New results about Solitaire Clobber 2

Eric Duchêne*, Sylvain Gravier†, Julien Moncel‡
ERTé "Maths à modeler"

Abstract

Solitaire Clobber is a one-player game that was introduced by Demaine et al. in 2002. In the variant Solitaire Clobber 2, the aim is to delete the maximum number of stones that are initially placed on the vertices of a graph, according to specified rules. In this paper, we present new results when one play on multipartite graphs, and we then consider extremal questions about the maximum number of stones one can delete.

1 Introduction and definitions

We consider the game *Solitaire Clobber 2* introduced in [3]. This game is a variant of *Solitaire Clobber* defined by Demaine et al. in [1], which is itself a solitaire variant of the two-player game *Clobber* defined by Albert et al. in [2].

Solitaire Clobber 2 (*SC2*, in its abbreviated form) is a one-player game defined by the following rules: black and white stones are placed on the vertices of a given graph G (one per vertex). A move consists in picking a stone and clobbering another one of the opposite color located on an adjacent vertex. The clobbered stone is removed from the graph and is replaced by the picked one. The goal is to find a succession of moves that minimizes the number of remaining stones. A game configuration of *SC2* is said to be *k-reducible* if there exists a succession of moves that leaves at most k stones on the graph. The *reducibility value* of a game configuration C is the smallest integer k for which C is k-reducible.

In [3, 4], *Solitaire Clobber 2* was investigated on paths, cycles, and trees. The reducibility value of any configuration on a path or a cycle of size n can be computed in linear time $O(n)$. On a tree, there exists an algorithm working in $O(n^9)$ operations. In [5], it was proved that any game configuration on a Hamming graph is 1-reducible, except for hypercubes which are 2-reducible.

For convenience for the reader, we may often mix up a vertex with the stone it supports. The *label* or *color* of a vertex will define the color of the stone located on it. We may also say that "a vertex clobbers another one", instead of talking of the corresponding stones.

*Post-Doc in the University of Liège, Belgium, email:educhene1@yahoo.fr
†CNRS, email:sylvain.gravier@ujf-grenoble.fr
‡INPG Professor, Grenoble, France email:julien.moncel@g-scop.inpg.fr

Given a game configuration C on a graph G, we say that a label/color c is *rare* on a subgraph S of G if there exists a unique vertex $v \in S$ such that v is labeled c. On the contrary, c is said to be *common* if there exist at least two vertices of this color in S. A configuration is said to be *monochromatic* if all the vertices have the same color. It is said *proper* otherwise. In all this paper, only proper configurations will be considered.
If c is a color, denote by $\overline{c}$ the opposite color. If v is a vertex of G, the color of v will be denoted by $c(v)$.

According to a probabilistic result of Ruszinkó (cf. [6]), almost all graphs are 1-reducible. This means that the structure of game configurations which are not 1-reducible is an interesting problem. In the next section of this paper, we study the case of complete multipartite graphs, and we decide when a configuration cannot be 1-reducible. For these graphs, we give a general formula which computes the reducibility value of any game configuration in linear time. In section 3, we consider extremal questions about *SC2*: given a graph G, we investigate the proper configurations that minimize and maximize the reducibility value on G. In [4], it was showed that the reducibility value of a cycle C_n is at most $\lfloor n/3 \rfloor$. Moreover, for all n there exists a configuration for which this bound is tight. In the present paper, we claim that on any graph with n vertices, the reducibility value is at most equal to $n - \delta$, where δ is the minimum degree of the graph. Therefore, for any fixed δ, we find out the only configurations for which the reducibility value is equal to $n - \delta$.

2 SC2 played on complete multipartite graphs

2.1 SC2 on complete bipartite graphs

Let $G = (V, E)$ be a complete bipartite graph. Therefore, G can be splitted into two stable sets, say S_0 and S_1. A game configuration on a complete bipartite graph will be represented by a "two rows" layout, each row refering to a stable set S_i with $i \in \{0, 1\}$. Figure 1 shows an example of a configuration on a complete bipartite graph. One can move any stone of a row to any stone of the other row having the opposite color. Moves inside a row are not allowed.

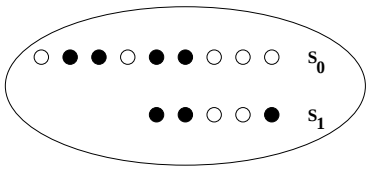


Figure 1: A game configuration on a complete bipartite graph

In [1], Demaine et al. define an invariant δ on bipartite graphs and available for both games *SC* and *SC2*. The decription of δ follows.
Let C be a game configuration on a bipartite graph G. We allocate the color *white* the to set S_0, and the color *black* to S_1. A stone of C is said to be *clashing* if its color differs from the

color of the stable set to which it belongs. Denote by $\delta(C)$ the following quantity:

$$\delta(C) = \text{number of stones} + \text{number of clashing stones}$$

In their paper, Demaine et al. proved that ($\delta(C) \bmod 3$) never changes during the game, where $a \bmod b$ is the remainder of the division of a by b. This implies that any 1-reducible configuration on a bipartite graph satisfies $\delta(C) \bmod 3 \neq 0$.

A complete bipartite graph is said to be *balanced* if S_0 and S_1 have the same size. It is said *well-colored* if S_0 and S_1 are both monochromatic. In a well-colored configuration, one do not lose generality by assuming that the stones of S_0 (resp. S_1) are white (resp. black). Under this assumption, there is no clashing stone in a well-colored configuration.

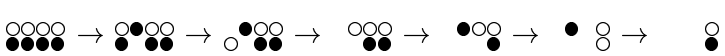
2.1.1 The reducibility value of balanced complete bipartite graphs

We now give a first result about the reduciblity value of a well-colored balanced complete bipartite graph.

Lemma 1. *A well-colored balanced complete bipartite graph is* 1*-reducible if* $\delta \bmod 3 \neq 0$. *Otherwise, the reducibility value is equal to 2.*

Proof. Let C be a game configuration on a well-colored balanced complete bipartite graph. Denote by n the size of each stable set. Then $\delta(C) = 2n$. According to the result of Demaine et al., a necessary condition for C to be 1-reducible is that $\delta(C) \bmod 3 \neq 0$, which means $n \bmod 3 \neq 0$.

