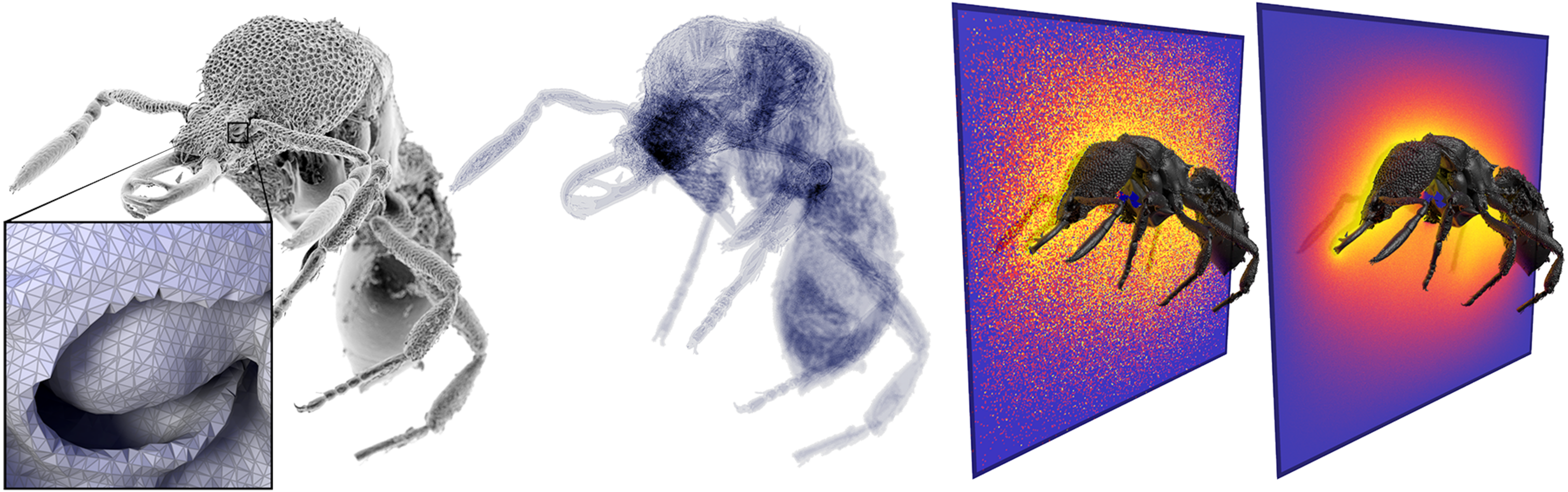
Monte Carlo Geometry Processing



Just a little bit of Laplace Equation

$$
\begin{aligned}
\Delta u &= 0 \quad \text{on } \Omega, \\
u &= g \quad \text{on } \partial\Omega, \quad g : \partial\Omega \to \mathbb{R}
\end{aligned}
$$

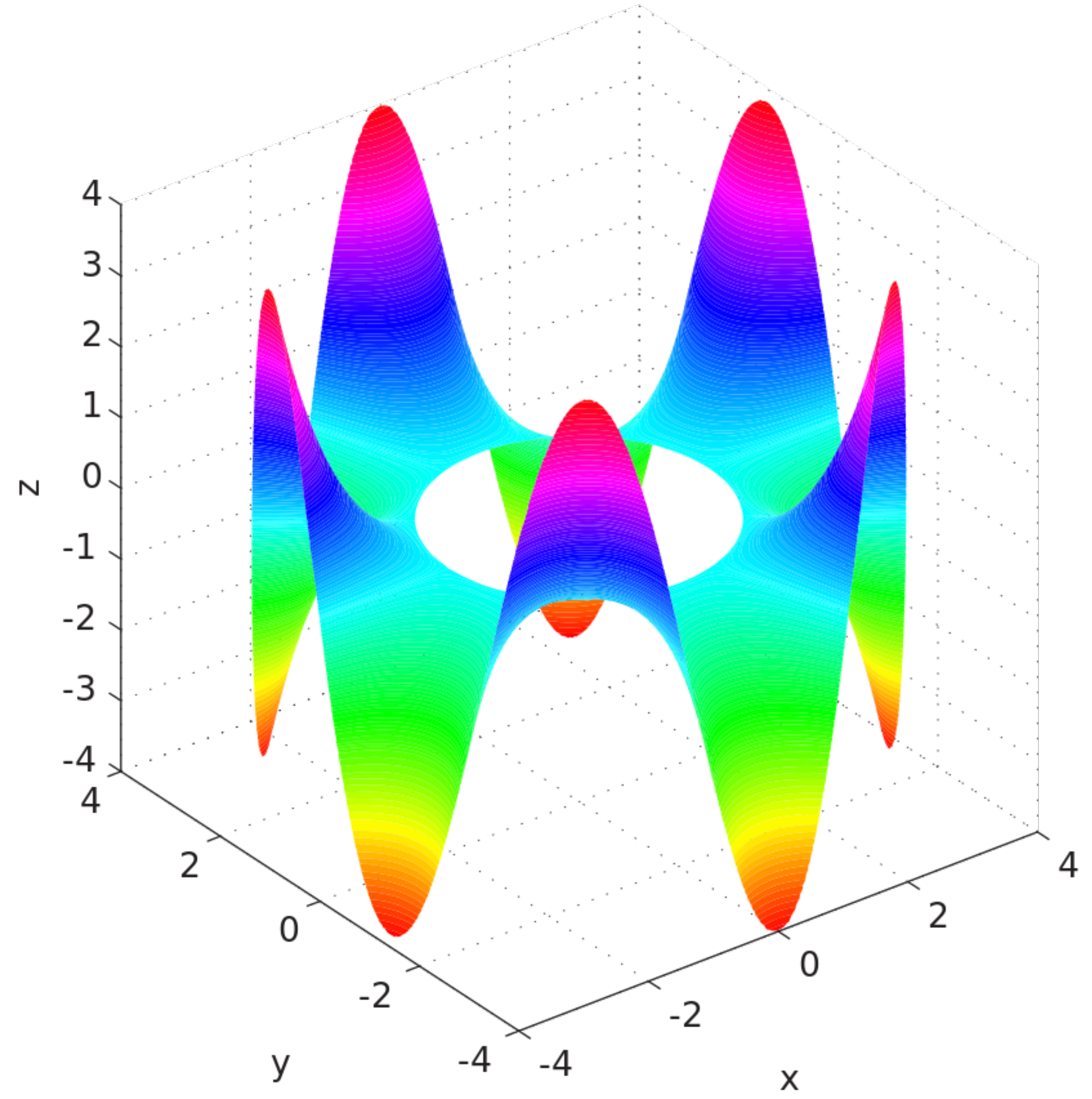
Just a little bit of Laplace Equation

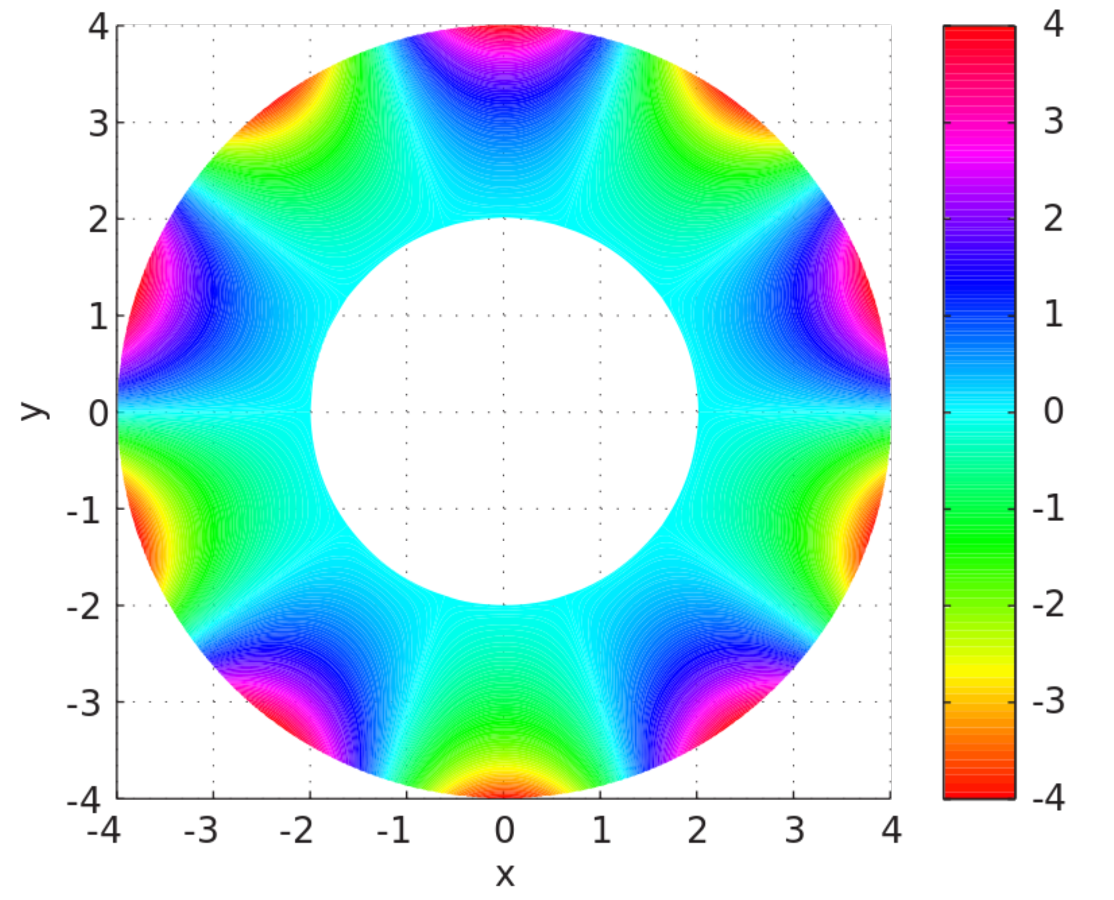
$$\Delta u = 0 \quad \text{on } \Omega,$$
$$u = g \quad \text{on } \partial\Omega, \qquad g : \partial\Omega \to \mathbb{R}$$

