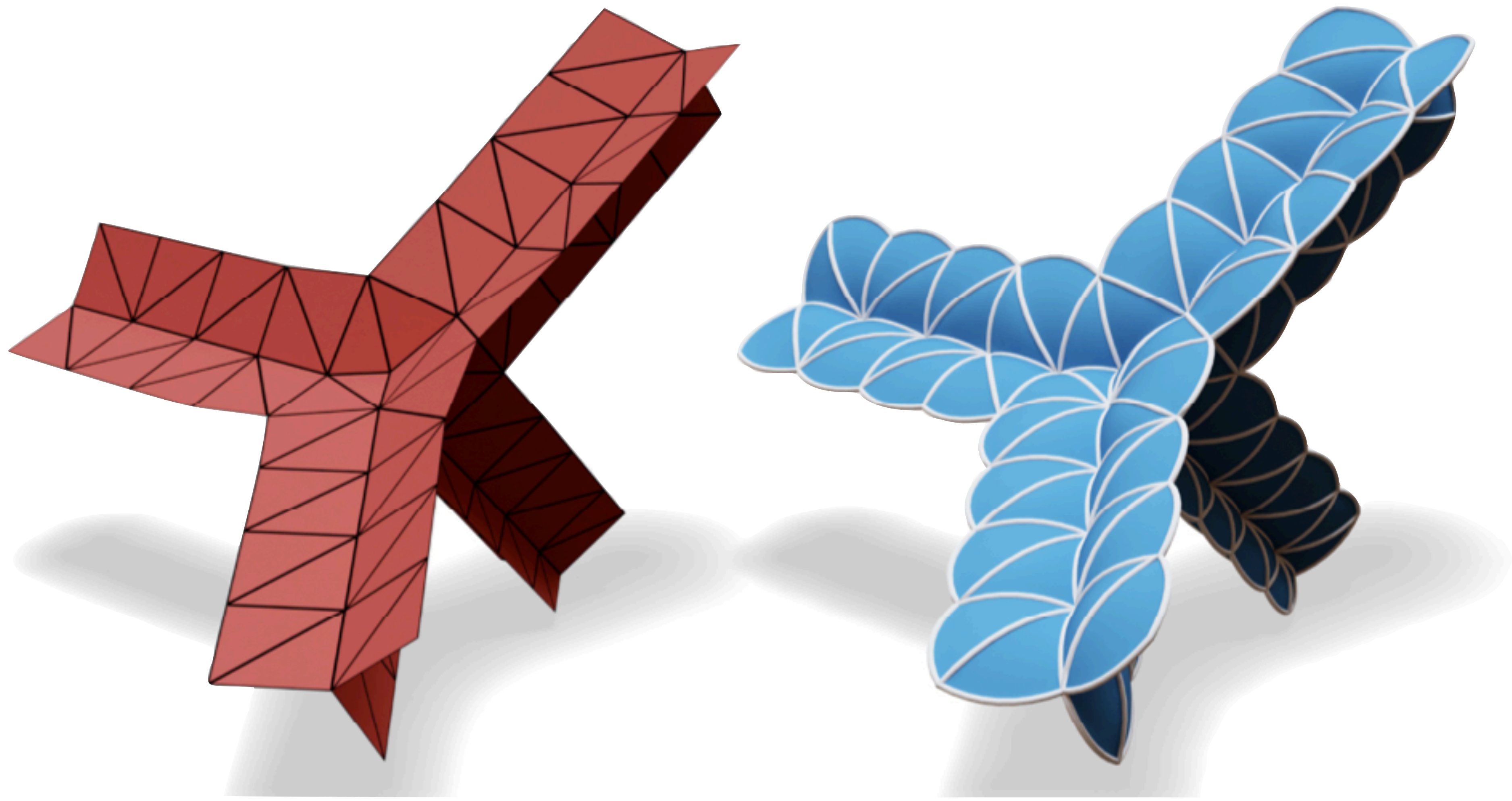


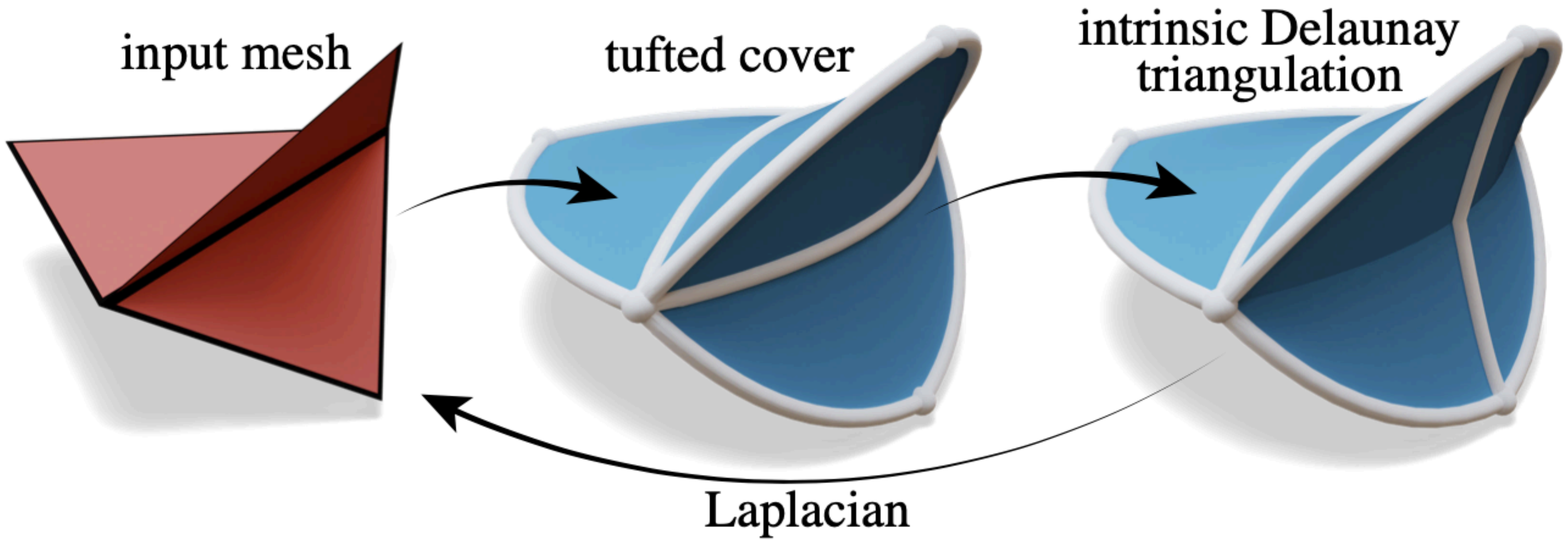
⇒ *The operator is not formally defined but we are looking for something that behaves similarly on the boundary of an « epsilon thickening »*


nonmanifold
Laplacian
(triangle mesh)
manifold
Laplacian
(tetrahedral mesh)