Let $n \geq 4$. Then C can be reduced to a well-colored balanced complete bipartite graph where each stable set has a size equal to $n - 3$. The reduction is the following:



By applying recursively this operation, and according to the value ($n \bmod 3$), C can be reduced to one of these three configurations:

$n \bmod 3 = 1$ $\quad n \bmod 3 = 2$ $\quad n \bmod 3 = 0$

One clearly see that when $n \bmod 3 \neq 0$, the above configurations are 1-reducible. When $n \bmod 3 = 0$, the above configuration is 2-reducible and we can not improve this reduction because of the invariant. □

Remark 1. *When* $n \bmod 3 \neq 0$*, the value* $\delta(C)$ *sets a correlation between the color and the location of the remaining stone. When* $n \bmod 3 = 1$*, the last stone will be clashing. It will not be the case when* $n \bmod 3 = 2$.

Lemma 2. *A balanced complete bipartite graph is 1-reducible if* $\delta \bmod 3 \neq 0$. *Otherwise, the reducibility value is equal to 2.*

Proof. For more convenience, we arrange "graphically" S_0 and S_1 as follows: on S_0 the white stones are placed on the left, and the black ones on the right. We proceed conversely on S_1. This operation splits G into three complete bipartite subgraphs, of respective lengths L_1, L_2 and L_3, as shown on Figure 2.

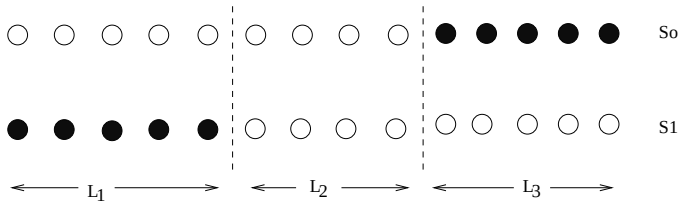


Figure 2: Splitting a balanced complete bipartite graph

The values L_1 and L_3 are the lengths of both well-colored balanced parts, whereas L_2 is dedicated to the monochromatic part.

According to this notation, $\delta(C) = 2L_1 + 3L_3 + 4L_3$. Hence $\delta(C) \mod 3 = 0$ if and only if $L_1 \bmod 3 = L_3 \bmod 3$. Without loss of generality, assume that the monochromatic part L_2 is white. Since there exists a black stone somewhere, we have it clobber the whole monochromatic part of size L_2, as shown on Figure 3:

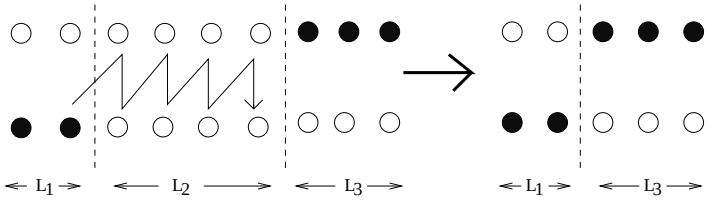


Figure 3: Clobbering the monochromatic part of length L_2

We now consider the resulting graph as the union of two well-colored balanced complete bipartite graphs, of respective sizes L_1 and L_3. We distinguish two cases:

* $L_1 \bmod 3 \neq L_3 \bmod 3$.

  Now suppose that either $L_1 \bmod 3$ or $L_3 \bmod 3$ is equal to 1. Without loss of generality, assume that $L_1 \bmod 3 = 1$. Then from Lemma 1 and Remark 1, we reduce the left part to a single clashing stone. If $L_3 = 0$, then it is done. Otherwise, we know that L_3 is at least equal to 2. By playing as shown on Figure 4, we reduce the graph to a well-colored balanced complete bipartite of length $L_3 - 1$. Since $L_3 - 1 \neq 0 \bmod 3$ and from Lemma 1, this resulting graph is 1-reducible.

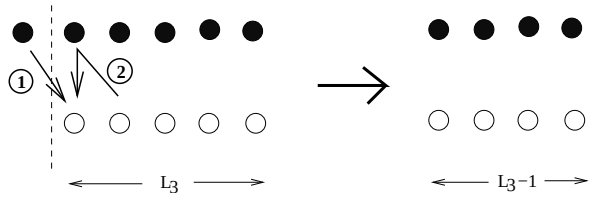


Figure 4: Reduction in a special case

Now suppose that neither $L_1 \bmod 3$ nor $L_3 \bmod 3$ is equal to 1. Therefore and without loss of generality, we can assume $L_1 \bmod 3 = 2$ and $L_3 \bmod 3 = 0$. By lemma 1, we reduce the left part to a single non clashing stone. If $L_3 = 0$, then it is done. Otherwise, $L_3 \geq 3$. We proceed as depicted by Figure 4 to reduce the configuration. This operation yields a well-colored balanced complete bipartite graph of length $(L_3 - 1)$. Since $(L_3 - 1) \bmod 3 = 2$, we conclude to the 1-reducibility thanks to Lemma 1.

* $L_1 \bmod 3 = L_3 \bmod 3$.

  If L_1 and L_3 are not multiples of 3, then from Lemma 1, both parts are 1-reducible. Therefore, the whole configuration is 2-reducible. If $L_1 \bmod 3 = L_3 \bmod 3 = 0$, we play as depicted by Figure 5.

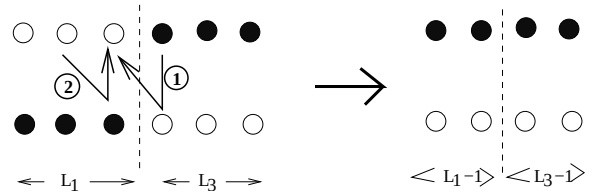


Figure 5: The case $L_1 \bmod 3 = L_3 \bmod 3 = 0$

The resulting configuration is a balanced complete bipartite graph with

$L_1 \bmod 3 = L_3 \bmod 3 = 2$, which is 2-reducible.

□

2.1.2 The reducibiliy value of non-balanced complete bipartite graphs

We now consider the general case on $K_{n,m}$ with $|S_0| = n > |S_1| = m > 0$. Unlike balanced complete bipartite graphs, we will show that the reducibility value may no more be bounded by a constant. For example, it suffices to choose $|S_1| = 1$ and $|S_0| > |S_1|$, put a black stone on S_1 and only white stones on S_0. In such a configuration, only one move is available, leading to a reducibility value equal to $|S_0|$.