Ω is an annulus

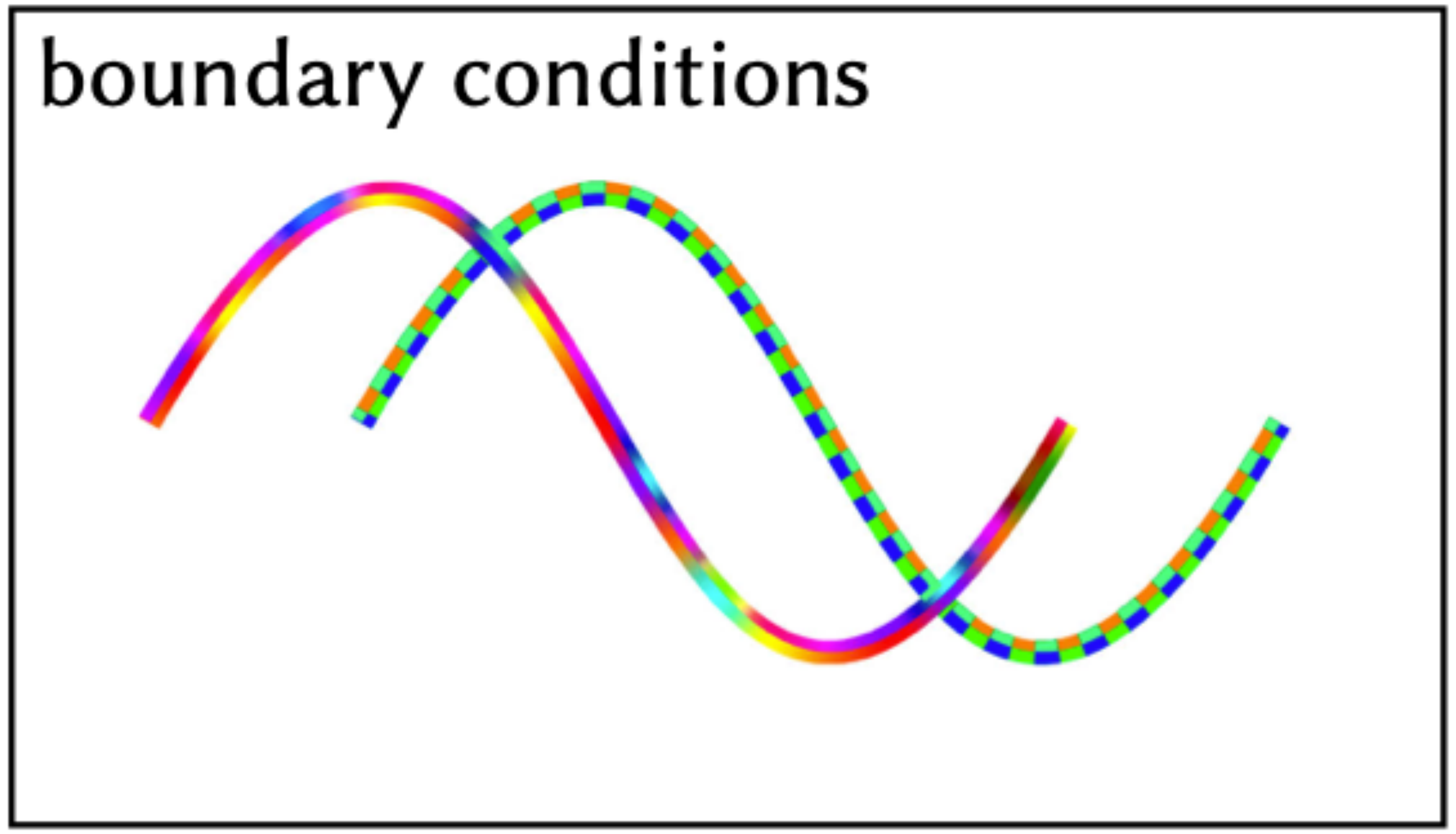
$u = 0$ inner boundary of $\partial\Omega$

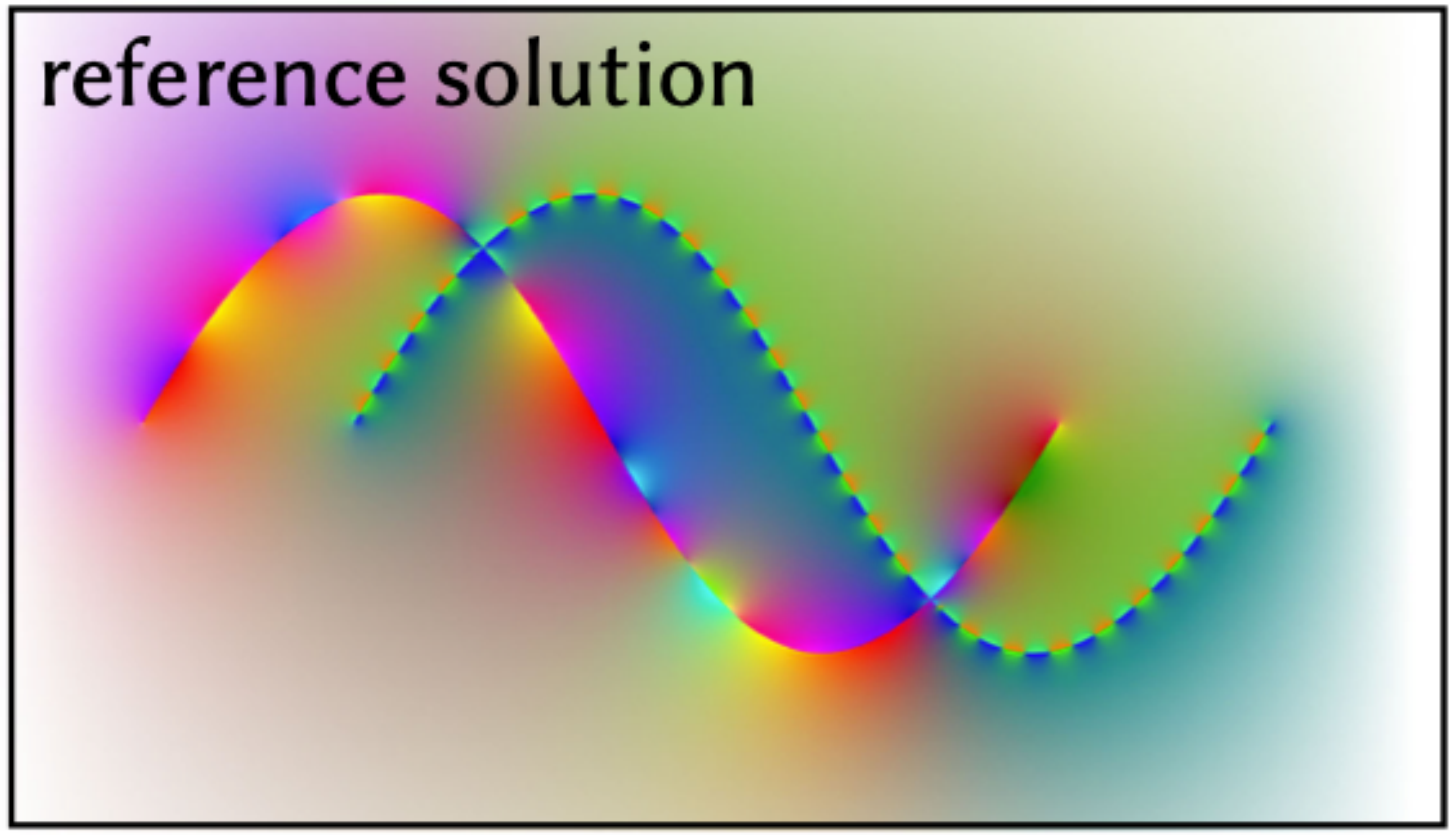
$u = 4\sin(\theta)$ exterior boundary of $\partial\Omega$





Laplace equation as a diffusion process



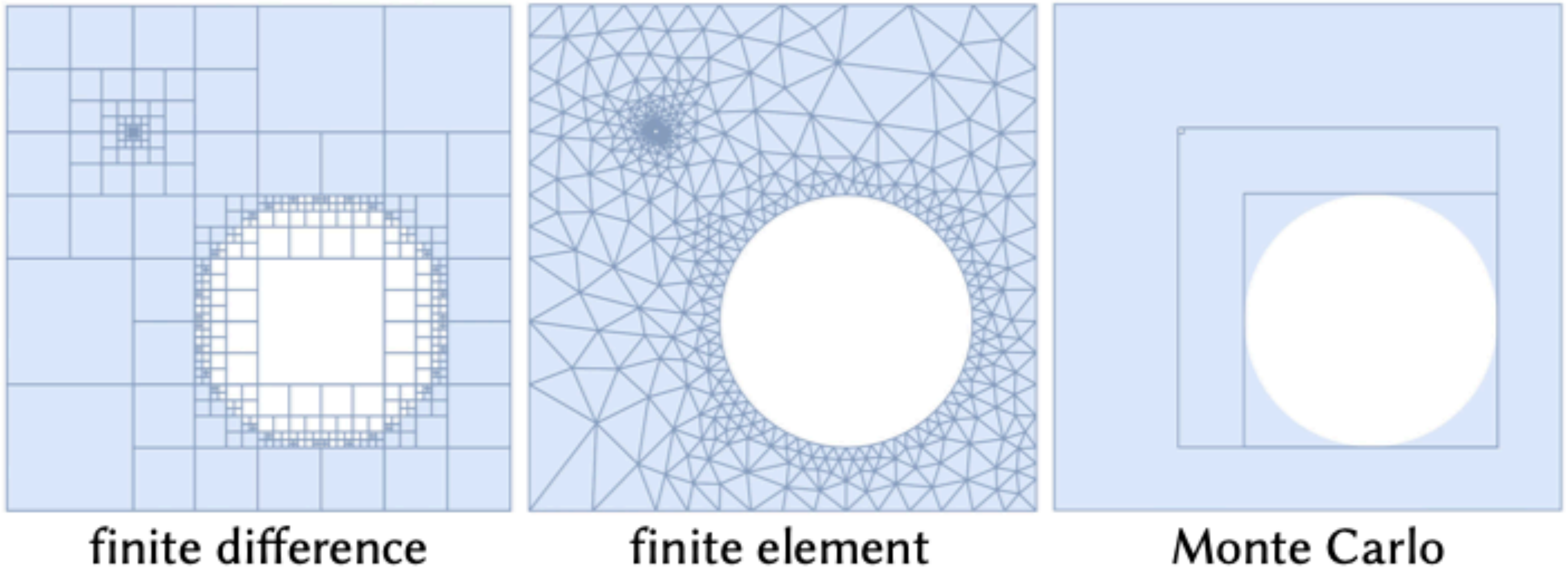


Laplace / Laplace-Beltrami operator

$$\Delta u = -\, div(\nabla u)$$

On Cartesian domains $$\Delta f(x, y) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$$

