Geometrical extensions of Wythoff's game

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Abstract

In 1905 Bouton gave the complete theory of a two player combinatorial game: the game of Nim. Two years later, Wythoff defines his game as "a modification" of the game of Nim. In this paper, we give the sets of the losing positions of geometrical extensions of Wythoff's game, where allowed moves are considered according to a set of vectors $(v_1, \ldots, v_n)$. When $n = 3$, we present algorithms and algebraic characterizations to determine the losing positions of such games. In the last part, we investigate a bounded version of Wythoff's game, and give a polynomial way to decide whether a game position is losing or not.

Key words: Wythoff's sequence, Nim, combinatorial games, game graph

1 Introduction

In a game of Nim with n heaps, two players alternately move from a configuration made up of n heaps of tokens. By turn, each player removes any number of tokens from a single heap. A player is not allowed to miss his turn. The winner is the player taking the last token, the other loses as he is unable to move again (see [5]).
As defined in [15], Wythoff's game is played with two heaps of tokens. Each player can either remove any number of tokens from a single heap (the Nim rule), or remove the same number of tokens from both heaps (Wythoff's rule). A position of Wythoff's game is denoted by a pair (a, b), where a and b are the number of tokens in each heap.

Definition 1 *A $\mathcal{N}$-position is a position from which there exists a winning move for the first player. A $\mathcal{P}$-position is a position from which there exists no good move for the first player.*

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Note that in the context of two-player games, a $\mathcal{P}$-position is also winning for the second player.

Definition 2 *Let $U \subset \mathbb{Z}_{\geq 0}$. We define the Minimum Excluded value of U by the smallest nonnegative integer not in U. It will be denoted by $Mex(U)$. In particular, $Mex(\emptyset) = 0$.*

The set of the $\mathcal{P}$-positions of Wythoff's game is described in e.g. [1,2,15]. The symmetry of the game implies that each pair (a, b) has its symmetrical (b, a) of the same type, i.e., $\mathcal{N}$- or $\mathcal{P}$-position. The following algorithm computes the sequence (a_n, b_n) (with $a_n \leq b_n$) of the $\mathcal{P}$-positions of Wythoff's game:

- $(a_0, b_0) = (0, 0)$ is the first $\mathcal{P}$-position.
- Assume that (a_i, b_i) is known for all $0 \leq i < n$. Then (a_n, b_n) is defined as follows: $a_n = Mex(a_0, \dots, a_{n-1}, b_0, \dots, b_{n-1})$, and $b_n = a_n + n$.

Note that each positive number appears exactly once in the sequence (a_n, b_n). This is also true for the sequence of the differences $(b_n - a_n)$. In the literature, the sequence (a_n, b_n) is often called Wythoff's sequence.

Each impartial combinatorial game is associated with a digraph $G = (V, E)$, called the game graph. The set of the vertices V are the positions of the game. Given two vertices v_1 and v_2, there is an edge from v_1 to v_2 if there exists a move from the position v_1 to the position v_2.

Definition 3 *Given a digraph $G = (V, E)$, a set $S \subseteq V$ is said stable if there is no edge between any two vertices of S. A set $A \subseteq V$ is said absorbent if for any $v \in V \setminus A$, there exists $a \in A$ such that $(v, a) \in E$.*

Definition 4 *Given a digraph $G = (V, E)$, a kernel of G is both a stable and an absorbent set of G.*

The $\mathcal{P}$-positions of a game constitute a kernel of its game graph. If the game graph does not contain any circuit, such a kernel exists and is unique (see [1] for details).

Definition 5 *For a game $G = (V, E)$ and a position $v \in V$, let $\mathrm{Op}(v) = \{w \in V/(v, w) \in E\}$ be the set of the options of v. That is $\mathrm{Op}(v)$ is the set of all the positions that can be reached from v in one move.*

To each game position v of an impartial game we associate a nonnegative integer value $\mathcal{G}(v)$, called the $\mathcal{G}$-value of v. This function $\mathcal{G}$ is called the *Grundy function*, and can be defined recursively as follows:

$$\mathcal{G}(v) = Mex(\{\mathcal{G}(u) : u \in \mathrm{Op}(v)\}).$$

If the game graph is acyclic, then this function exists and is unique. It is well-known that the zeros of the Grundy function are the $\mathcal{P}$-positions of the game. See [1,6,13] for more information on the Grundy function.

In [15], it is proved that the $\mathcal{P}$-positions Wythoff's game can be characterized with the golden ratio $\tau = (1+\sqrt{5})/2$. Each $\mathcal{P}$-position (a_n, b_n) can also be written $(\lfloor n\tau \rfloor, \lfloor n\tau^2 \rfloor)$. With this characterization, one can decide in polynomial time whether a game position is $\mathcal{P}$ or $\mathcal{N}$ (cf. [12] for information about the complexity of a combinatorial game). In [2], Blass et al. detail some other properties of the $\mathcal{P}$-positions of Wythoff's game, and investigate the case of the other $\mathcal{G}$-values.

The $\mathcal{P}$-positions of a game of Nim with n heaps satisfy $a_0 \oplus a_1 \oplus \ldots \oplus a_{n-1} = 0$, where a_i is the number of tokens in the i^{th} heap, and $\oplus$ is the "Nim-sum" operator, i.e., the binary addition without carrying (the first proof was given in [5]). It is well known that $(\mathbb{Z}_{\geq 0}, \oplus)$ defines an additive group, where the identity element is 0 and where the inverse of an element is the element itself.

2 A geometrical extension of Wythoff's game

In their book (see [1]), Berlekamp, Conway, and Guy describe Wythoff's game with a chessboard on which a queen is placed. Both players alternately move the queen according to chess rules. The queen must be closer to the $(0,0)$ position after each move (otherwise the game could never end). A player wins when he moves the queen on the square $(0,0)$ (see Figure 1).