In order to describe a game configuration on $K_{n,m}$, we consider the following set of parameters:

The values n_b and n_w denote respectively the numbers of black and white stones in S_0. Obviously we have $n = n_b + n_w$. Similarily, m_b and m_w denote respectively the numbers of black and white stones in S_1.
Without loss of generality, we consider configurations satisfying $n_b \leq n_w$. Under this condition, we define a nonnegative integer $q = n - 2n_b$.

With these parameters, a game configuration belongs to one of these two forms:
(F1): Configurations for which $q - m \leq m_b - 1$.
(F2): Configurations for which $q - m > m_b - 1$.

Figure 6 illustrates the form $(F1)$ through three examples, according to the sign of $(q-m)$ and the parity of $(n-m)$. We will understand why further.

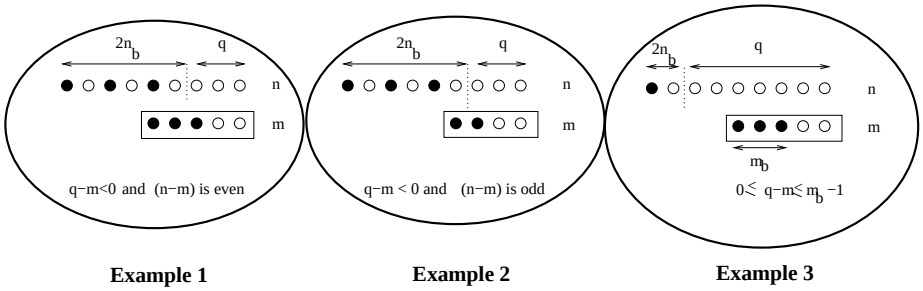


Figure 6: Three examples of configurations in the form $(F1)$

Figure 7 illustrates the second form. In that case, the stable set S_0 can be partitionned into three parts:
L_1 contains as many black stones than white ones and satisfies $|L_1| = 2n_b$.
L_2 and L_3 are monochromatic parts, of respective sizes $(q-m)$ and m.

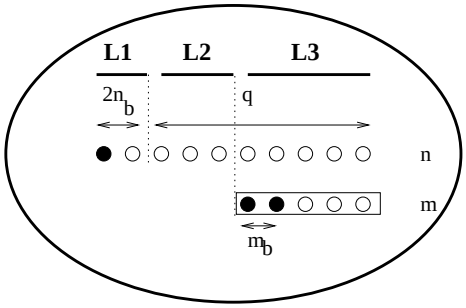


Figure 7: A configuration in the form $(F2)$

We will prove that configurations in the form $(F1)$ have a reducibility value equal to 1 or 2. The results will be similar to the "balanced case". For the configurations in the form $(F2)$, we will present a formula depending on the values q, m and m_b. For such configurations, we

introduce new results that will help us to conclude.

Let C_2 be a game configuration in the form $(F2)$. We define the integer function f as:

$$f(C_2) = q - m - m_b$$

By definition of $(F2)$, we have $f(C_2) \geq 0$.

The following lemma asserts that f never decreases during the game. Moreover, it adds that any move from a configuration in $(F2)$ yields a resulting configuration also in $(F2)$.

Lemma 3. *Let C_2 be a game configuration in $(F2)$, and let R be the resulting configuration after an arbitrary move from C_2. If R is not a stable set, then R is also in $(F2)$ and satisfies $f(C_2) \leq f(R)$.*

Proof. Denote by q', m' and m'_b the values of q, m and m_b after a move from C_2 to R. These values are defined since R is supposed to be a configuration on a complete bipartite graph.

There are four moves to consider:

1. One play a black stone from S_0 to S_1. Then $m'_b = m_b + 1$, $q' = q + 1$ and m is unchanged. This implies $f(R) = f(C_2)$.
2. One play a white stone from S_0 to S_1. We get $q' = q - 1$, $m'_b = m_b - 1$, and m is unchanged. We still have $f(R) = f(C_2)$.
3. One play a black stone from S_1 to S_0. Then $m'_b = m_b - 1$, $m' = m - 1$, and $q' = q - 2$. Therefore $f(R) = f(C_2)$.
4. One play a white stone from S_1 to S_0. Then $m' = m - 1$, $q' = q + 2$ and m_b is unchanged. Therefore $f(R) = f(C_2) + 3$.

Moreover, since f is monotonic, we clearly have $q' - m' > m'_b - 1$, which means that R is in the form $(F2)$. □

We now have collected all the necessary results to prove our main theorem about the reducibility of $K_{n,m}$.

Theorem 4. *Let C be a game configuration on $K_{n,m}$ with $n > m > 0$.*
If $q - m \leq m_b - 1$, i.e., C is in $(F1)$, then C is 1-reducible if $\delta(C) \bmod 3 \neq 0$, and 2-reducible otherwise.
If $q - m > m_b - 1$, i.e., C is in $(F2)$, then C has a reducibility value equal to $(q - m - m_b + 2)$.

Proof. **C is in the form (F1)**. We will prove that we can reduce C to a non-monochromatic balanced complete bipartite graph, so as to apply Lemma 2. We distinguish three cases (see Figure 6).

* $q - m < 0$ and $(n - m)$ is even. Let s be any stone of S_1. Since $(n - m)$ is even, this means that we can play exactly $(n - m)$ stones (precisely $(n - m)/2$ black stones, and $(n - m)/2$ white ones) from S_0 to s by alternating black and white moves. The resulting configuration R is thus a balanced complete bipartite graph. Note that S_1 is unchanged after this set of operations. Since $q - m < 0$, there exists at least a black and a white stone in the stable set S_0 of R, which is consequently not monochromatic.

* $q - m < 0$ and $(n - m)$ is odd. As previously, we play exactly $(n - m - 1)$ stones (precisely $(n - m - 1)/2$ black stones, and $(n - m - 1)/2$ white ones) from S_0 to any vertex s of S_1 by alternating black and white moves. In the resulting configuration R, S_0 contains exactly $(m + 1)$ stones and is not monochromatic, whereas S_1 is unchanged. We now play any move from S_0 to S_1. This is possible since S_0 is not monochromatic. This operation yields a non-monochromatic balanced $K_{m,m}$.

* $q - m \geq 0$. We first play $2n_b$ stones (precisely n_b stones of each color) from S_0 to any vertex s of S_1 by alternating white and black moves. We then play $(q - m)$ white stones from S_0 to S_1. This is possible since $m_b \geq (q - m + 1)$ for a configuration in $(F1)$. The resulting configuration R contains at least a black stone in S_1, and at least a white one in S_0. Moreover, R is balanced.