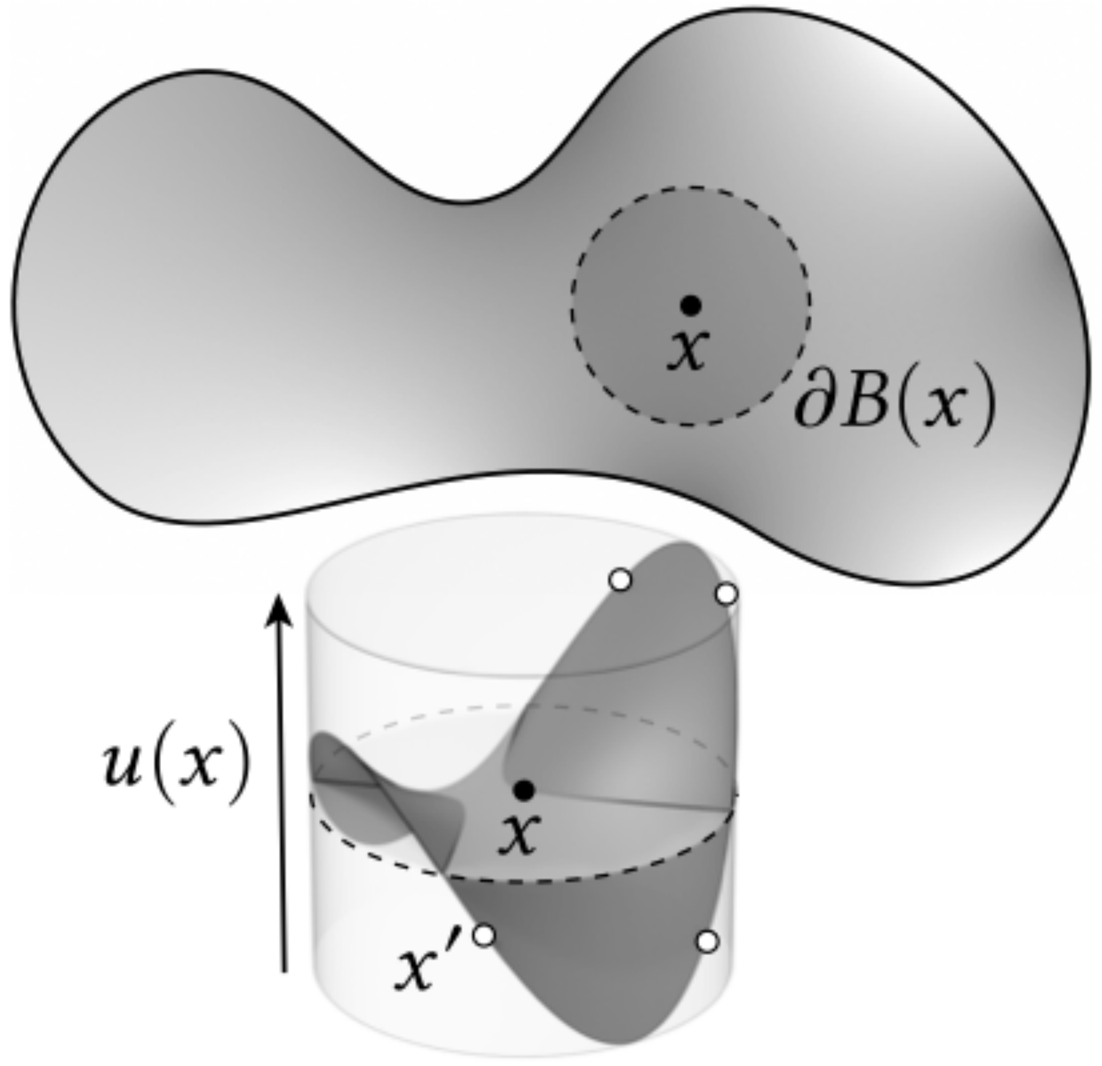
Many discretisation toolboxes: Finite difference, finite element, finite volume, discrete exterior calculus…



$$\Delta u = 0 \quad u = g \quad \text{on } \partial\Omega .$$

- ***mean value property***

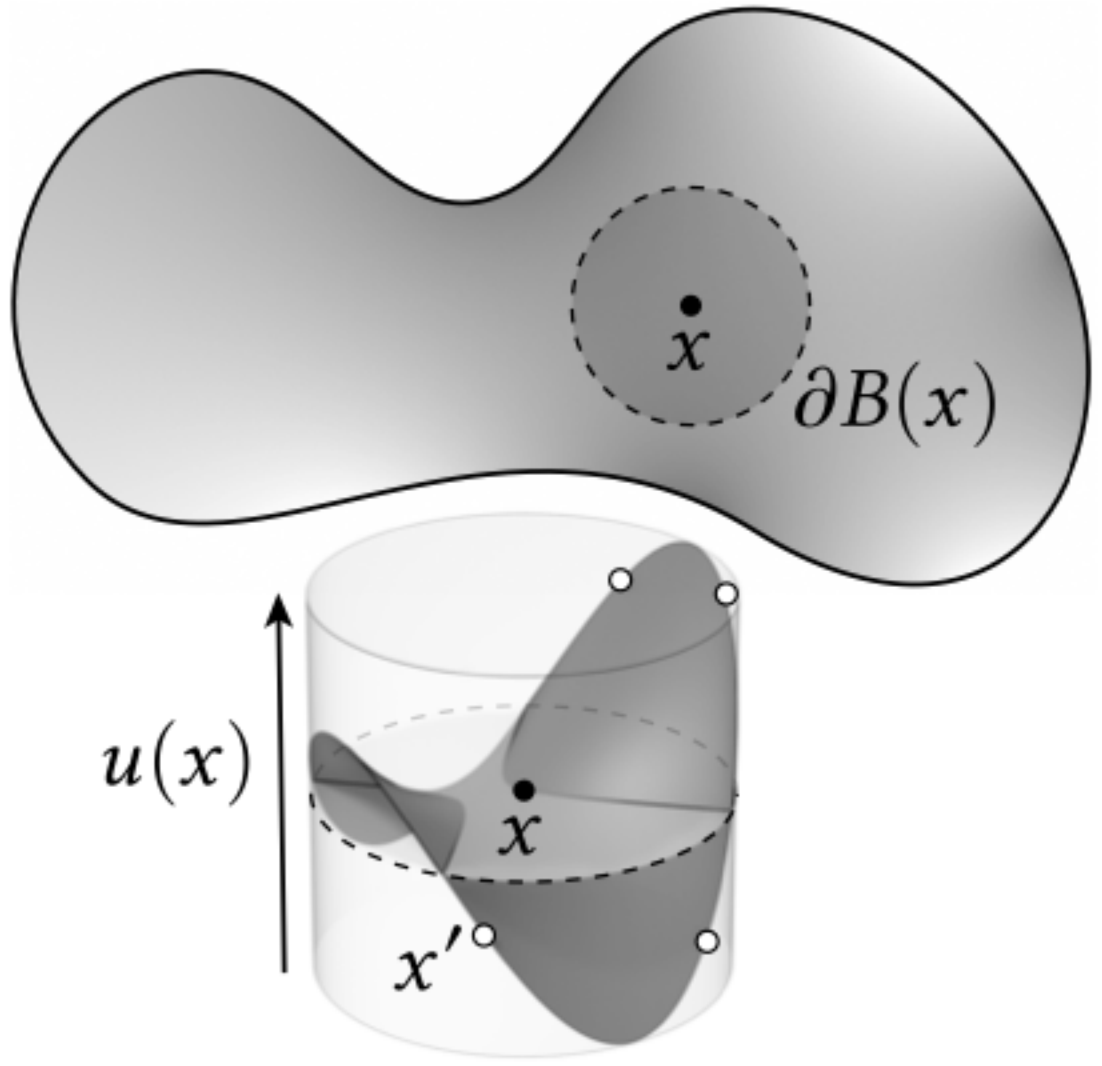
$$u(x) = \frac{1}{|\partial B(x)|} \int_{\partial B(x)} u(y) dy$$