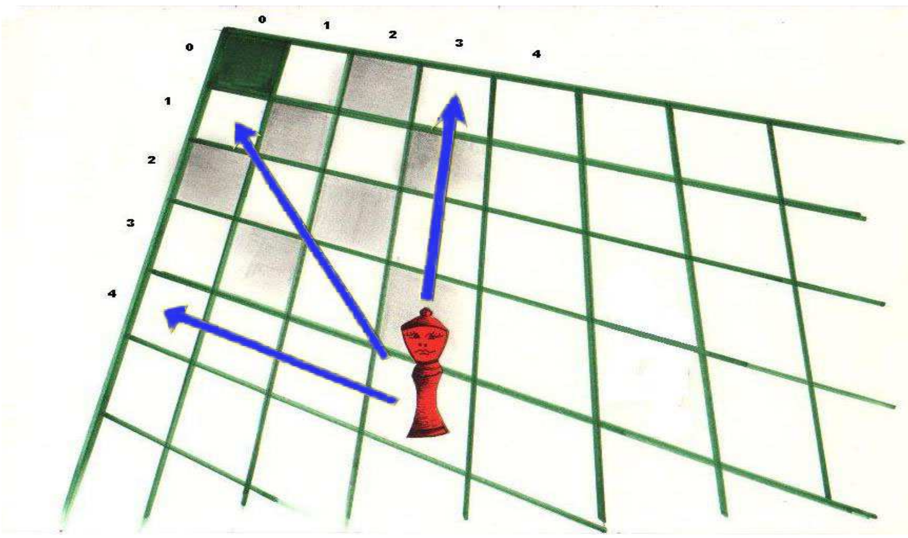

Figure 1. Wythoff's game played on a chessboard

In the literature, lots of variations of Wythoff's game were investigated, in terms of $\mathcal{P}$-positions or Grundy function. The modifications often concern Wythoff's rule: instead of removing $k = l$ tokens in both heaps, new constraints are considered for the pair (k, l). For example, in the generalized Wythoff's game (see [11]), one can remove k and l tokens in both heaps provided $|k - l| < a$, where a is a fixed positive integer. In [10], the condition $k \leq l < 2k + 2$ must be satisfied.

We now define another extension of Wythoff's game, called the *n vectors game.*

Let n and p be two positive integers. Let $v_1, \ldots, v_n$ be n vectors of $\mathbb{R}^p$, with nonnegative coordinates.

A game position of the *n vectors game* is a vector $v = (a_1 v_1 + \ldots + a_n v_n)$, with $a_i \in \mathbb{Z}_{\geq 0}$. A move consists in choosing a vector v_i and removing it k times from v, provided the vector $(v - k v_i)$ can also be written $(a'_1 v_1 + \ldots + a'_n v_n)$, $a'_i \in \mathbb{Z}_{\geq 0}$. In other words, we have:

$$\begin{aligned} \mathrm{Op}(a_1 v_1 + \ldots + a_n v_n) = \{ & w = (a_1 v_1 + \ldots + a_n v_n - k v_i) : \\ & 1 \leq i \leq n,\ 0 < k, \\ & \exists (a'_1, \ldots, a'_n) \in \mathbb{Z}^n_{\geq 0} \,/ w = (a'_1 v_1 + \ldots + a'_n v_n) \} \end{aligned}$$

The final position is the null vector. Note that the game has an end, since the n vectors are chosen with nonnegative coordinates.

With this formalism, Wythoff's game can be seen as the three vectors game with $v_1 = (1, 0)$, $v_2 = (0, 1)$, and $v_3 = (1, 1)$. However, the same position of the *n vectors game* may be described with two distinct notations. For example in Wythoff's game with $v_1 = (1, 0)$, $v_2 = (0, 1)$, and $v_3 = (1, 1)$, the game positions $(1.v_1 + 1.v_2 + 0.v_3)$ and $(0.v_1 + 0.v_2 + 1.v_3)$ are identical, although the coefficients a_i are different. To get through this difficulty and make the set of the options more accessible, one key is to find a canonical representation of a game position.

Definition 6 *$v_1, \ldots, v_n$ are said $\mathbb{Z}$-independent vectors if*

$$\nexists\, \alpha_1, \ldots, \alpha_n \in \mathbb{Z}, \ \textit{non all zero} \ / \alpha_1 v_1 + \ldots + \alpha_n v_n = 0$$

Fact 7 *In the n vectors game, if the n vectors are $\mathbb{Z}$-independent, then there exists a unique set of nonnegative integers $(a_1, \ldots, a_n)$ to define each game position. Suppose indeed that $(a_1 v_1 + \ldots + a_n v_n)$ and $(a'_1 v_1 + \ldots + a'_n v_n)$ define the same vector. Then we would have $(a_1 - a'_1) v_1 + \ldots + (a_n - a'_n) v_n = 0$, where $\forall i$, $(a_i - a'_i) \in \mathbb{Z}$.*

Therefore, when the n vectors are $\mathbb{Z}$-independent, we choose the n-tuple $(a_1, \ldots, a_n)$ as the canonical representation of the game position $(a_1 v_1 + \ldots + a_n v_n)$. The following proposition is thus deduced:

Proposition 8 *The n vectors game played with $\mathbb{Z}$-independent vectors is equivalent to the game of Nim with n heaps.*

PROOF. With the above canonical representation, from a game position $(a_1, \ldots, a_n)$, a move consists in decreasing one of the n values $a_1, \ldots, a_n$ by a positive integer until reaching the $(0, \ldots, 0)$ position. □

In the rest of this paper, we consider instances of the *n vectors game* with non $\mathbb{Z}$-independent vectors. The search of a canonical representation for the game positions will be required. We will present a characterization of the $\mathcal{P}$-positions in a particular case with an odd number of vectors (section 3), or more specifically when n is equal to three (section 4). Section 5 details another variation of Wythoff's game, when the number of tokens one can remove is bounded by a constant.

3 Odd number of $\mathbb{Z}$-independent vectors

We consider instances of the $(2n+2)$ *vectors game* such that $v_1, \ldots, v_{2n+1}$ are $\mathbb{Z}$-independent vectors, and $v_{2n+2} = \sum_{i=1}^{2n+1} v_i$. The following canonical representation will be chosen: denote a game position $(a_1 v_1 + \ldots + a_{2n+1} v_{2n+1} + a_{2n+2} v_{2n+2})$ by the $(2n+1)$-tuple $(a_1, \ldots, a_{2n+1})$. Then we have:

$$
\begin{aligned}
\mathrm{Op}(a_1, a_2, \ldots, a_{2n+1}) = & \{(a_1 - k, a_2, \ldots, a_{2n+1}) : 1 \le k \le a_1\} \\
& \cup \{(a_1, a_2 - k, \ldots, a_{2n+1}) : 1 \le k \le a_2\} \\
& \cup \ldots \\
& \cup \{(a_1, a_2, \ldots, a_{2n+1} - k) : 1 \le k \le a_{2n+1}\} \\
& \cup \{(a_1 - k, a_2 - k, \ldots, a_{2n+1} - k) : 1 \le k \le min(a_i)\}.
\end{aligned}
$$

This game can also be described on $(2n+1)$ heaps of tokens. A move consists in removing any number of tokens from a single heap (the Nim rule), or removing the same number of tokens from all the heaps, on condition that each of them is a non empty heap (extended Wythoff's rule).

