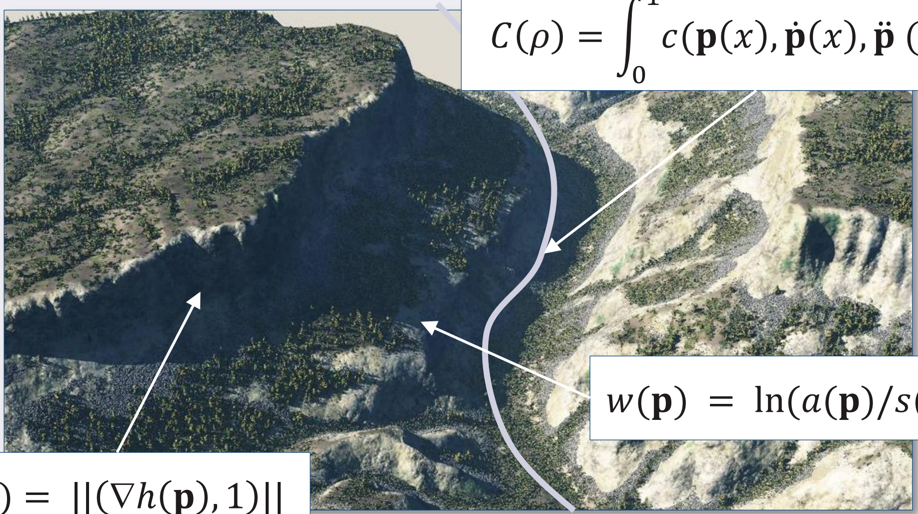
Digital World Modeling

From mathematics …

$$C(\rho) = \int_0^1 c(\mathbf{p}(x), \dot{\mathbf{p}}(x), \ddot{\mathbf{p}}\,(x))dx$$



$$w(\mathbf{p}) \;=\; \ln(a(\mathbf{p})/s(\mathbf{p}))$$

$$s(\mathbf{p}) = \;||(\nabla h(\mathbf{p}), 1)||$$

… to the screen

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Digital World Modeling

Terrain descriptors

Procedural Modeling

Erosion Simulation

Procedural Road Generation

Vegetation and Ecosystems

Growth models

Aging and weathering

Local descriptors

Heightfield

A continous function $z = h(\mathbf{q})$, $\mathbf{q} \in \mathbf{R}$^2
(possibly from elevations z_{ij} on a grid)

Gradient and Slope

Local descriptors

Hydrology

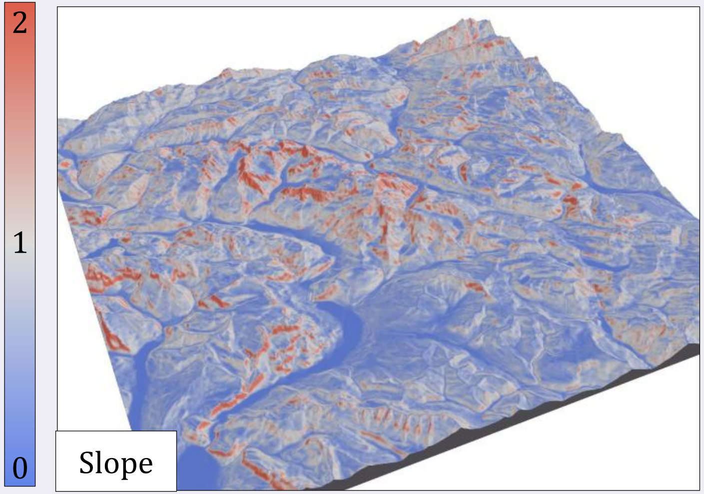
Mathematical definition

We consider $h: \Omega \subset \mathbf{R}^2 \to \mathbf{R}$

The slope and normal at $\mathbf{p}$ are defined as

$$s(\mathbf{p}) = \|\nabla h(\mathbf{p})\|$$

$$n(\mathbf{p}) = \frac{(-\nabla h(\mathbf{p}), 1)}{\sqrt{1 + \nabla h(\mathbf{p})^2}}$$



Average slope

The directional slope is the slope between two points

The average slope $\bar{s}$ is the average of directional slopes in many directions

$$s(\mathbf{p}, \mathbf{q}) = \frac{|h(\mathbf{p}) - h(\mathbf{q})|}{\|\mathbf{p} - \mathbf{q}\|}$$

$$\bar{s}(\mathbf{p}) = \frac{1}{n} \sum_{i=0}^{n-1} s(\mathbf{p}, \mathbf{p} + \varepsilon \mathbf{d}_i)$$