$$\Delta u = 0 \quad u = g \quad \text{on } \partial\Omega\,.$$

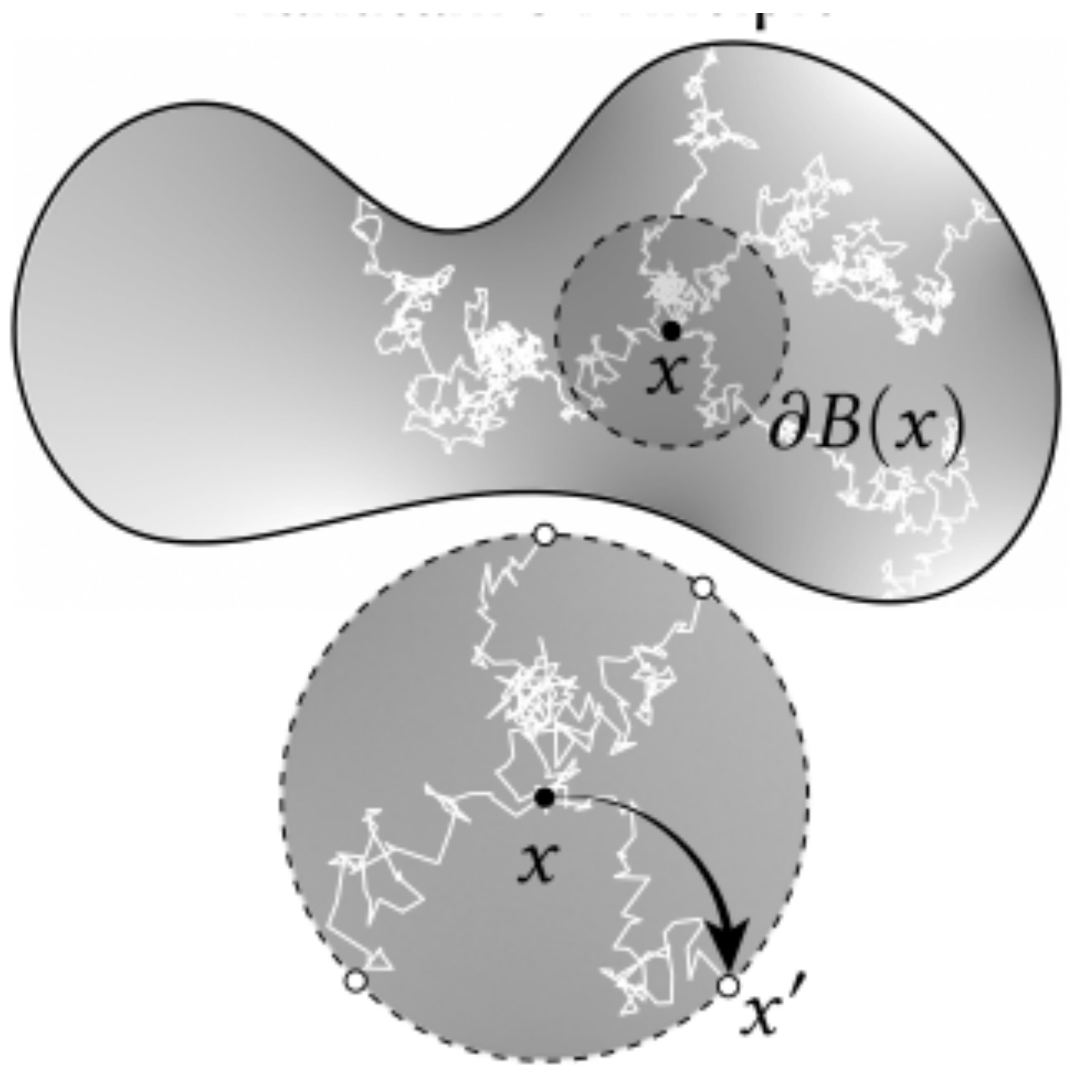
- ***mean value property***

$$u(x) = \frac{1}{|\partial B(x)|}\int_{\partial B(x)} u(y)dy$$



- ***Random walker***

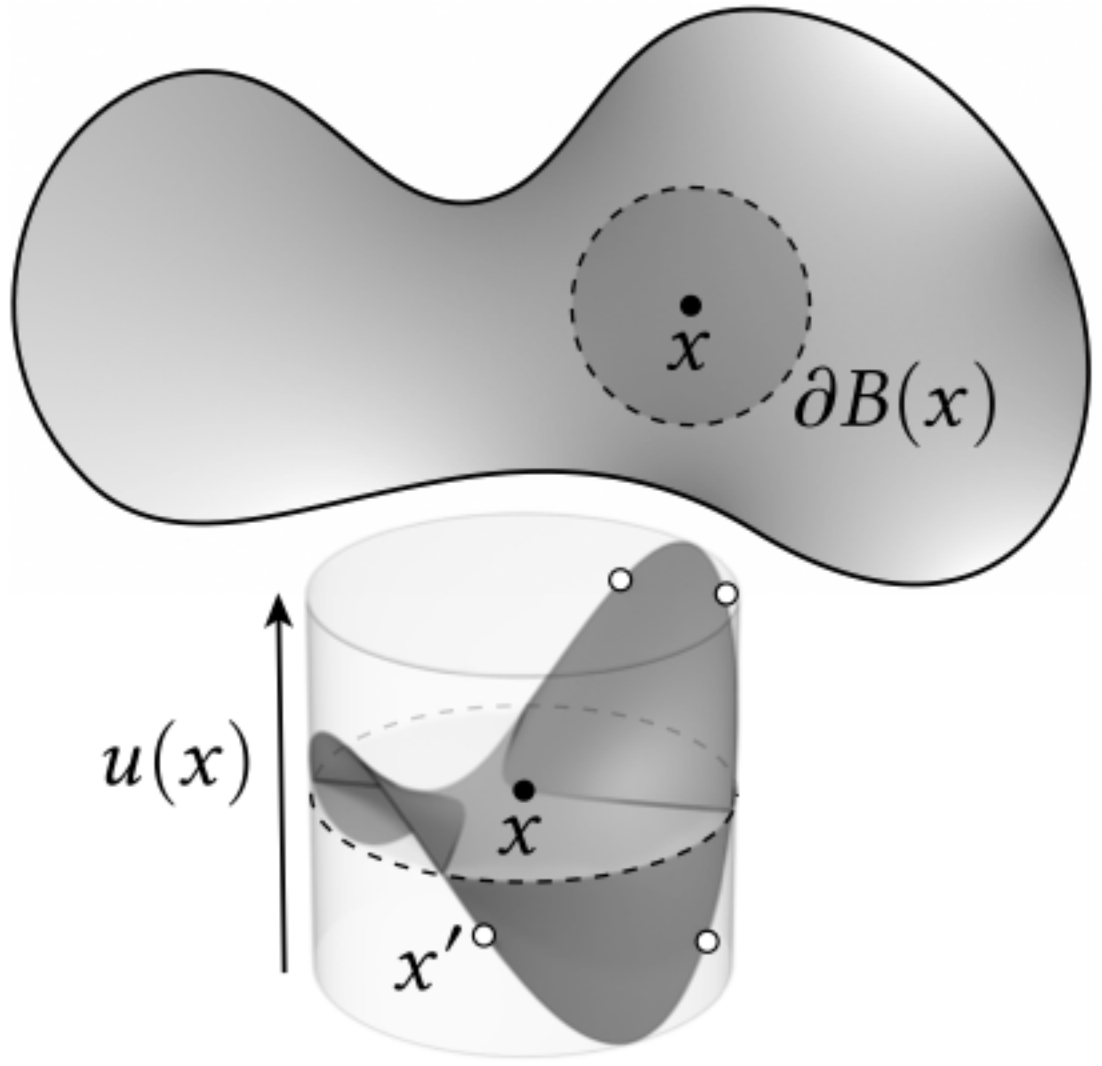
$$u(x) = E[g(y)], \quad y \in \partial\Omega$$