Theorem 9 *The $\mathcal{P}$-positions of this game are identical to those of the game of Nim with $(2n+1)$ heaps.*

PROOF. It suffices to show that if $(a_1, a_2, \ldots, a_{2n+1})$ is a $\mathcal{P}$-position of Nim, then $(a_1 - x, a_2 - x, \ldots, a_{2n+1} - x)$ with $1 \leq x \leq min(a_i)$ is a $\mathcal{N}$-position of Nim.

Let $(a_1, a_2, \ldots, a_{2n+1})$ be a $\mathcal{P}$-position of Nim. It satisfies $a_1 \oplus a_2 \oplus a_{2n+1} = 0$. Now consider the position $(a_1 - x, a_2 - x, \ldots, a_{2n+1} - x)$ with $1 \leq x \leq min(a_i)$. Denote by $x = x_r \ldots x_1$ $(x_r = 1)$ the binary writing of x. Let x_k be the smallest nonzero bit of it, i.e. $x_k = 1$ and $\forall\ 1 \leq x_i < k,\ x_i = 0$.
On Figure 2, consider the Nim-sum of the a_i's before the move.

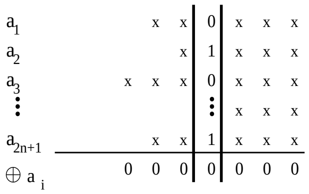

Figure 2. Nim-sum before the move

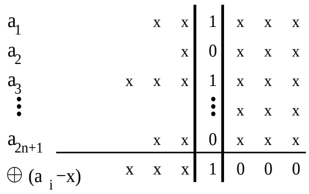

Figure 3. Nim-sum after the move

This sum is equal to 0 on each bit. There is thus an even number of $1's$ and an odd number of $0's$ in each column. This property is of course true for the k^{th} column.
Each a_i is decreased by x after the move. By definition of x_k, a_i and $(a_i - x)$ have the same first $(k-1)$ bits of their binary writing. As $x_k = 1$, the k^{th} bit of each a_i will be changed after the substraction of x (see Figure 3). Then the Nim sum of the resulting position contains an odd number of $1's$ in its k^{th} column, which implies that it differs from 0. $\square$

Remark 10 *In [3], Blass et al. proved a more general result. They showed that this game and the game of Nim with* $(2n+1)$ *heaps have the same Grundy function. By adapting our proof, we obtain the same result. However, the purpose of Theorem 9 is not to present a new result, but to illustrate the* n *vectors game with a large value for* n.

Now consider the even case: $v_1, \ldots, v_{2n}$ are $\mathbb{Z}$-independent vectors, and $v_{2n+1} = \sum_{i=1}^{2n} v_i$. For such games, it is proved in [3] that the $\mathcal{P}$-positions are different from those of the game of Nim. When the number of vectors is three, this is Wythoff's game. In the other cases, finding a polynomial characterization of the $\mathcal{P}$-positions remains a tricky problem. The following table contains the first few $\mathcal{P}$-positions of this game with five vectors.

a_n	b_n	c_n	d_n
0	b	c	$b \oplus c$
1	1	1	2
1	1	3	4
1	1	5	6
1	2	2	2
1	2	4	4
1	2	5	5
1	2	6	6
1	3	3	3
1	3	5	8
1	4	6	8
1	4	7	9

Table 1. The first $\mathcal{P}$-positions of the 5 vectors game with $v_5 = \sum_{i=1}^{4} v_i$

4 The three vectors game

In this section, we consider instances of the *3 vectors game* with non $\mathbb{Z}$-independent vectors. Let v_1, v_2, and v_3 be three vectors of $\mathbb{R}^p$ such that

$$\exists\ \alpha, \beta, \gamma\ \in \mathbb{Z}\, / \alpha v_1 + \beta v_2 + \gamma v_3 = 0$$

Since v_1, v_2 and v_3 have positive coordinates, α, β, and γ can not be of the same sign. Withous loss of generality, assume we can write

$$\gamma v_3 = \alpha v_1 + \beta v_2,$$

where α, β, and γ are nonnegative integers. An instance of the *three vectors game* will be denoted by a triplet $[\alpha, \beta, \gamma]$. In order to determine a canonical respresentation for the game positions, we impose the condition $\alpha, \beta, \gamma > 0$ in the instances.

4.1 $\mathcal{P}$-positions of $[\alpha, \beta, 1]$ games, with $\alpha \neq \beta$

We consider instances of the *three vectors game* with $\gamma = 1$.

Thus $v_3 = \alpha v_1 + \beta v_2$. A canonical form for a game position will be a pair $(a, b) \in \mathbb{Z}^2_{\geq 0}$. Therefore, the options of a game position are the following:

$$
\begin{aligned}
\mathrm{Op}(a,b) = &\{(a-k,b) : 1 \leq k \leq a\} \\
&\cup\{(a,b-k) : 1 \leq k \leq b\} \\
&\cup\{(a-k\alpha, b-k\beta) : 1 \leq k\alpha \leq a,\ 1 \leq k\beta \leq b,\ 0 < k\}
\end{aligned}
$$

As for Wythoff's game, this game can be described on a rectangular grid. A piece is placed on a square of the grid. Both players alternately move the piece according to three allowed directions : vertically (according to v_1), horizontally (according to v_2), or on squares corresponding to multiples of v_3 (see Figure 4).

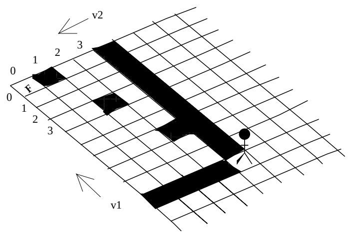

Figure 4. Allowed moves when $v_3 = 3v_1 + v_2$ (the $[3,1,1]$ game)

Theorem 11 *The $\mathcal{P}$-positions of $[\alpha,\beta,1]$ games with $\alpha \neq \beta$ are in the form (a,a) $\forall a \geq 0$.*

They correspond to the diagonal of the grid (cf. Figure 5).