Basic Idea: (implicit) tufted cover + intrinsic edge flips

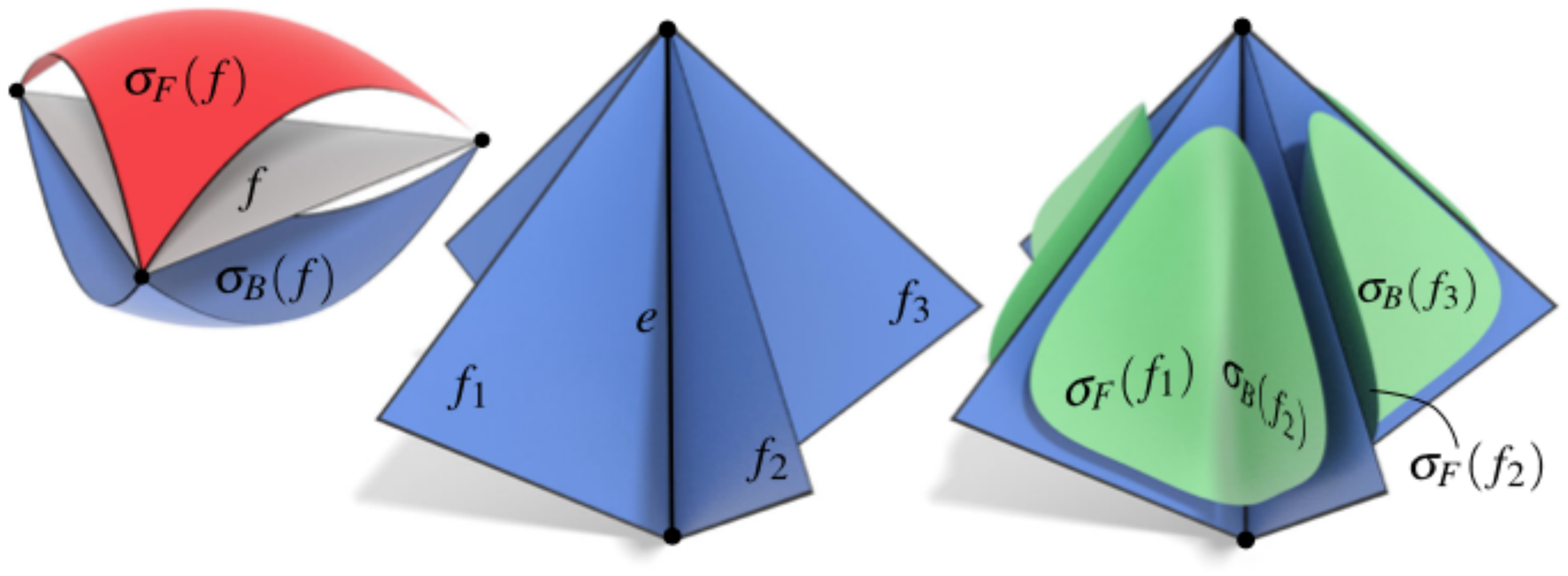
- Tufted cover has the same number of vertices
- Laplacian is constructed locally per face of the Tufted cover, and summed up to define $L \in \mathbb{R}^{|V|\times|V|}$