$$\Delta u = 0 \quad u = g \quad \text{on } \partial\Omega\,.$$

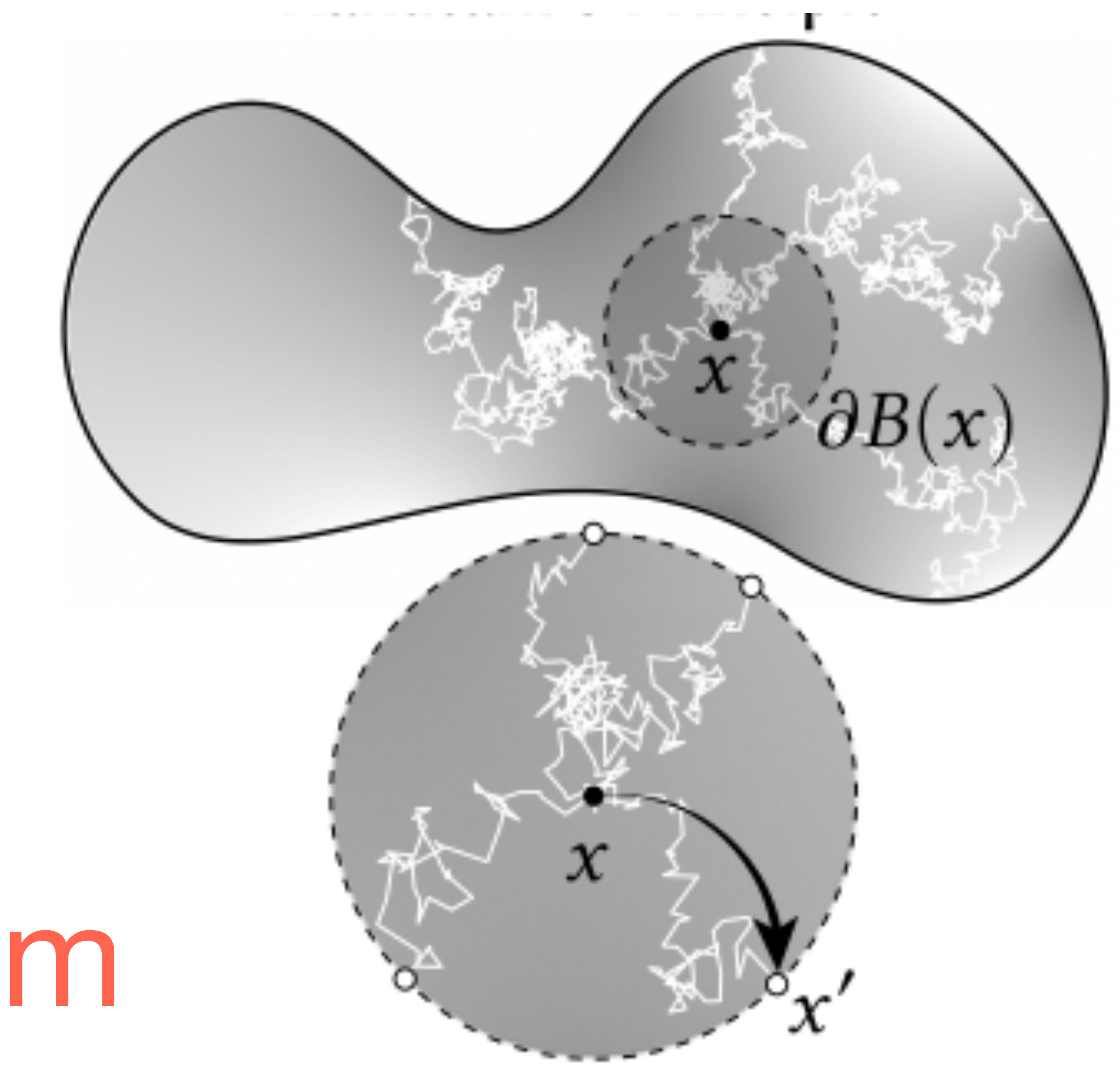
- ***mean value property***

$$u(x) = \frac{1}{|\partial B(x)|}\int_{\partial B(x)} u(y)dy$$



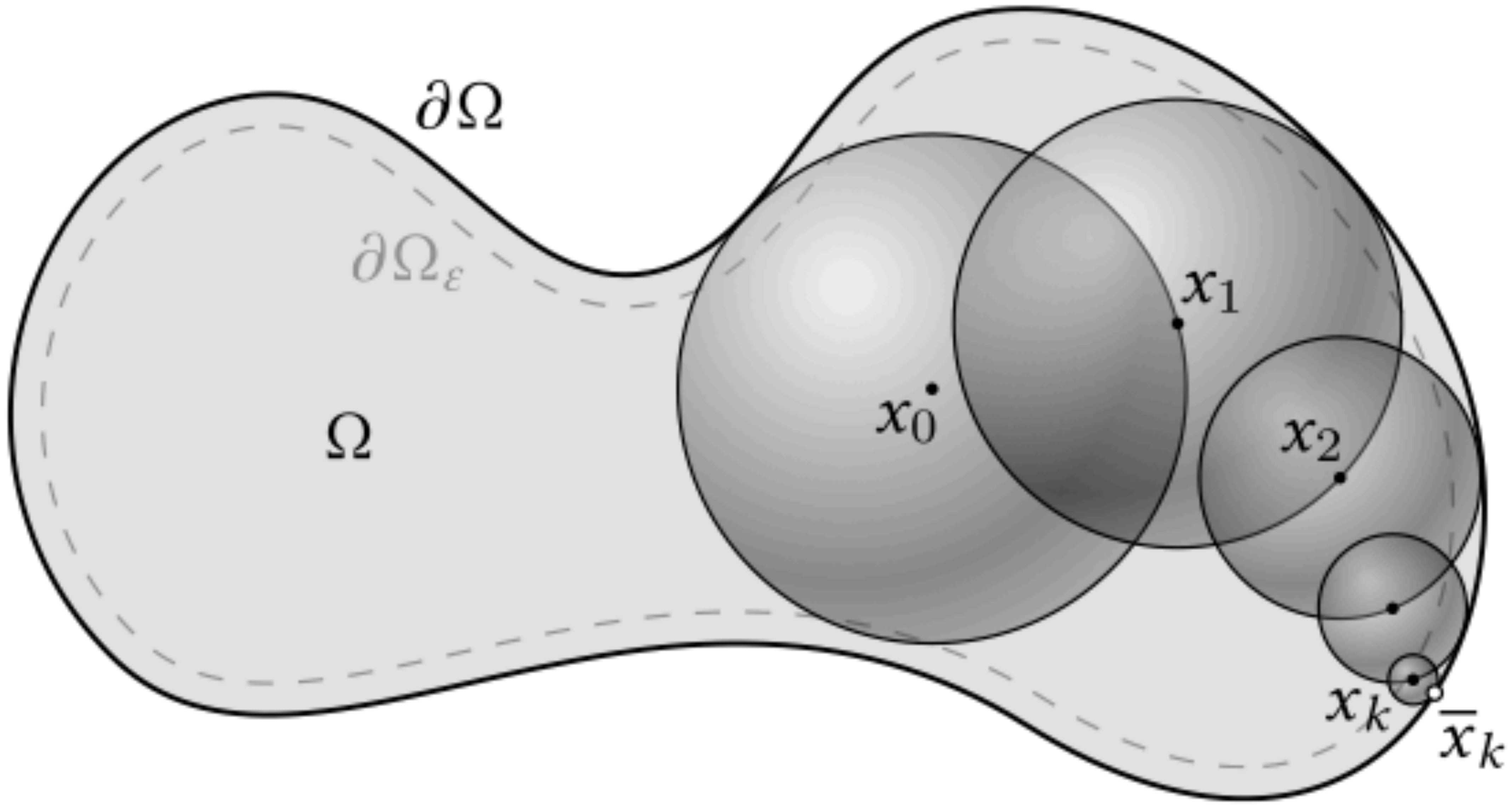
- ***Random walker***

$$u(x) = E[g(y)], \quad y \in \partial\Omega$$



⇒ Solving linear elliptic PDE as a MC integration problem

Walk Of Sphere Algorithm



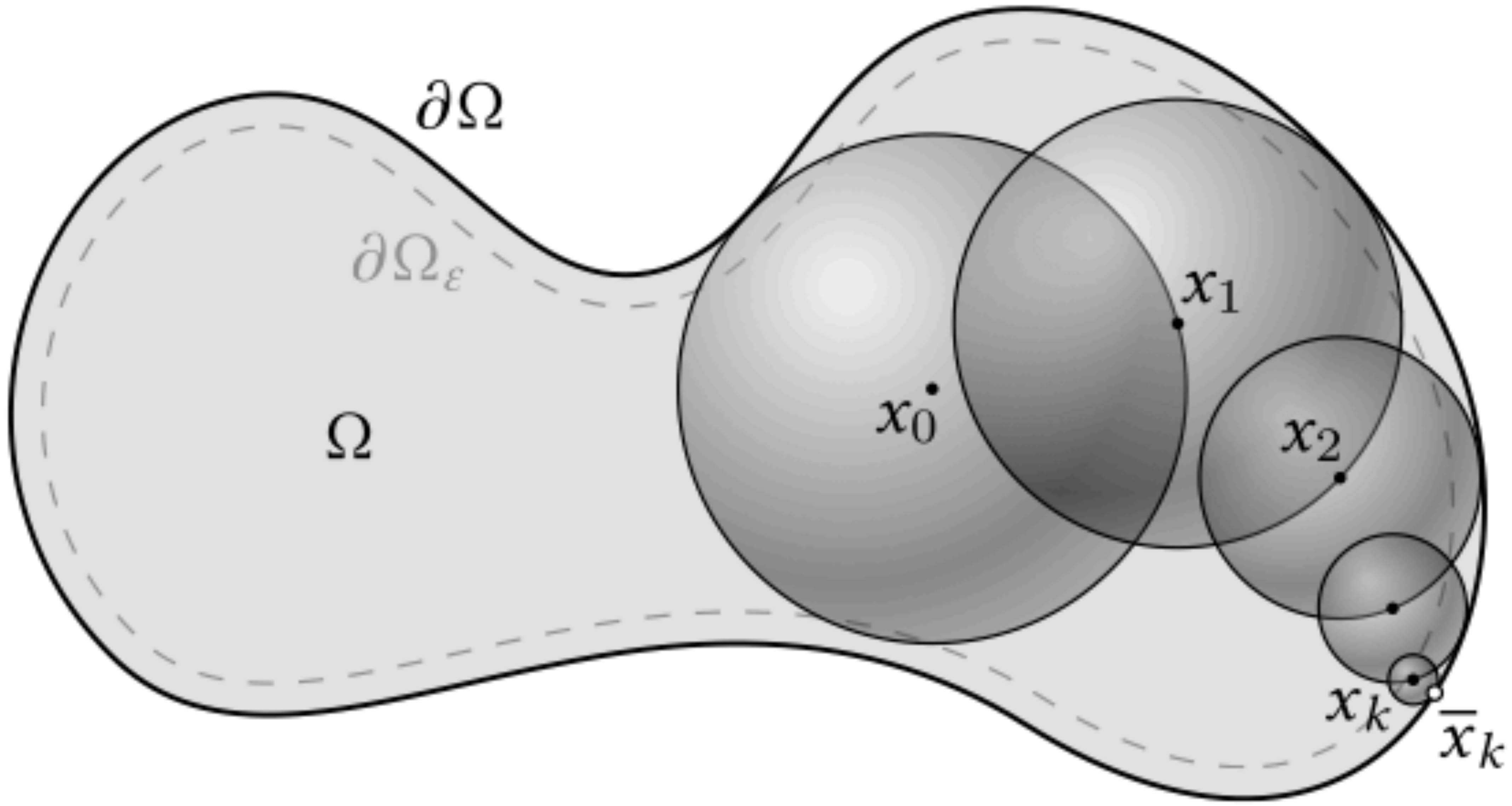
Path construction

To estimate $u(x_0)$, compute the inscribed largest sphere centered at x_0,

Sample the boundary of the sphere to define x_1, compute its largest inscribed ball

Repeat until we get a sample x_k close to the boundary, accumulate contributions $g(\bar{x}_k)$

Walk Of Sphere Algorithm



Path construction

Nearest Neighbor Search

To estimate $u(x_0)$, compute the inscribed largest sphere centered at x_0,

Sample the boundary of the sphere to define x_1, compute its largest inscribed ball

Repeat until we get a sample x_k close to the boundary, accumulate contributions $g(\bar{x}_k)$

Poisson, Screened Poisson and biharmonic equations

$$\Delta u = f \quad on\, \Omega$$
$$u = g \quad on\, \partial\Omega$$

$$u(x) = \frac{1}{|\partial B(x)|}\int_{\partial B(x)} u(y)dy + \int_{B(x)} f(y)G(x,y)dy$$

Poisson, Screened Poisson and biharmonic equations

$$\Delta u = f \quad on\, \Omega$$
$$u = g \quad on\, \partial\Omega$$

$$u(x) = \frac{1}{|\partial B(x)|}\int_{\partial B(x)} u(y)dy + \int_{B(x)} f(y)G(x,y)dy$$

$\Rightarrow$ we need to sample $\partial B(x)$ and $B(x)$

Poisson, Screened Poisson and biharmonic equations

$$\Delta u = f \quad on\, \Omega$$
$$u = g \quad on\, \partial\Omega$$

$$u(x) = \frac{1}{|\partial B(x)|}\int_{\partial B(x)} u(y)dy + \int_{B(x)} f(y)G(x,y)dy$$

$\Rightarrow$ we need to sample $\partial B(x)$ and $B(x)$

$$\Delta u - cu = f \quad on\, \Omega$$
$$u = g \quad on\, \partial\Omega$$

Poisson, Screened Poisson and biharmonic equations

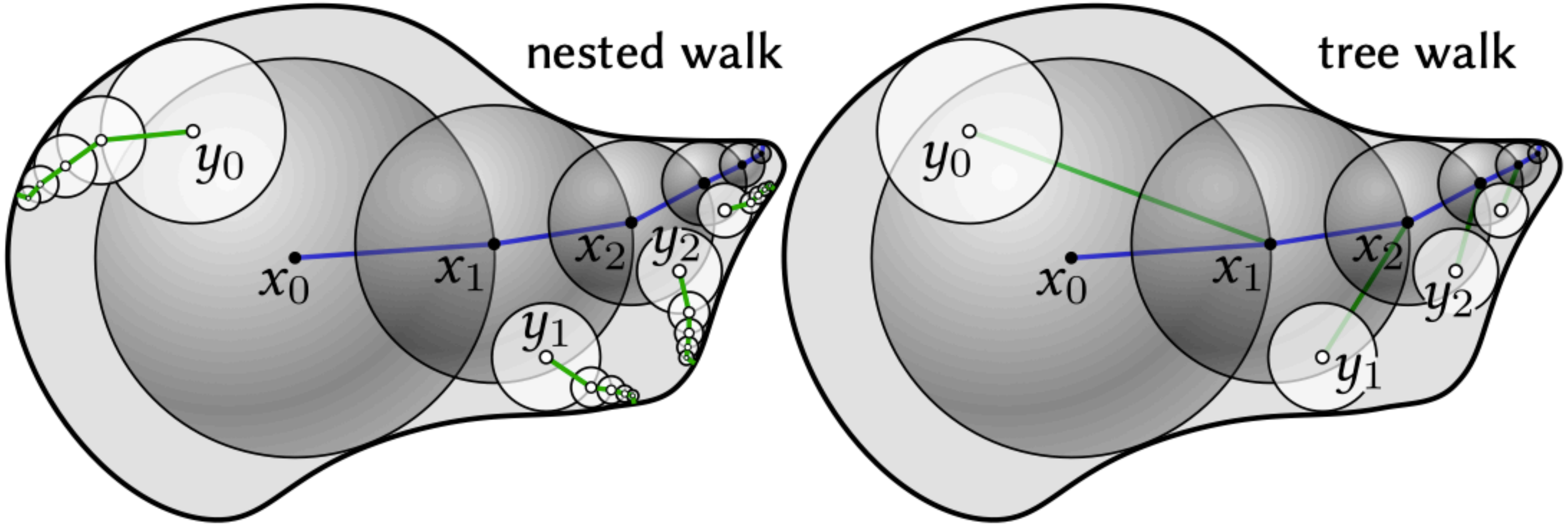
$$\Delta u = f \quad on\, \Omega$$
$$u = g \quad on\, \partial\Omega$$