In each of these three cases, we yield a proper configuration R which is balanced. We then apply Lemma 2 on R to conclude.

C is in the form (F2). We first prove that there exists a way of play that leaves exactly $(q - m - m_b + 2)$ stones on the graph.
If $m_b > 0$, we play the $2n_b$ stones from L_1 to any vertex s of S_1 by alternating white and black moves. After this operation, S_0 is monochromatic (of color white). We then play $(m_b - 1)$ white stones from L_2 to S_1. The single remaining black stone (which belongs to S_1) then walks in zigzags to clobber all the white stones of L_1 and S_1. The resulting configuration is a stable set containing a black stone and $(q - m - m_b + 1)$ white ones (coming from L_2).
If $m_b = 0$, then L_1 is not empty, since we do not consider monochromatic configurations. Therefore, we play a black stone from L_1 to S_1. We get a resulting configuration C' in $(F2)$ with parameters $m'_b = m_b + 1$ and $q' = q + 1$. We now apply the same method as above, which leaves $(q' - m - m'_b + 2) = (q - m - m_b + 2)$ stones.

We now prove that any succession of moves leaves at least $(q - m - m_b + 2)$ stones on the graph. We consider an optimal succession of moves on a configuration C in $(F2)$, i.e., a succession of moves that leaves the minimum number of stones. Denote by C' the final configuration after such a way of play. No move is playable from C'.

We first suppose that C' is not a stable set. According to Lemma 3, C' belongs to $(F2)$ and satisfies $f(C') \geq f(C)$. Denote by $q', m', m'_b, L'_1, L'_2, L'_3$ the parameters previously defined and relative to C'.
Since $m' > 0$, L'_3 is not empty, which means that there exists a white stone on S_0. Since no move is playable from C', the set S_1 is monochromatic of color white, i.e., $m'_b = 0$, and L'_1 is

empty.
Moreover, the monotonicity of the function f ensures that

$$q' - m' - m'_b = q' - m' \geq q - m - m_b \tag{1}$$

Since there is at least a stone in L'_3 and another one in S_1, the number of remaining stones on C' is at least $|L'_2| + 2$. Then $|L'_2| + 2 = q' - m' + 2 \geq q - m - m_b + 2$ according to (1).
Now assume that C' is a stable set. This means that the last move of the optimal way of play was done from S_1 to S_0. It remains optimal if we consider this last move in the opposite direction (i.e., from S_0 to S_1). Now consider this new optimal way of play where C' is not a stable set and refer to the previous case to conclude. □

2.2 SC2 played on complete multipartite graphs (with at least three parts)

Let $P > 2$. Denote by $S_0, S_1, \ldots, S_{P-1}$ the P stable sets of a complete P-partite graph G. Among these P stable sets, denote by $M_0, \ldots, M_{i-1}$ the i stable sets whose size is maximum, i.e., $|M_j| \geq |S_l|$ for all $j = 0 \ldots i-1$ and $l = 0 \ldots P-1$. We call them the *maximum stable sets* of G.
Denote by G' the induced subgraph $G \setminus \{M_0, \ldots, M_{i-1}\}$. For more convenience, we graphically define "raws" of stones on $G \setminus G'$: if $t = |M_0| = \ldots = |M_{i-1}|$, then a raw R_j with $1 \leq j \leq t$ defines a set of i stones (one stone per maximum stable set) such that $\bigcup_{1 \leq j \leq t} R_j$ is the set of all the stones belonging to $G \setminus G'$. Of course, the number of partitions of this form is very large (each new "graphical" arrangement of the stones in $G \setminus G'$ defines a new set of raws).
Finally, denote by s_j^k (with $1 \leq j \leq t$ and $0 \leq k < i$) the vertex that both belongs to M_k and R_j.
Figure 8 illustrates these notations.

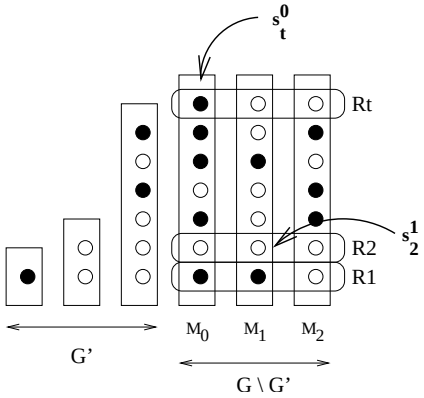


Figure 8: An example of a configuration on a P-partite graph ($P = 6$ and $i = 3$)

The reducibility value of P-partite graphs depends on the number i of maximum stable sets. When $i > 1$, any game configuration is 1-reducible. The reducibility value may be far larger when there is a unique maximum stable set.

Theorem 5. *Let C be a game configuration on a P-partite graph (with $P > 2$) having at least two maximum stable sets. Then C is 1-reducible.*

Proof. Denote by t the value $|M_0|$, i.e. the size of each maximum stable set.
If $t = 1$, then the graph is a clique, which is clearly 1-reducible (see [5] for more details).
Now consider $t > 1$ and assume that the result is true for all $1 \leq t' < t$. We consider two cases:

1. $G \setminus G'$ is monochromatic of color c. Since G is not monochromatic, there exists a vertex x in G' whose color is $\overline{c}$. Now have x clobber the monochromatic raw R_t, and then any stone of the raw R_{t-1}. In the resulting non-monochromatic configuration, the size of the maximum stable sets is $(t-1)$, so that we can apply the induction hypothesis.

2. $G \setminus G'$ is not monochromatic and $i \geq 3$. We consider a partition of $G \setminus G'$ into t raws, such that R_t is not monochromatic. Such a partition is possible since $G \setminus G'$ is not monochromatic. Besides, since $i \geq 3$, there exists a stone of R_t whose color is not rare in R_t. Without loss of generality, assume that s_t^0 satisfies this property. For the same reasons, either s_{t-1}^1 or s_{t-1}^2 has a common color. Assume that $c(s_{t-1}^1)$ is not rare.

   The raw R_t is now seen as a clique. From [5], we know that cliques are strongly 1-reducible, which means that we can choose the location and the color of the unique remaining stone (provided some conditions are fulfilled, which will be the case here). Hence we reduce R_t to a single stone of color $\overline{c(s_{t-1}^1)}$ and located on s_t^0. We then play from s_t^0 to s_{t-1}^1, the resulting configuration is not monochromatic, so that we can apply the induction hypothesis.