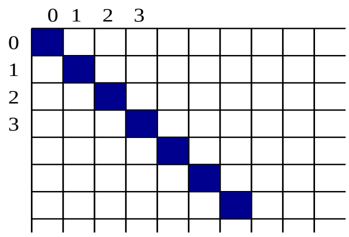

Figure 5. $\mathcal{P}$-positions when $\alpha \neq \beta$

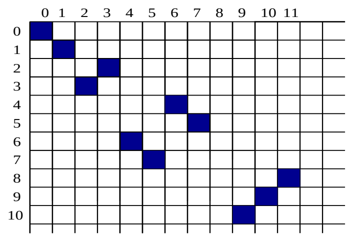

Figure 6. $\mathcal{P}$-positions when $\alpha = \beta = 2$

PROOF. It suffices to show that the set of positions (a,a), $a \geq 0$ is stable and absorbent for the game graph.

From a game position (a,b) with $a < b$, remove $k = (b-a)$ tokens from the larger heap, and thus land in a (a,a) position.

From a game position (a,a), one can not reach another position in the form (k,k), since none of the three vectors is directed according to the diagonal direction $(\alpha \neq \beta)$. $\square$

The case $\alpha = \beta = 1$ is Wythoff's game. In this game, there is exactly one $\mathcal{P}$-position per diagonal. When $\alpha > 1$, since the diagonal moves are played according to multiples of α, there will be exactly α $\mathcal{P}$-positions on each diagonal. We now give a recursive algorithm that computes the sequence of the $\mathcal{P}$-positions (a_n, b_n) of a $[\alpha, \alpha, 1]$ game, with $a_n \leq b_n$. By symmetry of the game, if (a_n, b_n) is a $\mathcal{P}$-position, then (b_n, a_n) is also a $\mathcal{P}$-position.

Definition 12 *Given two integers $a \geq 0$ and $b > 0$, denote by $r(\frac{a}{b})$ the remainder of the euclidean division of a by b.*

Algorithm computing the $\mathcal{P}$-positions of $[\alpha, \alpha, 1]$ games

- $(a_0, b_0) = (0, 0)$
- Assume that (a_i, b_i) is known for all $0 \leq i < n$.
  Then $a_n = Mex(a_0, \ldots, a_{n-1}, b_0, \ldots, b_{n-1})$. Choose b_n as the smallest integer not in $\{a_0, \ldots, a_{n-1}, b_0, \ldots, b_{n-1}\}$ such that there exists no other previous pair $(a_i, b_i)_{0 \leq i < n}$ for which both conditions $b_i - a_i = b_n - a_n$ and $r(\frac{a_i}{\alpha}) = r(\frac{a_n}{\alpha})$ are simultaneously satisfied.

Figure 6 illustrates the first $\mathcal{P}$-positions of the case $\alpha = \beta = 2$.
This algorithm means that for a given nonnegative difference d, all the $\mathcal{P}$-positions $\{(a_{i1}, b_{i1}), \ldots, (a_{i\alpha}, b_{i\alpha})\}$ such that $b_{ik} - a_{ik} = d$ for all $1 \leq k \leq \alpha$ (i.e., the α $\mathcal{P}$-positions on the d^{th} diagonal) satisfy
$\{r(\frac{a_{i1}}{\alpha}), \ldots, r(\frac{a_{i\alpha}}{\alpha})\} = \{0, \ldots, \alpha - 1\}$.

Theorem 13 *The algorithm above constructs the set of the $\mathcal{P}$-positions of the $[\alpha, \alpha, 1]$ games.*

PROOF. Denote by S the set of positions built by the algorithm. In order to show that S is the set of the $\mathcal{P}$-positions of the $[\alpha, \alpha, 1]$ game, it suffices to show that S is a stable and absorbent set of the game graph.

S is a stable set
Let (a, b) and (a', b') be two distinct positions of S. We will prove that there exists no move from (a, b) to (a', b').

By construction, there is exactly one position of S in each row and each column of the chessboard. A move according to v_1 or v_2 from (a, b) to (a', b') implies $a' = a$ or $b' = b$, contradicting the previous remark.
Playing according to v_3 implies $(a', b') = (a - q\alpha, b - q\alpha)$ with $q > 0$. The differences $(b - a)$ and $(b' - a')$ are identical, and $r(\frac{a}{\alpha}) = r(\frac{a - q\alpha}{\alpha})$. According

to the algorithm, positions (a,b) and (a',b') cannot be in S.

S is an absorbent set

On the grid, mark each position of S with the symbol "S". Mark "*" each position absorbed by S. We will prove that each position of the chessboard is marked.

By definition, a S-marked position P generates a set of positions marked with stars (these are positions from which there exists a move leading to P). See Figure 7 for an example with $\alpha = 2$.

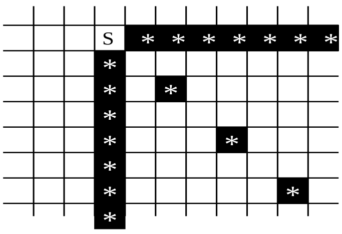

Figure 7. Absorbed positions from a S-marked position when $\alpha = 2$

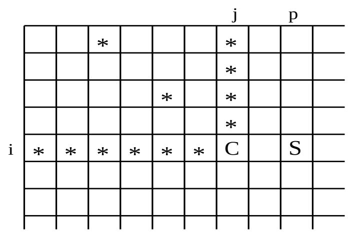

Figure 8. Proof of the absorption when $\alpha = 2$

Suppose there exists a position $C = (i,j)$ which is not marked. Choose C as the "smallest" one, i.e., there exists no other position $C' = (i',j')$ not marked such that $i' \leq i$ and $j' \leq j$. By symmetry of the game and without loss of generality, assume that $i \leq j$. The positions on the left of C, above C, and on the north-west diagonal at a distance multiple of α are necessarily marked with stars (see Figure 8).

Since there is exactly one S-marked position by row, there exists a position marked S in the i^{th} row. Since C is not marked and positions on the left of C are marked with stars, there exists $p > j$ such that (i,p) belongs to S (see Figure 8). Since $j < p$, there are two possible reasons explaining why (i,p) was selected by the algorithm instead of (i,j):

- The integer j has already been selected by the algorithm. Hence there would be a S-marked position in the column above C, but it is not the case.
- There exists a S-marked position (k,l) with $k < i$, $l < j$, such that $(l-k) = (j-i)$ and $r(\frac{k}{\alpha}) = r(\frac{i}{\alpha})$. This means that (k,l) is on the same diagonal as (i,j) and at a distance which is multiple of α. But such a position is marked $*$, as mentioned above.