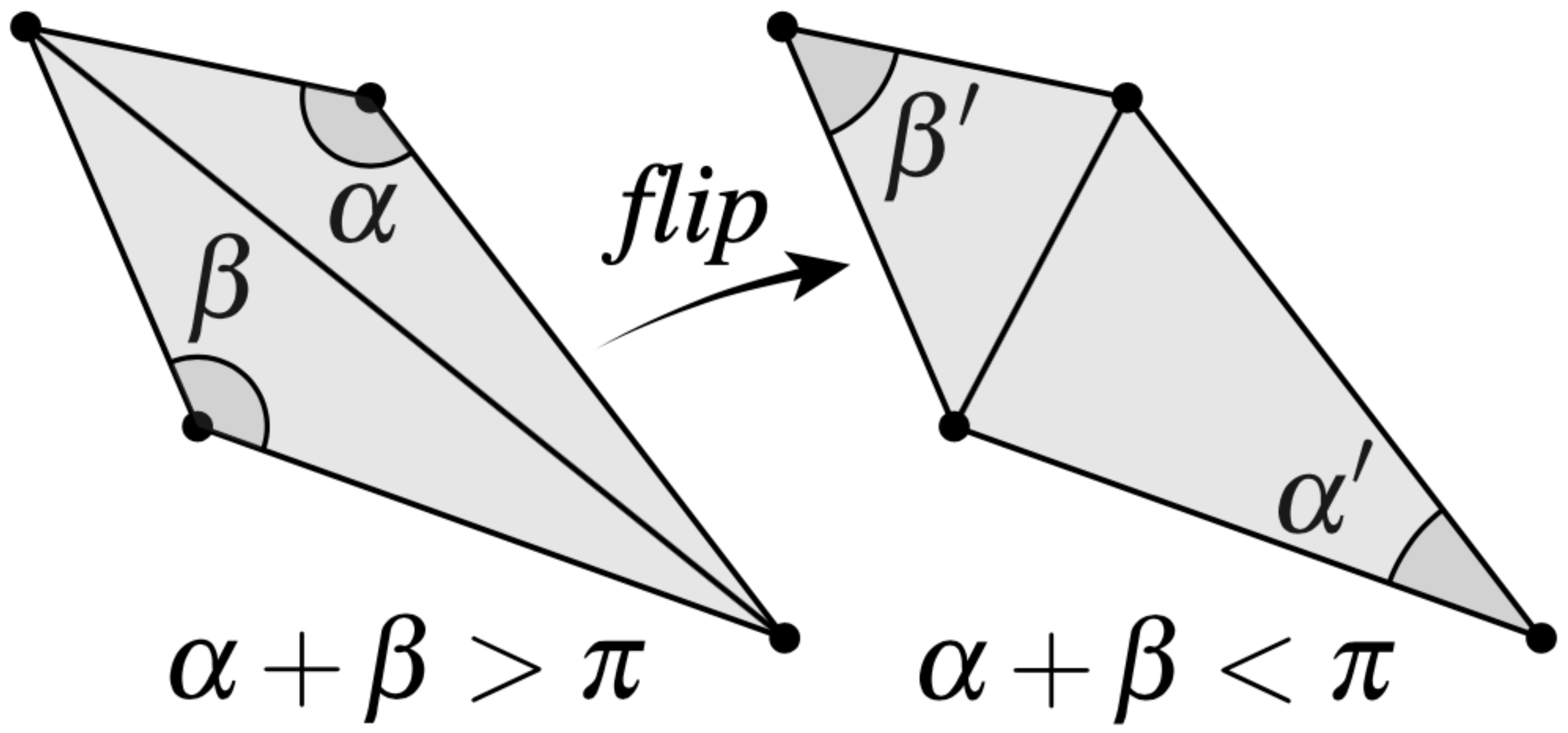


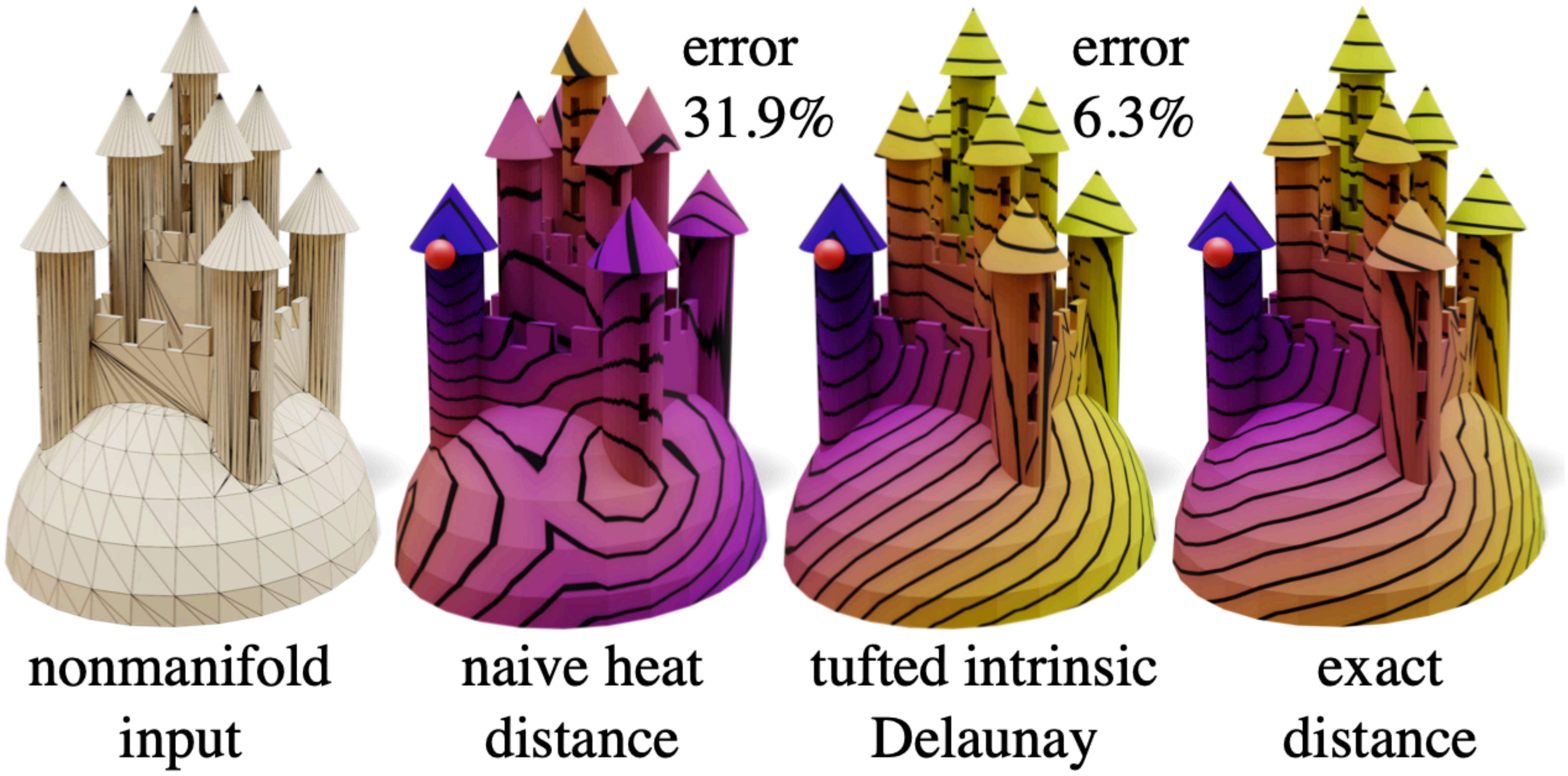


Basic Idea: (implicit) tufted cover + intrinsic edge flips

- Implicit construction requires « triangle ordering » at nonmanifold edges
- Efficient data structure (just flag on the intrinsic representation of M)
- Intrinsic Delaunay flip on the tufted to ensure positive « cotan » weights







⇒ *Laplace-Beltrami on nonmanifold meshes that « makes sense » (exactly cotan+intrinsic on manifold, mimicking an extrinsic thickening around nonmanifold edges)*

Laplace-Beltrami on point clouds

Laplace-Beltrami on point could

- General idea: estimate the underlying manifold locally using k-nearest neighbours [Belkin et al, Liu et al]:
    - project neighbours onto the estimated tangent plane,
    - construct a planar Delaunay triangulation to estimate the « mass matrix » from the local triangulation
    - Heat kernel based laplacian (~ edge weight via a Gaussian function of distances) (cf later)

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Pointwise convergence proof for some « fairly uniform » point distributions