3. $G \setminus G'$ is not monochromatic and $i = 2$. As previously, choose a partition into t raws such that R_t is not monochromatic. Since $P > 2$, at least a color, say black, appears in G'. According to the colors of R_{t-1}, we play as follows to get a non monochromatic resulting graph, whose size of the maximum stable sets is $(t-1)$. We conclude to 1-reducibility by applying the induction hypothesis.

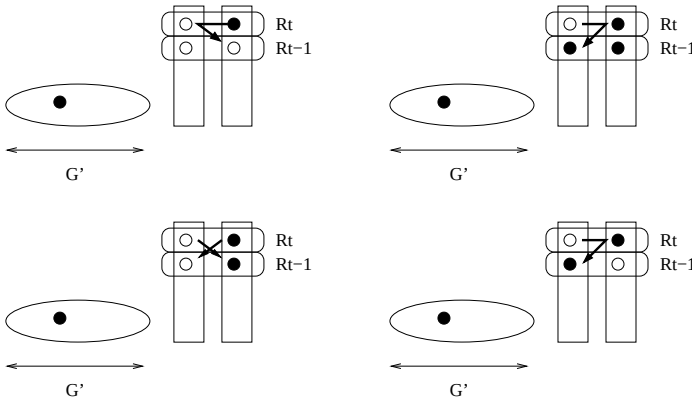


Figure 9: Playing when $i = 2$ and $G \setminus G'$ is not monochromatic

□

When the graph admits an unique maximum stable set M_0 of size t, we define the values $n = t$, m as the number of stones in G' and m_b the number of black stones in G'. The values n_b and n_w are respectively the numbers of black and white stones in M_0. Without loss of generality, assume that $n_b \leq n_w$. We thus define q as the value $n - 2n_b$. We denote by M' a stable set of G' whose size is maximum (i.e, a "second maximum" stable set in G). We call its size m'. Finally, denote by G'' the graph $G' \setminus M'$. Figure 10 shows a P-partite graph with an unique maximum stable set.

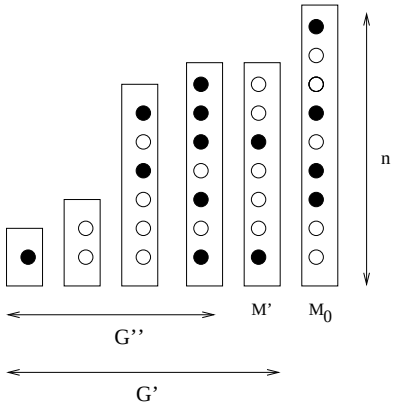


Figure 10: An example of a P-partite graph with an unique maximum stable set

With the above notations, we extend the function f to a complete P-partite graph having an unique maximum stable set (with $P \geq 2$). We define f as $f(C) = q - m - m_b$. Lemma 3 is thus extended to P-partite configurations:

Lemma 6. *Let C be a game configuration on a complete P-partite graph having an unique maximum stable set (with $P > 2$) satisfying $q - m > m_b - 1$, and let R be the resulting configuration after an arbitrary move from C. Then R is also P-partite (with $P \geq 2$), has an unique maximum stable set and still satisfies $f(C) \leq f(R)$.*

Proof. If is straightfoward to see that R is P-partite with $P \geq 2$.
By hypothesis, the configuration C satisfies

$$q = n - 2n_b \geq m + m_b = m_b + |M'| + |G''| \geq m_b + |M'| + 1$$

Hence we have

$$n \geq m_b + 2n_b + |M'| + 1$$

Since C is a proper configuration, the values m_b and n_b can not be simultaneously equal to 0. This implies that $n \geq |M'| + 2$. Therefore, any move from C leads to a configuration R where M_0 remains the unique maximum stable set. This result ensures that the function f is defined on R.
If we consider any move between G' and M_0, then we refer to Lemma 3 to assert that $f(C) \leq f(R)$. And if we consider a move inside of G', we remark that q is unchanged and the values m and m_b are not increased. This also implies $f(C) \leq f(R)$. □

Theorem 7. *Let G be a P-partite graph (with $P > 2$) having an unique maximum stable set. Any game configuration on G such that $q - m \leq m_b - 1$ is 1-reducible. Any game configuration on G such that $q - m > m_b - 1$ has a reducibility value equal to $q - m - m_b + 2$.*

Proof. **Any game configuration satisfying $q < m$ is 1-reducible:**
According to n, we choose a "minimal" counter-example C, i.e. C satisfies $q < m$, is not 1-reducible, and there is no other configuration of this type whose size of M_0 is smaller than n. We first suppose that $n_b \neq 0$ and we consider three cases for C:

* $n - m' = 1$. We play any authorized move from M_0 to M': this is possible since $n_w \geq n_b > 0$. Therefore, the resulting configuration is not monochromatic, and admits at least two maximum stable sets of size $n - 1$ (M_0 and M'). From Theorem 5, it is 1-reducible, which ensures the contradiction.

* $n - m' \geq 2$ and $n_b \geq 2$. We play successively a black and a white stone from M_0 to any stone of G'. The resulting configuration is not monochromatic since there remains at least a stone of each color in M_0, and still satisfies $q < m$ (both values are unchanged). If $|M_0| = |M'|$ after the operation, then it is 1-reducible from Theorem 5. If not, this means that M_0 remains the unique maximum stable set, but its size is now smaller than n, which yields a contradiction by minimality of C.

* $n - m' \geq 2$ and $n_b = 1$. If $m_b > 0$, we play any white stone from M_0 to any black one of G'. The resulting configuration is not monochromatic, has an unique maximum stable set, still satifies $q < m$ (since m is unchanged and q is decreased by 1), and is smaller than C. It thus contradicts the minimality of C. Hence we have $m_b = 0$, which means that G' is monochromatic white. With the unique black stone of the configuration, we clobber alternately stones of M_0 and G", until M' and M_0 have the same size. This is possible, since the number of stones in G" is equal or greater than $q + 1 - m'$. Note that in the case where $|G"|$ exactly equals $q + 1 - m'$, the black stone will delete the entire G" and then alternately clobber all the white stones between M' and M_0. In any case, we reduce C to a single stone, which contradicts its existence.