Hence C should have been selected by the algorithm instead of (i,p), leading to a contradiction. $\square$

In the case $\alpha = 2$, we will detail another algorithm to compute the $\mathcal{P}$-positions of the game. It looks like the algorithm of Wythoff decribed in the introduction of this paper, but there is no algebraic characterization known for it until today.

Define (P_n, Q_n) as the sequence of integers such that
$P_n = Mex(P_0, \dots, P_{n-1}, Q_0, \dots, Q_{n-1})$, and $Q_n = P_n + 2\lfloor n/4 \rfloor$. Initialize it with $(P_0, Q_0) = (0, 0)$. Hence P_n and Q_n are two sequences for which each even difference $(Q_n - P_n)$ appears exactly four times in an increasing order. Given two positive integers θ and μ, we can actually define a set of integer sequences $(P_n^{\theta,\mu}, Q_n^{\theta,\mu})$, where $P_n^{\theta,\mu}$ is defined by the Mex rule, and the $(n+1)^{th}$ difference $(Q_n^{\theta,\mu} - P_n^{\theta,\mu})$ is equal to $\theta\lfloor n/\mu \rfloor$. This means that differences that are multiples of θ appear exactly μ times in an increasing order. When $\theta = \mu = 1$, this defines Wythoff's sequence. According to the theorem detailed in [4], a characterization of P_n or Q_n as a spectrum sequence (i.e., in the form $\lfloor n\alpha + \beta \rfloor$, $\alpha, \beta \in \mathbb{R}$, as for Wythoff's game) does not exist. Moreover, C. Kimberling has recently added one of these sequences (with $\theta = 1$ and $\mu = 2$) to Sloane's encyclopedia[2]. In agreement with him, it seems that there is no known characterization for it.

Consider now the integer sequence (p_n, q_n) such that $p_n = P_n$, $q_n = Q_n$ if $n \equiv 0(4)$ or $n \equiv 1(4)$, $q_n = Q_{n+1}$ if $n \equiv 2(4)$, and $q_n = Q_{n-1}$ if $n \equiv 3(4)$.

The array below describes the first values of both sequences (P_n, Q_n) and (p_n, q_n). To obtain q_n from Q_n, it suffices to exchange values Q_{4k+2} and Q_{4k+3} for each positive integer k.

[2] http://www.research.att.com/~njas/sequences/index.html

$Q_n - P_n$	P_n	Q_n	p_n	q_n	$q_n - p_n$	$r(\frac{p_n}{\alpha})$
0	0	0	0	0	0	0
0	1	1	1	1	0	1
0	2	$\boxed{2}$	2	3	1	0
0	3	$\boxed{3}$	3	2	-1	1
2	4	6	4	6	2	0
2	5	7	5	7	2	1
2	8	$\boxed{10}$	8	11	3	0
2	9	$\boxed{11}$	9	10	1	1
4	12	16	12	16	4	0
4	13	17	13	17	4	1
4	14	$\boxed{18}$	14	19	5	0
4	15	$\boxed{19}$	15	18	3	1
6	20	26	20	26	6	0
6	21	27	21	27	6	1

Lemma 14 *For each nonnegative integer k, there exist two nonnegative integers i and j such that each block of pairs $(p_{4k+p}, q_{4k+p})_{p=0\ldots3}$ can be written in the form:*

$$\begin{aligned}(p_{4k}, q_{4k}) &= (2i, 2i + 2k)\\ (p_{4k+1}, q_{4k+1}) &= (2i + 1, 2i + 1 + 2k)\\ (p_{4k+2}, q_{4k+2}) &= (2j, 2j + 2k + 1)\\ (p_{4k+3}, q_{4k+3}) &= (2j + 1, 2j + 2k)\end{aligned}$$

PROOF. By construction of (p_n, q_n) from (P_n, Q_n), each of the four pairs satisfies the right difference $(q_{4k+p} - p_{4k+p})_{p=0\ldots3}$. Therefore, it suffices to show that the values p_{4k+p} are correct.

According to the table above, the lemma is true for $k = 0$ (choose $i = 0$ and $j = 1$). Assume it is true until a certain rank k and consider the $(k + 1)^{th}$ rank. As p_n is a strictly increasing sequence, we have $p_{4k+4} > p_{4k+3} = (2j+1)$. By definition of p_n with the Mex rule, the unique reason for which p_{4k+4} could not be equal to $(2j + 2)$ is that this integer has already appeared in the q_n sequence. By induction hypothesis and since $(2j + 2)$ is even, this value could have appeared only as $q_{4k'}$ or $q_{4k'+3}$, with $k' \leq k$. If it is the case, the consecutive integer $(2j + 3)$ would have also appeared before, respectively as $q_{4k'+1}$ or $q_{4k'+2}$. Hence we conclude that if an even value already appeared before, the consecutive odd value appeared too. Therefore, there exists l such

that $p_{4k+4} = 2l$. Hence $p_{4k+5} = 2l + 1$ (If not, $(2l + 1)$ would have appeared in the q_n sequence. By induction hypothesis, its predecessor $2l$ would have also appeared in q_n, yielding a contradiction).
Since $p_{4k+6} > (2l + 1)$, and with the same argument as previously, we prove that p_{4k+6} is an even positive integer, and that p_{4k+7} is the consecutive one. □

Proposition 15 *The sequence (p_n, q_n) describes the $\mathcal{P}$-positions of the $[2, 2, 1]$ game.*

PROOF. Denote by (a_n, b_n) the sequence of the $\mathcal{P}$-positions of the $[2, 2, 1]$ game. This sequence is obtained from the algorithm of Theorem 13. Define the sequence (x_n, y_n) as follows:

$(x_n, y_n) = (a_n, b_n)\ \forall\ 0 \leq n \leq 2$
$(x_3, y_3) = (y_2, x_2) = (3, 2)$
$(x_n, y_n) = (a_{n-1}, b_{n-1})\ \forall\ n > 3$

Now, if we apply the algorithm of Theorem 13 from the rank $n = 4$ and initiated with the values $(x_i, y_i)_{0 \leq i \leq 3}$, we construct precisely the sequence (x_n, y_n). Indeed, since the values x_3 and y_3 are the respective repetitions of y_2 and x_2, and since the difference $(y_3 - x_3)$ is negative, the insertion of the pair (x_3, y_3) does not affect the result of the algorithm. Therefore, we will consider that (x_n, y_n) is produced by the algorithm of Theorem 13.