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  - project neighbours onto the estimated tangent plane
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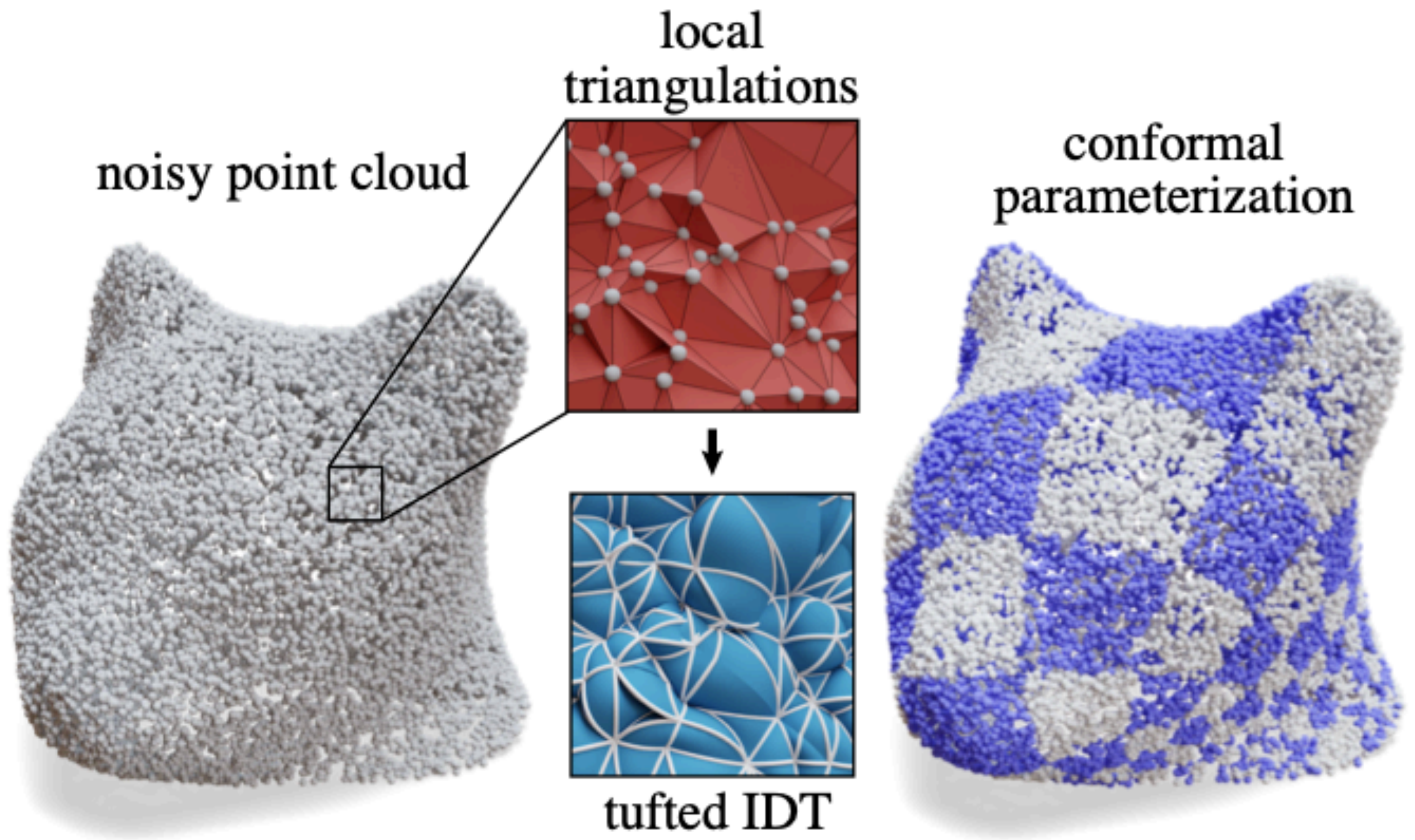
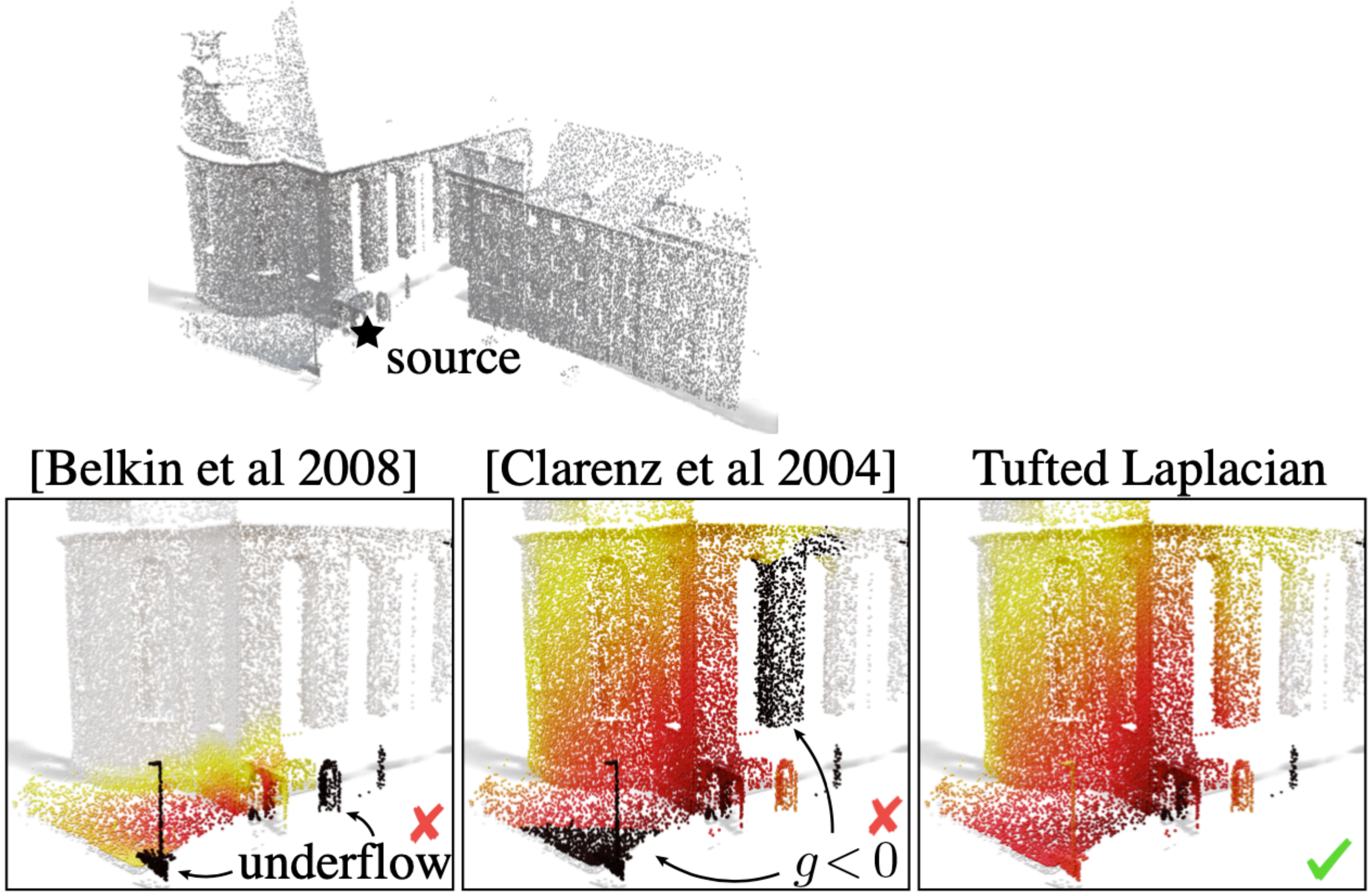
- Alternative: rough triangulation + tufted cover + edge flips