Therefore C satisifies $n_b = 0$. We consider two cases:

* G' is monochromatic black. Then play any white stone from M_0 to G'. The resulting configuration is not monochromatic (since there remains at least a black stone in G'), and still satisfies $q < m$ (m is unchanged and q is decreased by 1). In the resulting configuration, if M_0 and M' have the same size, we conclude to 1-reducibility thanks to Theorem 5. Otherwise, M_0 is the unique maximum stable set and it contradicts the minimality of C.

* G' is not monochromatic. Let s be any black stone of G'. Then s clobers any white stone of M_0 and then any other white stone of G'. The parameters q and m are both decreased by one after these moves. Once again, the minimality of C is contradicted. Note that the resulting configuration contains is still P-partite with $P \geq 3$. Indeeed, if it was bipartite, this would mean that C contains at least two maximum stable sets, since $q < m$.

Any game configuration satisfying $0 \leq q - m \leq m_b - 1$ is 1-reducible:
As previously, we choose C as a minimal configuration satisfying $0 \leq q - m \leq m_b - 1$, not 1-reducible, and such that there is no other configuration of this type whose size of M_0 is smaller than n. Note that this implies $m_b > 0$.
We first suppose that $n_b \neq 0$. As previously, if $n - m' = 1$, we play an authorized move from M_0 to G', so as to apply Theorem 5. If $n - m' > 1$, we play successively a black and a white stone from M_0 to any stone of G': the resulting configuration R has its parameters q, m and m_b unchanged, and is not monochromatic, since $m_b \geq 1$ and there are at least $m > 0$ white stones in M_0. As previously, we either conlude to the 1-reducibility with Theorem 5 (if $|M_0| = |M'|$ in R), or to the non-minimality of C.
Hence C satisfies $n_b = 0$. This means that $|M_0|$ is monochromatic white. We consider two cases:

* G' is not monochromatic black.

  If G' contains at least three non empty stable sets, then play any black stone s from G' to any white stone of M_0 and then to any white stone of G'. The resulting configuration R is still P-partite with $P \geq 3$, and is not monochromatic (there is at least a black stone in G' and a white one in M_0). Besides, q and m are both decreased by 1, and m_b is unchanged, which implies that we still have $0 \leq q - m \leq m_b - 1$. We either conlude to the 1-reducibility thanks to Theorem 5 (if $|M_0| = |M'|$ in R), or to the non-minimality of C.

  If G' contains exactly two stable sets and if $m_b \geq 2$, we apply the same method by choosing a black stone s such that s does not belong to a stable set of size one. It is always possible to find such a stone, since there are at least three stones in G' (two blacks and a white) and only two stable sets. With this choice, the resulting configuration remains tripartite.

  If G' contains exactly two stable sets and if $m_b = 1$, this implies $q = m$. There are as many stones in G' than in M_0. We use the only black stone to clobber alternately the white stones of M_0 and G'. This yields a unique stone on the graph in the end.

* G' is monochromatic black.

  If $q > m$, then play any white stone from M_0 to G'. After this operation, q and m_b are decreased by 1 and m is unchanged. Hence we still have $0 \leq q - m \leq m_b - 1$. The resulting configuration R is not monochromatic, since there are at least a black and a white stone in G'. We either conlude to the 1-reducibility with Theorem 5 (if $|M_0| = |M'|$ in R), or to the non-minimality of C.

  If $q = m$, we forget the edges inside G' and see $C = G' \cup M_0$ as a well-colored balanced complete bipartite graph. From Lemma 2, if $\delta(C) \bmod 3 \neq 0$, then C is 1-reducible. And if $\delta(C) \bmod 3 = 0$, any white stone of M_0 clobbers successively two black stones of G': this operation uses an inside edge of G', and thus changes the value of $\delta(C) \bmod 3$. Thanks to Lemma 2, we conclude to the 1-reducibility of C.

Any game configuration satisfying $q - m > m_b - 1$ has a reducibility value equal to $q - m - m_b + 2$:

Let C be a game configuration satisfying $q - m > m_b - 1$. If we forget the edges inside of G', C can be seen as a bipartite graph $G' \cup M_0$, where M_0 is the largest stable set. The parameters q, m and m_b have been extended from bipartite to multipartite graphs. From Theorem 4 about complete bipartite graphs, we conlude that the reducibility value of C is at most $q - m - m_b + 2$.

We now consider C', a final configuration after having played an optimal succession of moves from C. If C' is P-partite with $P \geq 2$, denote by m', n', n'_b, m'_b and q' the parameters of C'. We study two cases.
If C' is P-partite with $P \geq 3$, then necessarily $n'_b = m'_b = 0$. The number of remaining stones on C' is at least $|M_0| + |G'| \geq q' + 2$. Since f is monotonic from Lemma 6, we have $q' + 2 \geq q - m - m_b + 2$.
If C' is P-partite with $P < 3$, then there exists a configuration C" on a complete bipartite graph in the optimal way of play from C to C'. Since C" satisifes $q - m > m_b - 1$ (according to Lemma 6), we can apply Theorem 4 to prove that the reducibility value of C'' is greater than $q - m - m_b + 2$. □

3 Extremal questions

Let us denote by $\mathrm{rv}(G, C)$ the reducibility value of G with configuration C, and denote by $\mathrm{maxrv}(G)$ the maximum of these values taken over all proper configurations, and $\mathrm{minrv}(G)$ the minimum of these values taken over the same set.
The purpose of this section is to estimate the values of $\mathrm{minrv}(G)$ and $\mathrm{maxrv}(G)$ for all graph G.

3.1 Value of $\mathrm{minrv}(G)$

We determine the exact value of $\mathrm{minrv}(G)$ for all graph G, and prove that it is simply equal to the number of connected components of G.

Lemma 8. *Let T be a tree. Then we have*

$$\mathrm{minrv}(T) \;=\; 1.$$

Proof. The proof works by induction on the number of vertices of T. If T has one vertex, then the statement is trivial. If T has at least two vertices, then let u be a leaf of T. By induction, the tree $T' := T \setminus \{u\}$ admits a configuration C' such that $\mathrm{rv}(T', C') = 1$. Let v be the unique neighbour of u in T, and let C be the following configuration on T :

* $C(w) = C'(w)$ for all $w \neq u, v$,
* $C(u) = C'(v)$,
* $C(v) =$ black if $C'(v) =$ white, and $C(v) =$ white otherwise.