Define $(d_n) = (y_n - x_n)$ as the sequence of the differences. First observe that since x_n and p_n are both defined with the Mex rule, the sequences (x_n) and (p_n) are increasing. Moreover, according to Lemma 14 and the Mex rule, one can show that $q_j > q_i$ for all $j > i + 1$.

We will now show that $(x_n, y_n) = (p_n, q_n)$. The proof works as follows: proving that $x_n = p_n$ is easy since both values are computed with the Mex rule applied on the previous values (supposed equal by induction hypothesis).

In order to determine y_n, we study the possible values d^* for d_n. We start with the smallest possible value d^* available for d_n. This value is known by Lemma 14 and by induction hypothesis. If $d^* = y_j - x_j$ for some $j < n$ and if x_n has the same parity as x_j, then we consider the next available value d^* allowed by the algorithm and repeat the test. Otherwise, this means that d^* is chosen by the algorithm, and so $y_n = x_n + d^*$. By Lemma 14, it only remains to check that $x_n + d^* = q_n$.

We now give the proof in details:

One can easily check the equality of (x_n, y_n) and (p_n, q_n) for $n < 4$. Suppose this equality holds for the subsets (x_{4k+i}, y_{4k+i}) and $(p_{4k+i}, q_{4k+i})_{i=0\dots3}$, for all $k < t$ for some $t > 0$. This hypothesis ensures that $x_{4t} = p_{4t}$ since they are both defined with the Mex rule on the previous values. Note that x_{4t} is thus even by Lemma 14. By definition of the sequence (x_{4t}, y_{4t}) and from Lemma 14, $d_{4t} \geq (2t-1)$, since all the smallest differences have already appeared twice. The difference $(2t-1) = d_{4t-2}$ has been used once, but since $x_{4t-2} = p_{4(t-1)+2}$ (by induction hypothesis) and x_{4t} are both even, the algorithm does not allow $d_{4t} = (2t-1)$. The next difference $2t$ has not been used before and is greater than all the previous differences. Therefore, since (x_n) is an increasing sequence, the value $x_{4t} + 2t$ is strictly greater than all the previous values. It is chosen by the algorithm and thus $y_{4t} = x_{4t} + 2t$. By Lemma 14, one can check that $q_{4t} = y_{4t}$.

As previously, we have $x_{4t+1} = p_{4t+1}$, since the Mex rule is applied on the same set of integers. Hence $x_{4t+1} = x_{4t} + 1$ according to Lemma 14. Since $(x_{4t+1} + 2t - 1) = y_{4t}$, we have $d_{4t+1} \neq 2t - 1$, as an integer cannot appear several times in the sequence (x_n, y_n). Then we get $d_{4t+1} = 2t$, since x_{4t+1} and x_{4t} have different parities, and since $(x_{4t+1} + 2t)$ has not appeared before (greater than all other values). Therefore we have $y_{4t+1} = x_{4t+1} + 2t = p_{4t+1} + 2t = q_{4t+1}$ by Lemma 14.

By definition of the sequences x_n and p_n with the Mex rule, we have $x_{4t+2} = p_{4t+2}$. Lemma 14 ensures that x_{4t+2}, as x_{4t-2}, is even. Hence d_{4t+2} can not be equal to $(2t-1) = d_{4t-2}$. Since the difference $2t$ has been used twice, the next smallest available difference is $(2t+1)$. We proceed as in the first case ($i = 0$), which leads to the equality $y_{4t+2} = q_{4t+2}$.

We have $x_{4t+3} = p_{4t+3}$ for the same reasons as previously. By Lemma 14, this implies $x_{4t+3} = x_{4t+2} + 1$, which is an odd number. Now consider the integer $(x_{4t+3} + 2t - 1)$. According to Lemma 14, it is equal to $(y_{4t+2} - 1)$ and is strictly greater than y_i for any $i < (4t+2)$ (cf. preliminary remark about the sequence (q_n)). Since x_{4t+3} and x_{4t-2} have different parities, the algorithm chooses $d_{4t+3} = 2t - 1$. One can finally check that $y_{4t+3} = q_{4t+3}$, which concludes the proof. $\square$

As explained in [12], most of combinatorial games have an exponential strategy. Nim and Wythoff's games have a polynomial complexity thanks to their algebraic characterizations. In the case $\alpha = 2$, Theorem 15 improves the readability of the general algorithm and opens a door to a possible polynomial algebraic characterization (as a sum of spectrum sequences for example). A short investigation made us suppose that instances $[\alpha, \alpha, 1]$ have sets of $\mathcal{P}$-positions that can be similarly deduced from integer sequences. Such results combined with the obtention of formulas for these sequences would lead to a nice characterization of the $\mathcal{P}$-positions for the *three vectors game*.

Remark 16 *Although the sequence P_n cannot be written as a spectrum sequence (cf. [4]), it seems that it has the same progression as $\lfloor \frac{n(3+\sqrt{17})}{4} \rfloor$. We conjecture that $|P_n - \lfloor \frac{n(3+\sqrt{17})}{4} \rfloor| \leq 4$. In [9], Fraenkel investigates this conjecture and introduces the notion of probabilistic winning strategy.*

4.3 $\mathcal{P}$-positions of $[\alpha, \beta, \gamma]$ games, with $\gamma > 1$

We now consider instances $[\alpha, \beta, \gamma]$ of the three vectors game, with $\gamma > 1$. Note that one can assume that α, β and γ are prime together. Indeed, the instances $[\alpha, \beta, \gamma]$ and $[k\alpha, k\beta, k\gamma]$ define the same game.

Given a game position $(a_1 v_1 + a_2 v_2 + a_3 v_3)$, one associates the following canonical triplet:

$$(a_1 + \lfloor \frac{a_3}{\gamma} \rfloor \alpha,\ a_2 + \lfloor \frac{a_3}{\gamma} \rfloor \beta,\ r(\frac{a_3}{\gamma})).$$

Therefore, any game position will be defined by a triplet (a, b, i), with $0 \leq a, b$ and $0 \leq i < \gamma$. It can be checked that this representation uniquely describes the game positions.