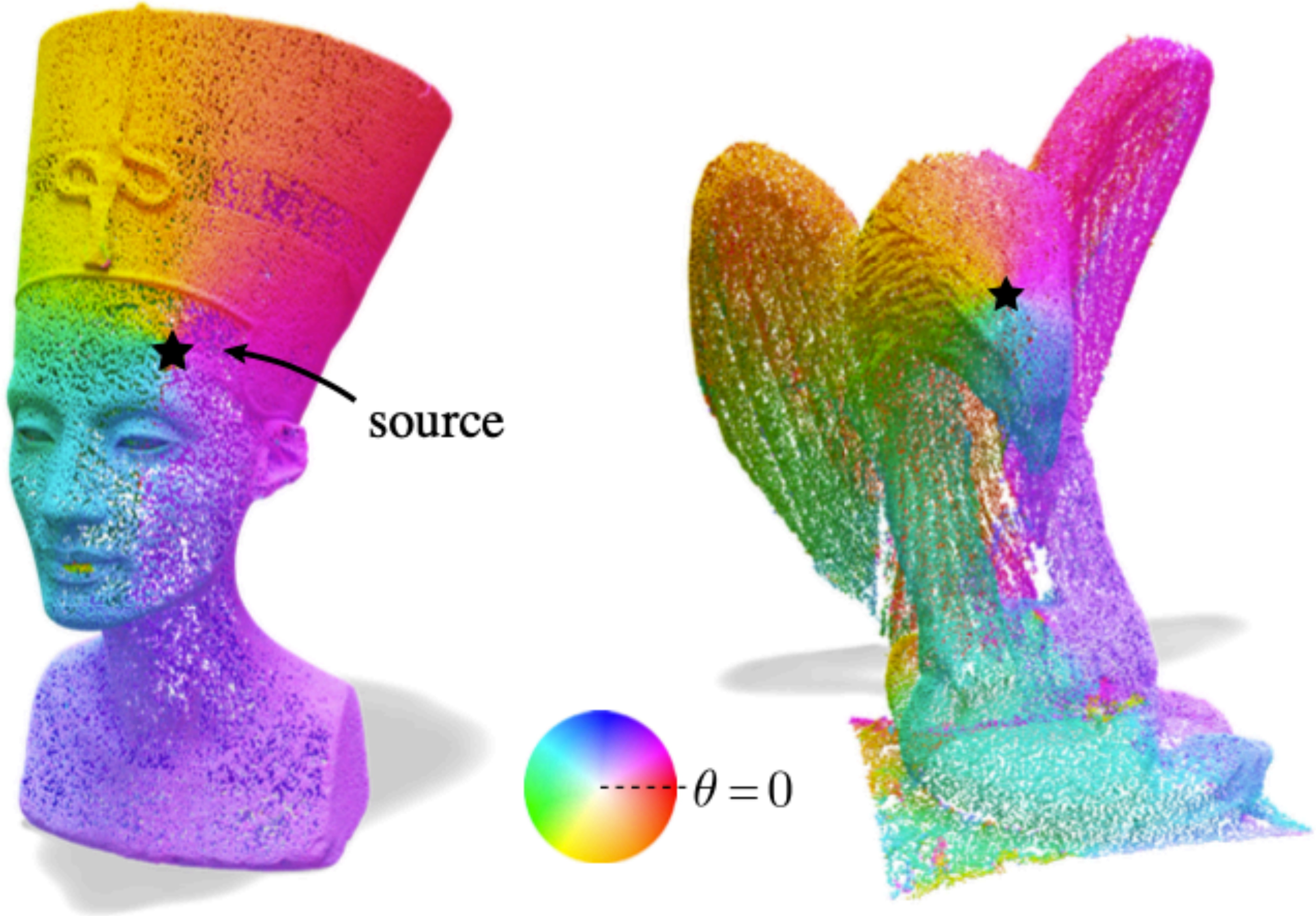
Pointwise convergence proof for some « fairly uniform » point distributions



Examples



source
[Belkin et al 2008]
[Clarenz et al 2004]
Tufted Laplacian
underflow
$g < 0$



source
$\theta = 0$

Laplace-Beltrami on digital surfaces

(definition, strong consistency…)

⇒ *HTML slide deck*