It is easy to check that $\mathrm{rv}(T, C) = 1$ (hint: the first move is u clobbering v). □

Theorem 9. *Let G be a connected graph. Then we have*

$$\text{minrv}(G) \;=\; 1.$$

Proof. This is a straightforward consequence of Lemma 8: since G is connected, then it admits a spanning tree T, that is a tree which is a subgraph of G having the same vertex set as G. Lemma 8 implies the existence of a configuration C such that $\text{rv}(T, C) = 1$, which implies $\text{rv}(G, C) = 1$. □

3.2 Bounds on $\text{maxrv}(G)$

3.2.1 Upper bound

We prove an upper bound related to the minimum degree of the graph, and we characterize the graphs for which this bound is tight.

Theorem 10. *Let G be a graph on n vertices of minimum degree δ, and let C be a proper configuration of G. Then we have*

$$\text{rv}(G, C) \leq n - \delta.$$

Furthermore, one can make at least δ moves by, step by step, greedily choosing any move which leaves at least one connected component with a proper configuration.

Proof. The proof works by induction on δ. If $\delta = 1$, then there exists at least an edge uv such that there are stones of different colors on u and v, hence one can make at least one move. If $\delta \geq 2$, then there exists at least an edge uv such that there are stones of different colors on u and v. Playing along this edge (either u clobbering v or v clobbering u) surely reduces to a graph of minimum degree at least $\delta - 1$, but the graph may not be connected, and the configurations on the connected components of this graph may not be proper. However, it is enough to get at least one connected component whose configuration is proper to apply the induction hypothetis. Now two cases follow.
If playing along the edge uv (either u clobbering v or v clobbering u) leads to a new graph for which at least one its connected component C has a proper configuration, then we can apply the induction hypothesis on C. If not, then let us assume that the stone on u is black (and then the stone on v is white). The fact that either u clobbering v or v clobbering u does not lead to a graph for which at least one its connected component has a proper configuration implies the following:

* the edge uv is such that $G \setminus \{uv\}$ has two connected components U and V, such that $u \in U$ and $v \in V$,
* all the stones on vertices of $U \setminus \{u\}$ are white,
* all the stones on vertices of $V \setminus \{v\}$ are black.

In this case, one can still apply the induction hypothesis by, for instance, clobbering a stone of U with the stone placed on u. □

For any $\delta \geq 1$, let us define $\mathcal{G}_\delta$, a set of connected graphs of degree minimum δ equipped with proper configurations, as follows:

* the complete graph on $\delta + 1$ vertices equipped with any proper configuration belongs to $\mathcal{G}_\delta$ for all $\delta \geq 1$;
* for any integer $k \geq 2$, let us define $S_k(K_{\delta+1})$ as the non-disjoint union of k copies of $K_{\delta+1}$, where one and only one vertex v belongs to all k copies of $K_{\delta+1}$, and no other vertex belongs to more than one copy of $K_{\delta+1}$ (one can think of $S_k(K_{\delta+1})$ as a star $K_{1,k}$, where each of the k leaves has been expanded into a K_δ, see Figure 11). Then $S_k(K_{\delta+1})$, equipped with a configuration such that the color of the vertex v is rare, belongs to $\mathcal{G}_\delta$ for all $k \geq 2$;
* for $\delta = 2$, the cycle on 4 vertices equipped with any configuration containing only two black stones, which are placed on consecutive vertices of the cycle, belongs to $\mathcal{G}_2$;
* no other graph belongs to $\mathcal{G}_\delta$.

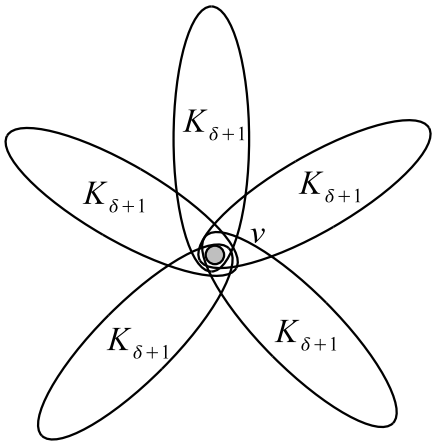


Figure 11: The graph $S_k(K_{\delta+1})$.

Theorem 11. *For all $\delta \geq 1$, the set of connected graphs G having minimum degree δ equipped with a proper configuration C such that*

$$\mathrm{rv}(G, C) = n - \delta$$

is exactly $\mathcal{G}_\delta$.

Proof. The proof works by induction on δ. The case $\delta = 1$ is easy to prove. Take an edge uv such that a black stone lies on u and a white stone lies on v. Assume that u has other neighbours than v. Since the reducibility value is $n - 1$, then all the other neighbours of u are white, and form a stable set. This implies that v has no other neighbours than u.

Now let $\delta \geq 2$, and let G be a connected graph on n vertices having minimum degree δ equipped with a proper configuration C such that $\mathrm{rv}(G, C) = n - \delta$. Let uv be the edge along which the first move of a game ending after exactly δ moves is made. We may assume that the stone lying on u is black, and that this stone clobbers a white stone placed on v. Let G' be the graph obtained after this first move. We then claim the following:

> The graph G' has minimum degree $\delta - 1$, and all the connected components of G' are of cardinality greater than or equal to δ. Moreover, there is one and only one of these components whose configuration is proper. (2)

The first part of the claim is obvious since G has minimum degree δ. The second part derives from Theorem 10. Indeed, if G' had at least two connected components equipped with proper configurations, then we could make at least $\delta - 1$ moves on one, and $\delta - 1$ moves on the other, which is a contradiction with $\mathrm{rv}(G, C) = n - \delta$ since $\delta \geq 2$. Hence claim (2) is true, and this implies the following:

> One and only one connected component of G' is a member of $\mathcal{G}_{\delta-1}$, and all the others are white monochromatic cliques on δ vertices. (3)

From claim (2), we know that G' has a connected component A equipped with a proper configuration. This component has minimum degree $\delta - 1$, and since $\mathrm{rv}(G, C) = n - \delta$, then one can make exactly $\delta - 1$ more moves on A, hence $A \in \mathcal{G}_{\delta-1}$. We know from (2) that all the other components are monochromatic, and that they have at least δ vertices. Let B be such a component. By definition, no vertex of B is adjacent to v. This implies that B is white monochromatic. Indeed, if B were black monochromatic, then v could clobber u, and this would lead to a graph equipped with a proper configuration of minimum degree δ. By Theorem 10, one could make at least δ more moves, a contradiction with $\mathrm{rv}(G, C) = n - \delta$. Now, let w be a neighbour of u in B. The second part of Theorem 10 implies that u clobbering w can be the first of a sequence of exactly δ moves, hence B is a member of $\mathcal{G}_{\delta-1}$. For all $\delta \geq 2$, it is easy to check that the only one possibility is that B is a complete graph on δ vertices, which proves the second part of (3). To complete the proof, it remains then to make a case study by applying (3) and using the induction hypothesis. The cases $\delta = 2$ and $\delta = 3$ have to be studied separately, because of the sporadic event of the cycle on four vertices belonging to $\mathcal{G}_2$. This (rather straightforward) case study is left to the reader. □