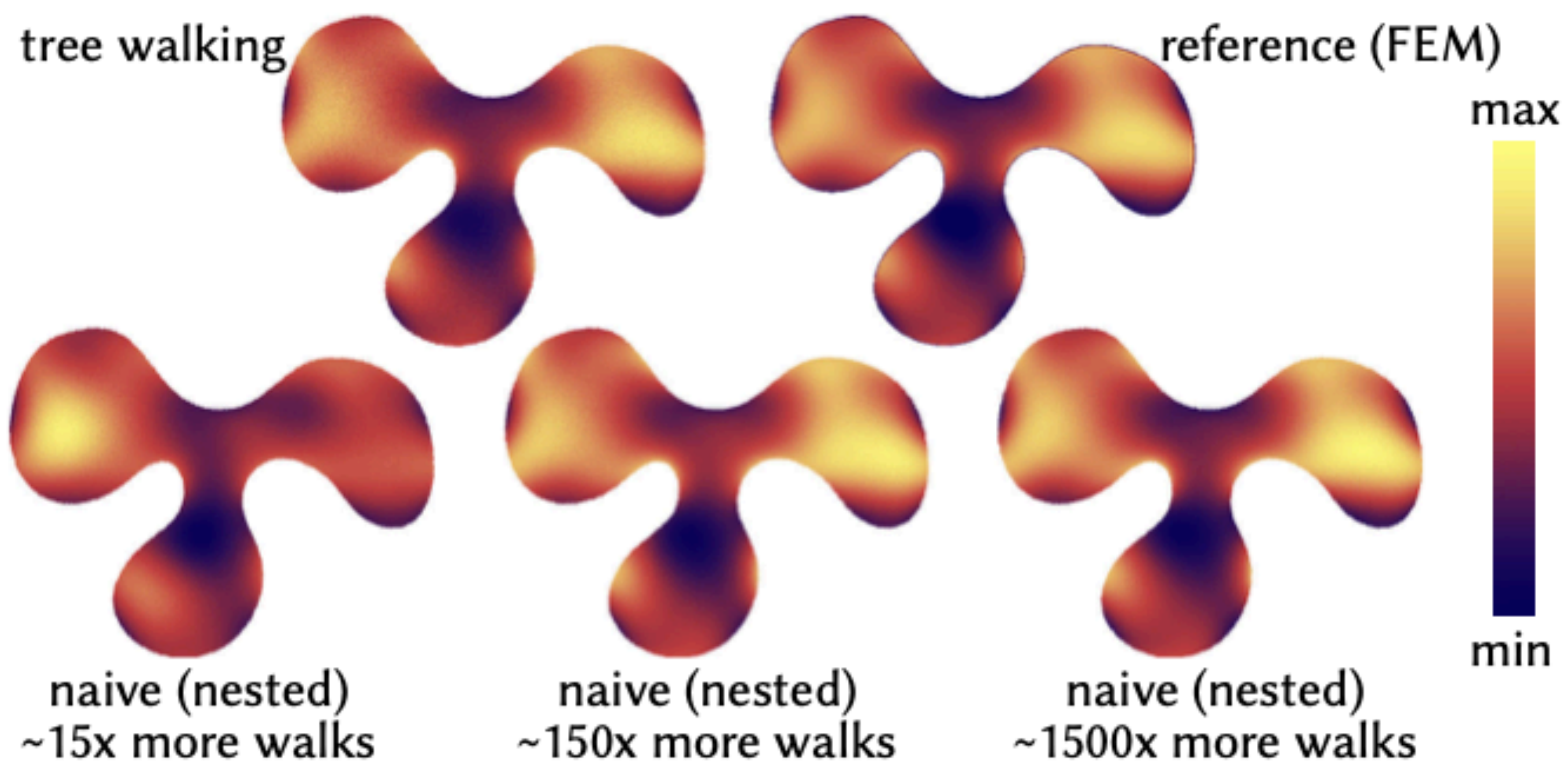
$$u(x) = \frac{1}{|\partial B(x)|}\int_{\partial B(x)} u(y)dy + \int_{B(x)} f(y)G(x,y)dy$$

$\Rightarrow$ we need to sample $\partial B(x)$ and $B(x)$

$$\Delta u - cu = f \quad on\, \Omega$$
$$u = g \quad on\, \partial\Omega$$

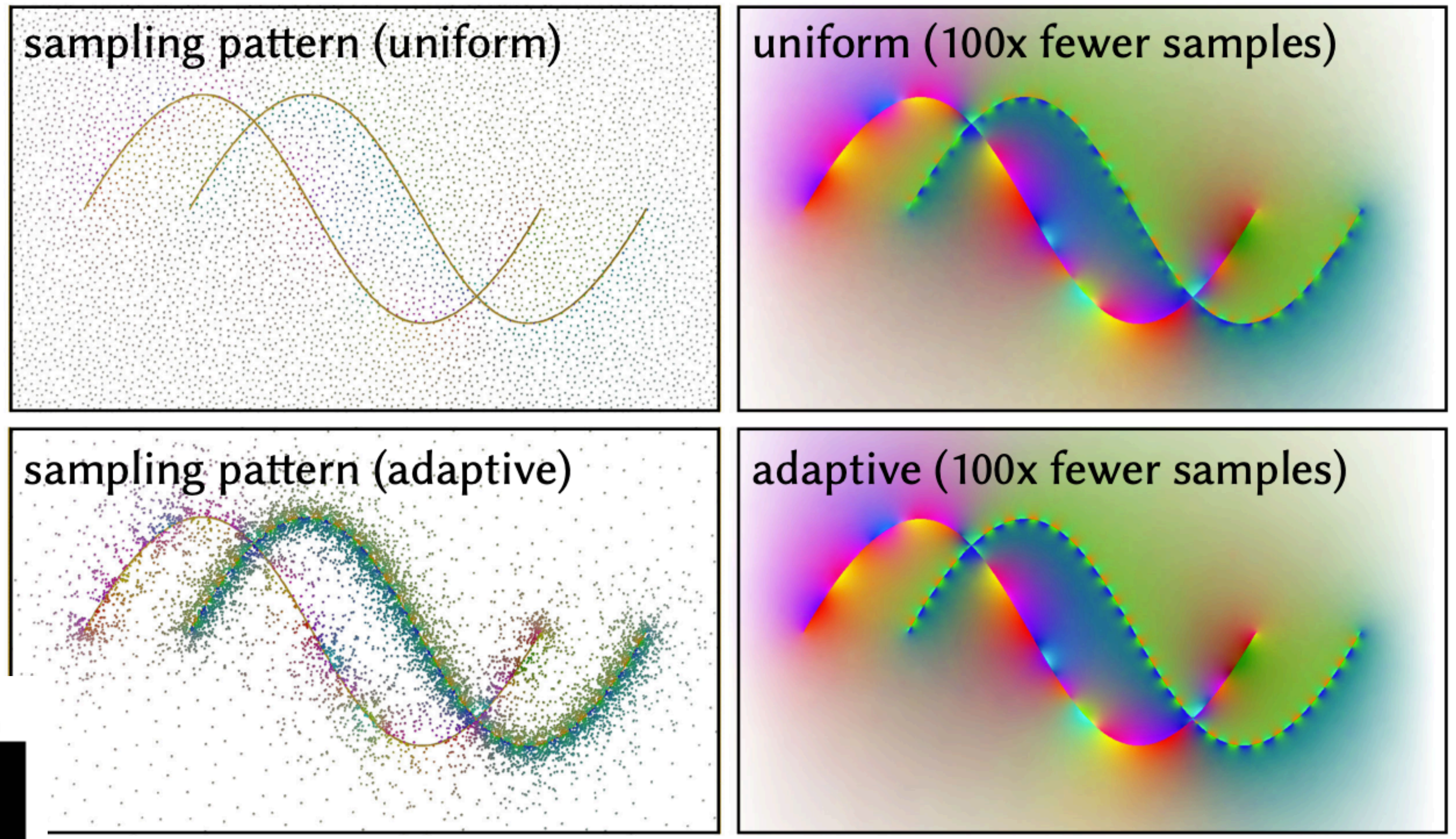
$$\Delta^2 u = 0 \quad on\, \Omega$$
$$u = g \quad on\, \partial\Omega$$
$$\Delta u = h \quad on\, \partial\Omega$$