Now, the options of a game position (a, b, i) can be defined as:

$$\begin{aligned}\mathrm{Op}(a,b,i) = &\{(a-k,b,i) : 1 \leq k \leq a\} \\ &\cup \{(a,b-k,i) : 1 \leq k \leq b\} \\ &\cup \{(a - \lfloor \frac{k}{\gamma} \rfloor \alpha, b - \lfloor \frac{k}{\gamma} \rfloor \beta, (i - r(\frac{k}{\gamma}))[\gamma]) : 0 < k,\ 0 \leq \lfloor \frac{k}{\gamma} \rfloor \alpha \leq a,\ 0 \leq \lfloor \frac{k}{\gamma} \rfloor \beta \leq b\}.\end{aligned}$$

where $[.]$ denotes the modulo operator.

Lemma 17 *Let (a, b, i) be a $\mathcal{P}$-position of Nim. Then $a - i \leq b \leq a + i$.*

PROOF. If (a, b, i) is a $\mathcal{P}$-position of Nim, then it satisfies $b = a \oplus i$. As the operator $\oplus$ defines the binary sum without carrying, it is clear that $b = a \oplus i \leq a + i$.

The second inequality is deduced from the first one, using the fact that if (a, b, i) is a $\mathcal{P}$-position of Nim, then (b, a, i) is also a $\mathcal{P}$-position of this game. $\square$

Theorem 18 *Let $[\alpha, \beta, \gamma]$ be an instance of the three vectors game, such that $\gamma > 1$ and $\beta > \alpha(2\gamma - 1)$. Then a game position (a, b, i) is a $\mathcal{P}$-position if and only if $a \oplus b \oplus i = 0$.*

PROOF. First observe that the options of a (a, b, i) position contain all the options of a game of Nim with three heaps of respective sizes a, b, and i. This result is straightforward for the options concerning the heaps of sizes a and b. For the heap of size i, consider the last option of the three vectors game with $\lfloor \frac{k}{\alpha} \rfloor = 0$.

Therefore, it suffices to show that for a position (a, b, i) such that $a \oplus b \oplus i = 0$, each position $(a', b', i') \in \mathrm{Op}(a, b, i)$ satisfies $a' \oplus b' \oplus i' \neq 0$.

Let (a, b, i) be a game position satisfying $a \oplus b \oplus i = 0$. Let (a', b', i') be an option of (a, b, i). Let $k > 0$ be an integer.

- Consider the options $(a', b', i') = (a - k, b, i)$ with $1 \leq k \leq a$. Since $(\mathbb{Z}_{\geq 0}, \oplus)$ is a group, we have $(a - k) \oplus b \oplus i \neq a \oplus b \oplus i$.
- The case $(a', b', i') = (a, b - k, i)$ with $1 \leq k \leq b$ can be treated as previously. Hence $a \oplus (b - k) \oplus i \neq 0$.
- Suppose that $(a', b', i') = ((a - \lfloor \frac{k}{\gamma} \rfloor \alpha, b - \lfloor \frac{k}{\gamma} \rfloor \beta, (i - r(\frac{k}{\gamma}))[\gamma])$ with $0 < k$, $0 \leq \lfloor \frac{k}{\gamma} \rfloor \alpha \leq a$, $0 \leq \lfloor \frac{k}{\gamma} \rfloor \beta \leq b$. Let $p = \lfloor \frac{k}{\gamma} \rfloor$. If $p = 0$, one can conclude as in the previous cases. Now assume that $p > 0$. With our notations, we have:
  $i' = (i - r(\frac{k}{\gamma}))[\gamma]$
  $a' = a - p\alpha$
  $b' = b - p\beta$
  One can notice that $(a - a')\beta = (b - b')\alpha$. By way of contradiction, assume that $a' \oplus b' \oplus i' = 0$. Then $b' = a' \oplus i'$ and from Lemma 17,

$$a' - i' \leq b' \leq a' + i'$$

  Similarly, since $a \oplus b \oplus i = 0$, we have

$$a - i \leq b \leq a + i$$

  From these two inequalities we get

$$b - b' \leq a - a' + i + i'$$

Moreover, since $i + i' \leq 2\gamma - 2$ and $b - b' = (a - a')\frac{\beta}{\alpha}$, we obtain

$$(a - a')(\frac{\beta}{\alpha} - 1) \leq 2\gamma - 2$$

Finally, since $a' < a$, we get

$$\beta \leq (2\gamma - 1)\alpha \qquad \square$$

Remark 19 *From Theorem 18, one can assert that for any values of α and γ, there exists β_α such that for all $\beta \geq \beta_\alpha$, the games $[\alpha, \beta, \gamma]$ and $[1, 2\gamma, \gamma]$ have the same $\mathcal{P}$-positions.*

Remark 20 *One can wonder whether the $[\alpha, \beta, \gamma]$ games and the game of Nim on three heaps have the same Grundy function. With the actual bound (i.e., $\beta > \alpha(2\gamma - 1)$), the $[\alpha, \beta, \gamma]$ games have a $\mathcal{G}$ function different from the Nim sum. However, this assumption might be true for some larger values of β.*

Therefore, a second question consists in comparing the $\mathcal{G}$ functions of the $[\alpha, \beta, \gamma]$ games, for $\beta > \alpha(2\gamma - 1)$. Even if α is fixed, there exists some games with different $\mathcal{G}$ functions (e.g. games $[1, 4, 2]$ and $[1, 5, 2]$).

Remark 21 *There exist $[\alpha, \beta, \gamma]$ games with $\beta \leq \alpha(2\gamma - 1)$ for which the $\mathcal{P}$-positions differ from those of the game of Nim. This is for example the case of the games $[1, 3, 2]$, $[1, 5, 3]$, or $[2, 14, 7]$. However, it seems that for some $\beta \leq \alpha(2\gamma - 1)$, the $\mathcal{P}$-positions of the corresponding $[\alpha, \beta, \gamma]$ games remain equal to those of Nim. This leads us to Conjecture 22 and Problem 23 below.*

Conjecture 22 *Let $\beta \geq 2\gamma + \alpha - 1$. Then a game position (a, b, i) is a $\mathcal{P}$-position of the corresponding $[\alpha, \beta, \gamma]$ game if and only if $a \oplus b \oplus i = 0$.*

Problem 23 *Find the smallest value $\beta(\alpha, \gamma)$ such that for all $\beta \geq \beta(\alpha, \gamma)$, all the games $[\alpha, \beta, \gamma]$ have a their $\mathcal{P}$-positions identical to those of Nim.*

5 The R-radius game

In Wythoff's game, one moves the queen on the chessboard according to one of the three directions, no matter the number of squares jumped. In the R-radius game, the length of the moves is bounded by a constant R. The directions allowed do not change. For example, playing the 1-radius game amounts to moving a king on the chessboard. The example of the 3-radius game is given by Figure 9. The original Wythoff's game can be considered as the ∞-radius game.