3.2.2 Lower bound

Theorem 12. *Let C_0 be a cutset of a simple, connected graph G, and let $C_1, C_2, \ldots, C_k$ be the connected components of $G \setminus C_0$. Let us assume that C_0 is small enough, that is to say $|C_0| < k$. Let C be the following configuration of stones on G : a black stone lies on all the vertices of C_0, and a white stone lies on all the vertices of $C_1 \cup C_2 \cup \ldots \cup C_k$. Then the reducibility value of G under the configuration C is such that*

$$\mathrm{rv}(G, C) \geq n - \left(\sum_{i=0}^{|C_0|} |C_i| - 1 \right).$$

Proof. The theorem is straightforward if C_0 is a single point. Indeed, in this case, we have a cutvertex v_0, on which lies the only black stone of the configuration. If this stone does not clobber any white stone, then one may make only one move, consisting in clobbering this stone with a white stone, and the game ends after 1 round. If the black stone clobbers some white stones, then it can clobber only white stones of a certain component C_i. Indeed, since v_0 is a cutvertex, then the black stone can visit only one of the components of $G \setminus C_0$.
If C_0 is a nontrivial cutset, then the situation is a little bit more complicated, since a black stone can visit more than one components of $G \setminus C_0$. The idea is the following. If a black stone visits more than one component, then it means that, at some points, this stone had to go back through C_0, hence clobbering a white stone. Since C_0 contains originally no white stones, it means that black stones of C_0 had to be clobbered without having visited any component of $G \setminus C_0$. Thus a black stone visiting p components implies that $p-1$ black stones did not visit any component, hence, *on average*, one black stone can visit at most one component.
More formally, given a vertex $v \in C_0$, let us define $s(v)$, the *score* of the black stone placed on v, as the number of components of $G \setminus C_0$ that this black stone will visit during the game. Clearly, we have

$$\sum_{v \in C_0} s(v) \ \leq \ |C_0|. \tag{4}$$

Now, the desired inequality derives from (4). Indeed, in the most favorable case, all the white stones of the $|C_0|$ largest components and of $|C_0|$, but for one, were clobbered, hence the $\sum_{i=0}^{|C_0|} |C_i| - 1$ in the formula. □

Remark 2. *This bound is tight for the path on n vertices, where the worst configuration is the one where the central vertex has a rare color (see Figure 12).*

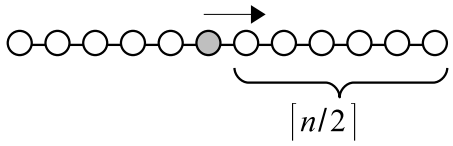


Figure 12: The path on n vertices has maximum reducibility value equal to $\lceil \frac{n}{2} \rceil$.

Indeed, the bound of the previous theorem says that $\mathrm{rv}(G,C) \ \geq \ n - \left(1 + \lfloor \frac{n}{2} \rfloor - 1\right) \ = \ \lceil \frac{n}{2} \rceil$, and it is easy to see that $\mathrm{maxrv}(\mathcal{P}_n) = \lceil \frac{n}{2} \rceil$ (where $\mathcal{P}_n$ denotes the path on n vertices). Let us briefly show this. Let $\{v_1, \ldots, v_n\}$ denotes the vertex set of $\mathcal{P}_n$ such that there is an edge between v_i and v_{i+1} for all $i = 1, \ldots, n-1$. Let us call 2-block a non-monochromatic interval $v_i, \ldots, v_j$, $1 \leq i < j \leq n$ such that there exists $i \leq k < j$ for which vertices $v_i, \ldots, v_k$ are of the same color, and vertices $v_{k+1}, \ldots, v_j$ are of the same color. Similarly, let us call 3-block a non-monochromatic interval $v_i, \ldots, v_j$, $1 \leq i < j \leq n$ such that there exists $i \leq k < l < j$ for which vertices $v_i, \ldots, v_k$ are of the same color, vertices $v_{k+1}, \ldots, v_l$ are of the same color, and

vertices $v_{l+1}, \ldots, v_j$ are of the same color. Now partition $\{v_1, \ldots, v_n\}$ into a certain number of 2-blocks and one 3-block. Clearly, it is always possible to do this. Now, let us look at a certain 2-block B, independently form the other blocks. It is clear that at least $\lfloor \frac{|B|}{2} \rfloor$ vertices of B can be clobbered by a vertex of B. Similarly, at least $\lfloor \frac{|B|}{2} \rfloor$ vertices of B can be clobbered by a vertex of a given 3-block B. Hence, we can laways clobber at least $\lfloor \frac{n}{2} \rfloor$ vertices of $\mathcal{P}_n$, which means $\operatorname{maxrv} \mathcal{P}_n \leq n - \lfloor \frac{n}{2} \rfloor = \lceil \frac{n}{2} \rceil$.
Note that the previous theorem does not apply for the case of cycles, because any cutset has at least as many vertices as the number of remaining components.

References

[1] Erik D. Demaine, Martin L. Demaine, and Rudolf Fleischer *Solitaire Clobber*. Theor. Comput. Sci. 313 (2004), 325-338.

[2] Michael H. Albert, J. P. Grossman, Richard J. Nowakowski, and David Wolfe *An introduction to Clobber*. Integers 5 (2005)

[3] L. Beaudou, E. Duchêne, L. Faria and S. Gravier *Solitaire Clobber 2 played on graphs*. Submitted

[4] Vincent D. Blondel, C. de Kerchove, Julien M. Hendrickx, and R. Jungers *Linear time algorithms for Clobber*. Submitted.

[5] P.Dorbec, E. Duchêne, and S. Gravier *Solitaire Clobber 2 played on Hamming graphs*. Submitted

[6] M. Ruszinkó, private communication.

[7] Ivars Peterson *Getting Clobbered*.
Science News 161 (2002), http://www.sciencenews.org/articles/20020427/mathtrek.asp