nested walk
tree walk
x_0
x_1
x_2
y_0
y_1
y_2

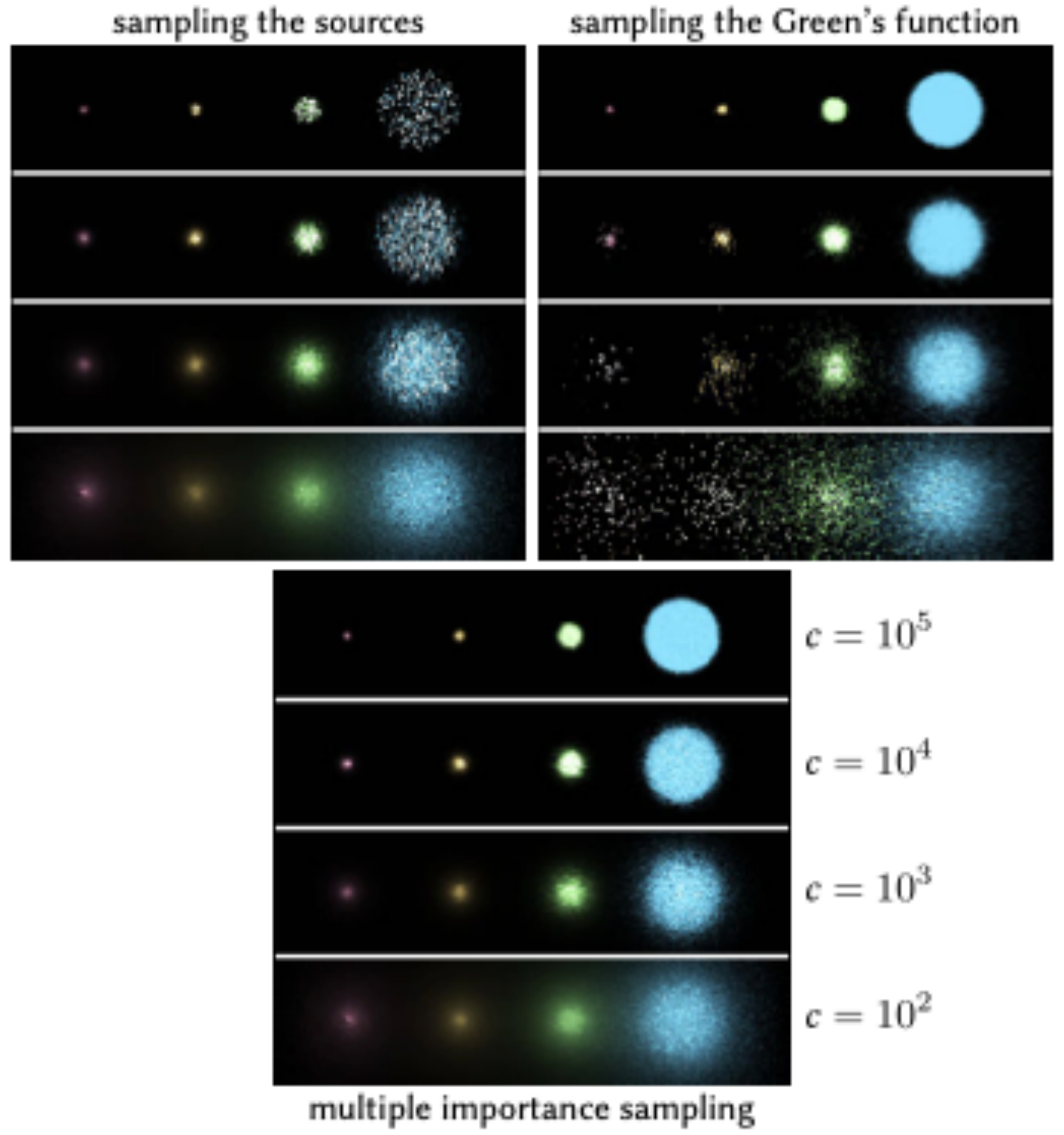

tree walking
reference (FEM)
max
min
naive (nested)
~15x more walks
naive (nested)
~150x more walks
naive (nested)
~1500x more walks