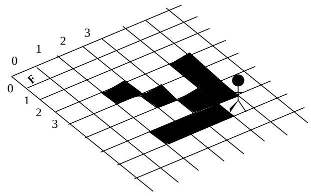

Figure 9. Allowed moves in the 3-radius game

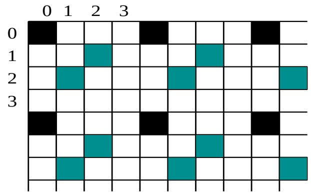

Figure 10. P-positions of the 3-radius game

In this section, we prove that the $\mathcal{P}$-positions of Wythoff's game are sufficient to determine those of the R-radius game.

Lemma 24 *If (a_n, b_n) is a $\mathcal{P}$-position of Wythoff's game with $a_n \leq b_n$, then $b_n \leq 2a_n$.*

PROOF. Wythoff proved in [15] that (a_n, b_n) can be written as the pair $(\lfloor n\tau \rfloor, \lfloor n\tau \rfloor + n)$. Since $\tau \geq 1$, it is easy to check that $n \leq \lfloor n\tau \rfloor$. $\square$

Theorem 25 *For any $R \geq 1$, the $\mathcal{P}$-positions of the R-radius game are described by :*

$$(a + (R+1)\,k,\ b + (R+1)\,l)\ \ \forall\, k,\, l \geq 0$$

for all the pairs (a, b) which are $\mathcal{P}$-positions of Wythoff's game with $a \leq R$ and $b \leq R$.

PROOF. Let $R \geq 1$ and (X, Y) be a position of the R-radius game. Then there exist a unique a, b and k such that

$$X = a + (R+1)k,\ 0 \leq a \leq R,\ 0 \leq k$$

$$Y = b + (R+1)l,\ 0 \leq b \leq R,\ 0 \leq l$$

- If (a, b) is not a $\mathcal{P}$-position of Wythoff's game, then there exists a move from (a, b) to a $\mathcal{P}$-position (a', b') of Wythoff's game with $a' \leq a$ and $b' \leq b$. Moreover, this move satisfies $|a - a'| < R$ and $|b - b'| < R$. Thus there exists a move of the R-radius game from (X, Y) to $(a' + (R+1)k, b' + (R+1)l)$.

- Now suppose that (a, b) is a $\mathcal{P}$-position of Wythoff's game. Let $(X', Y') \neq (X, Y)$ be a position of the R-radius game such that

$$X' = a' + (R+1)k', \ 0 \leq a' \leq R, \ 0 \leq k'$$

$$Y' = b' + (R+1)l, \ 0 \leq b' \leq R, \ 0 \leq l'$$

and where (a', b') is a $\mathcal{P}$-position of Wythoff's game.

We will prove that there is no move from (X, Y) to (X', Y').

By way of contradiction, assume that with $0 < m \leq R$, one of the three following conditions holds:

(1) $X' = X - m$ and $Y' = Y$
(2) $X' = X$ and $Y' = Y - m$
(3) $X' = X - m$ and $Y' = Y - m$

(1) The equality $Y' = Y$ implies $b = b'$. And since an integer cannot appear more than once in Wythoff's sequence, we have also $a' = a$. Since $X' \neq X$ we have $k > k'$, which implies that $m \geq R + 1$.
(2) Similarly by exchanging the roles of X and Y we get the same conclusion as in (1).
(3) The case $X' = X - m$ and $Y' = Y - m$. Since $m < (R + 1)$, we have one of the four following cases:
  - $k = k'$ and $l = l'$. Then $a = a' + m$ and $b = b' + m$. This implies $(b - a) = (b' - a')$. Since each difference appears exactly once in Wythoff's sequence, we must have $b = b'$ and $a = a'$, yielding a contradiction.
  - $k = k' + 1$ and $l = l' + 1$. Then $a = a' + m - R - 1$ and $b = b' + m - R - 1$. We conclude similarly as in the previous case.
  - $k = k'$ and $l = l' + 1$. Then $a = a' + m - R - 1$ and $b = b' + m$. We deduce the following equality:

$$a - b + R + 1 = a' - b' \tag{5A}$$

If $a' \leq b'$, then we have $a + R + 1 \leq b$ from (5A). By Lemma 24, this implies $a + R + 1 \leq 2a$ and finally $R + 1 \leq a$, yielding a contradiction.
Hence $a' > b'$. From (5A) and since $b < (R + 1)$, we obtain

$$a' - b' > a \tag{5B}$$

Since (b', a') is a $\mathcal{P}$-position of Wythoff's game and by Lemma 24, we have $a' \leq 2b'$. Thus $a < a' - b' \leq b'$ by using (5B). Now, from (5A) and since $a' < (R + 1)$, we get

$$a' - b' > a - b + a' \tag{5C}$$

Moreover, since $a' - b' \leq R$, we have $a < b$ from (5A). So by Lemma 24 we have $b \leq 2a$. Now we have

$$a' - b' < a' - a \leq a' + a - b,$$

which contradicts (5C).

- $k = k' + 1$ and $l = l'$. By symmetry, we conclude as in the previous case. $\square$

One may suppose that this kind of result is true for the "bounded" *three vectors game*, but this is not the case (this property is not satisfied for the instance $[2, 2, 1]$ for example).

Remark 26 *Theorem 25 asserts that the $\mathcal{P}$-positions of bounded Wythoff's game can be determined by translation of the $\mathcal{P}$-positions of Wythoff's game on a bounded chessboard. It is easy to check that this property is also true for the game of Nim with two heaps. Are there other games for which this result is true ?*

Problem 27 *One can investigate the case of Wythoff's game which is only bounded in the diagonal direction. Conversely, consider the case where the horizontal and vertical distances are bounded, but the diagonal distance is infinite.*

Problem 28 *In [11], Fraenkel defines a generalized Wythoff's game. If one removes k and l tokens in both heaps, then the condition $|k - l| < a$ must be fulfilled, where a is a fixed positive integer. One can investigate the bounded version of this game, and test whether some periodicity of the $\mathcal{P}$-positions appears.*

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