⇒ Uniform sampling patterns

⇒ Adaptive / importance sampling



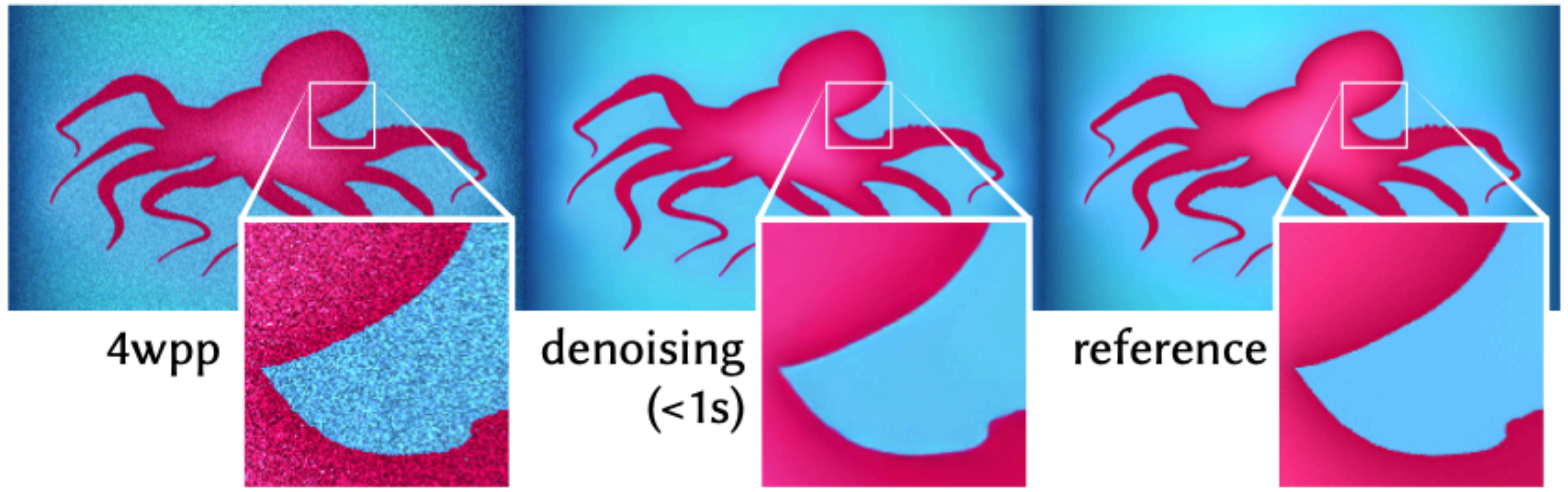
⇒ Multiple Importance Sampling



⇒ Path strategies

⇒ Russian roulette

⇒ Denoising



⇒ Uniform sampling patterns

⇒ Adaptive / importance sampling

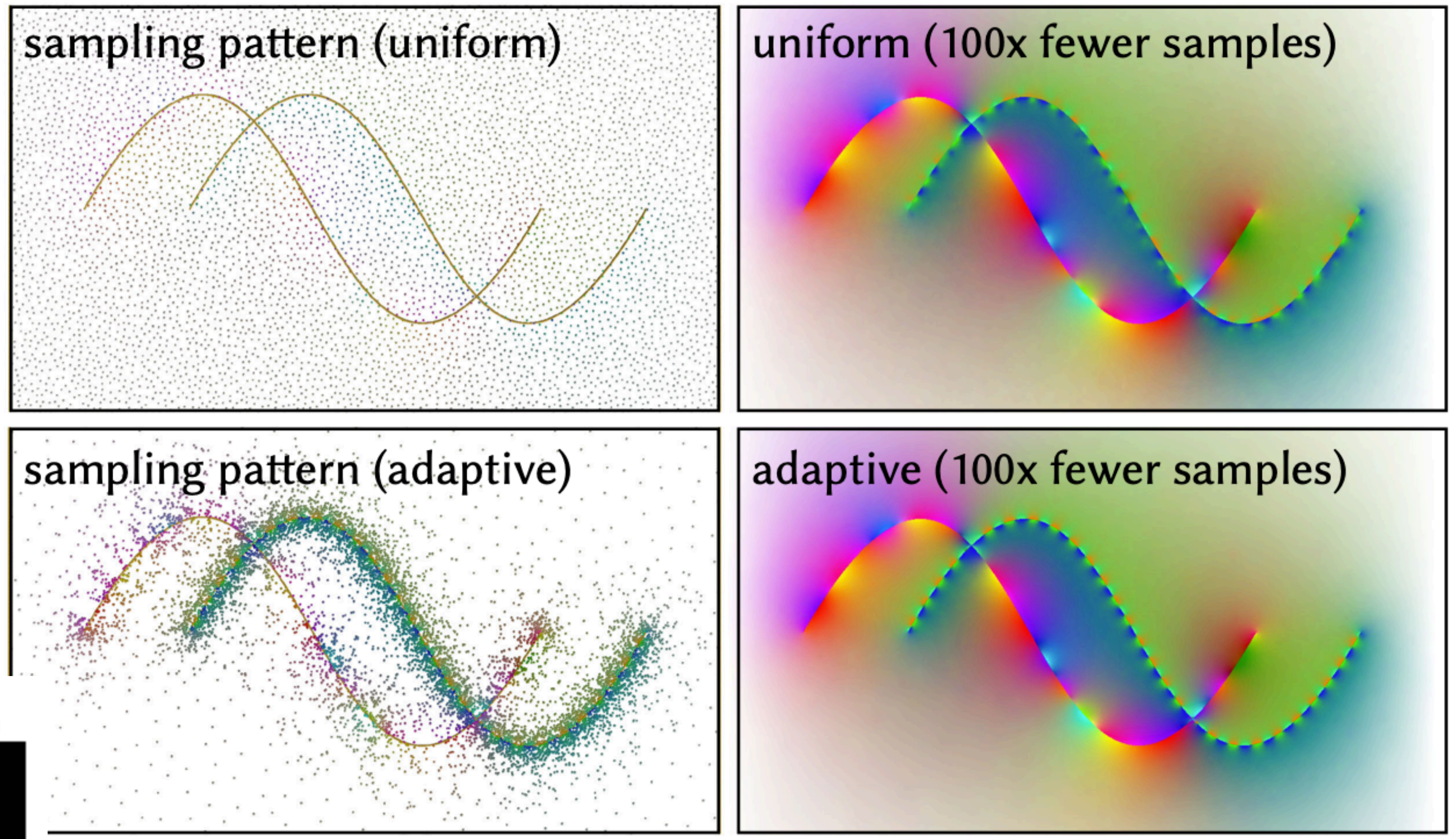
⇒ Control Variate

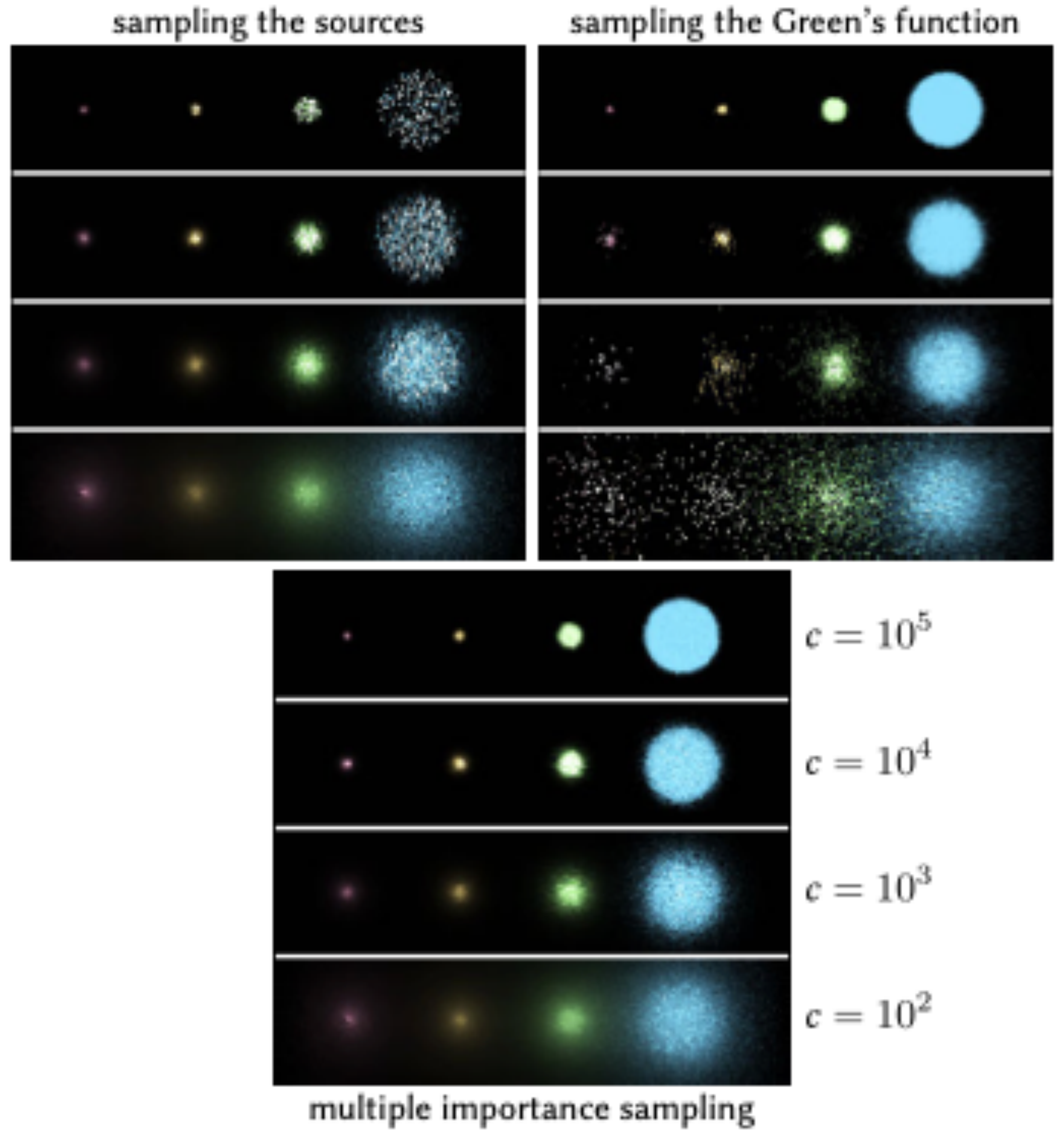
⇒ Multiple Importance Sampling

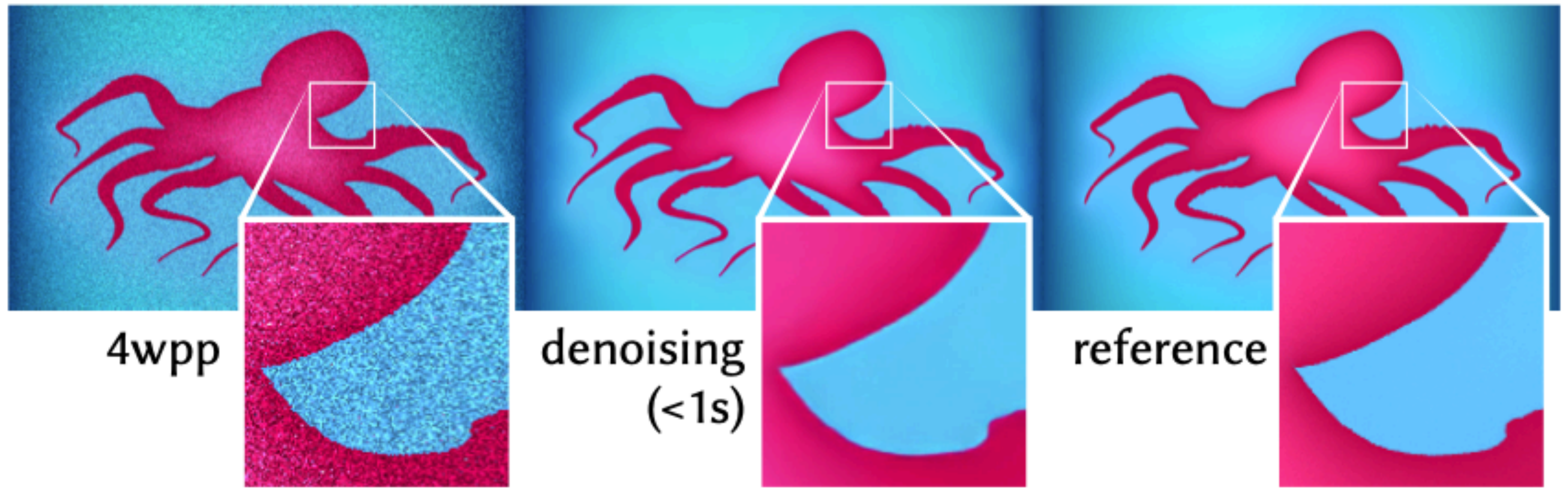
⇒ Path strategies

⇒ Russian roulette

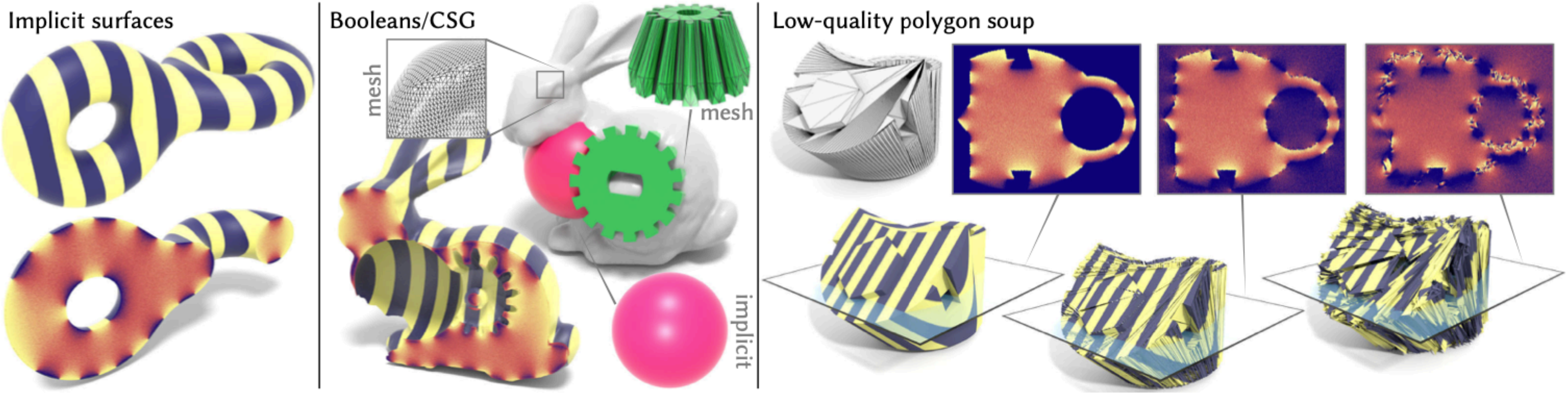
⇒ Denoising







PDE based *mesh-less* geometry processing



PDE based *mesh-less* geometry processing

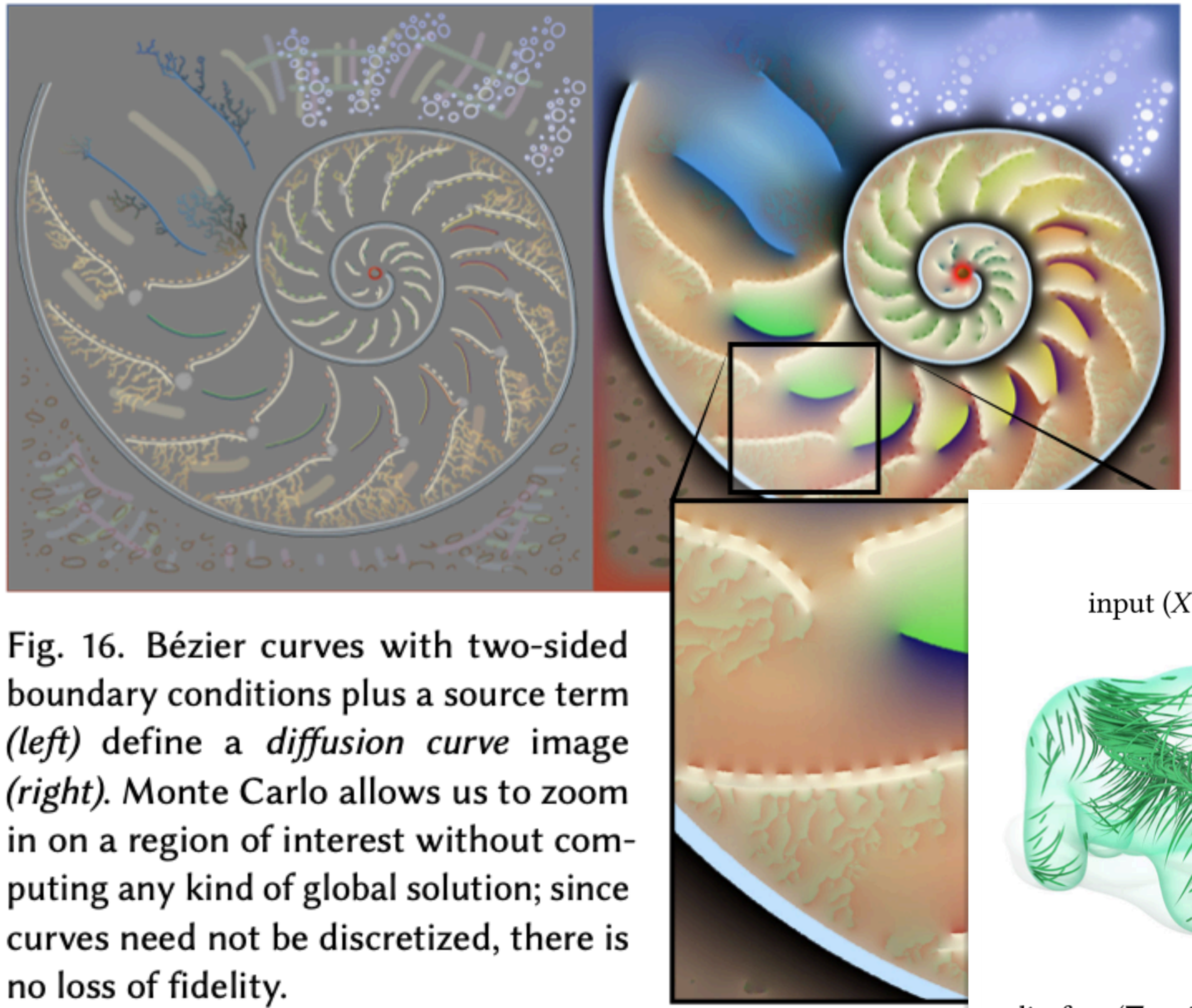


Fig. 16. Bézier curves with two-sided boundary conditions plus a source term *(left)* define a *diffusion curve* image *(right)*. Monte Carlo allows us to zoom in on a region of interest without computing any kind of global solution; since curves need not be discretized, there is no loss of fidelity.