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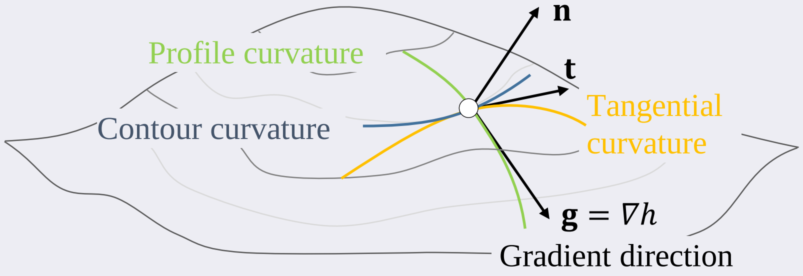
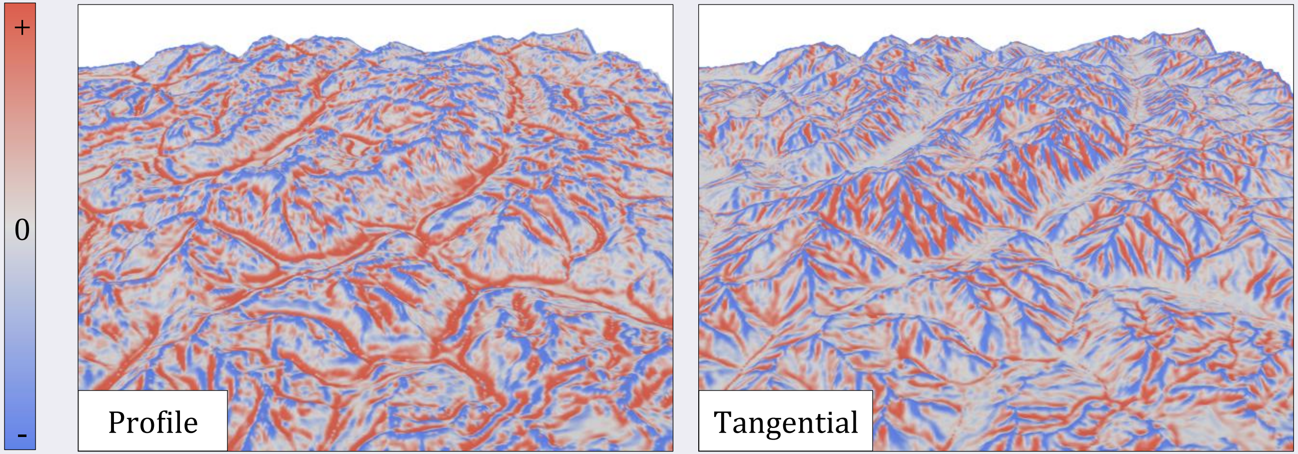
Bare terrain
Slope often used as a proxy for tree distribution

Definition

Several curvatures may be defined, depending on the direction
Curvature can be used to identify ridges and bottoms of valleys

$$\kappa(\mathbf{p}) = \frac{h_{xx}h_y^2 \pm h_{xy}h_x h_y + h_{yy}h_x^2}{(h_x^2 + h_y^2)(1 + h_x^2 + h_y^2)^{3/2}}$$





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Curvature

Curvature

For a general elevation function $z = h(x, y)$

Partial derivatives

$$\kappa = \frac{h_{xx}h_{yy} - h_{xy}^2}{\left(1 + h_x^2 + h_y^2\right)^2}$$

Gaussian curvature

$$\mu = \frac{\left(1 + h_y^2\right)h_{xx} - 2h_x h_y h_{xy} + \left(1 + h_x^2\right)h_{yy}}{2\left(1 + h_x^2 + h_y^2\right)^{3/2}}$$

Mean curvature

Profile and planform curvature

Profile curvature κ_{pro} is in the direction of the steepest slope ∇h

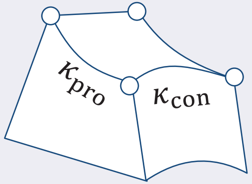
Planform or contour curvature κ_{con} is orthogonal to ∇h

$$\kappa_{\text{pro}} = -\frac{h_{xx}h_x^2 + 2h_{xy}h_x h_y + h_{yy}h_y^2}{\left(h_x^2 + h_y^2\right)\left(1 + h_x^2 + h_y^2\right)^{3/2}}$$

Rate of change of slope

$$\kappa_{\text{con}} = -\frac{h_{xx}h_y^2 - 2h_{xy}h_x h_y + h_{yy}h_x^2}{\left(h_x^2 + h_y^2\right)^{3/2}}$$

Divergence of a flow on the surface



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Background

Local descriptors

Hydrology

Profile and planform curvature

Formulas can be derived from the general equations $z = h(x, y)$

Recall that partial derivatives can be approximated as

$$h_x \approx \frac{h_{i+1,j} - h_{i-1,j}}{2\varepsilon}$$

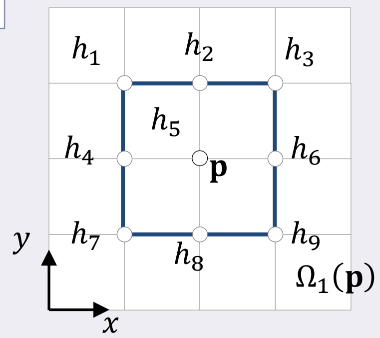
$$h_{xx} \approx \frac{h_{i+1,j} + h_{i-1,j} - 2h_{ij}}{\varepsilon^2}$$

$$h_{xy} \approx \frac{h_{i+1,j+1} + h_{i-1,j-1} - h_{i+1,j-1} - h_{i-1,j+1}}{4\varepsilon^2}$$

Surface S can be approximated [Evans 1980] by a bi-variate quadratic surface Q

$$z = a_{20}x^2 + a_{02}y^2 + a_{11}xy + a_{10}x + a_{01}y + a_{00}$$

$$\begin{aligned}
a_{20} &= \big(h_1 + h_3 + h_4 + h_6 + h_7 + h_9 - 2(h_2 + h_5 + h_8)\big)/6\delta^2 \\
a_{02} &= \big(h_1 + h_2 + h_3 + h_7 + h_8 + h_9 - 2(h_4 + h_5 + h_6)\big)/6\delta^2 \\
a_{11} &= (h_3 + h_7 - h_1 - h_9)/4\delta^2 \\
a_{10} &= (h_3 + h_6 + h_9 - h_1 - h_4 - h_7)/6\delta \\
a_{01} &= (h_1 + h_2 + h_3 - h_7 - h_8 - h_9)/6\delta \\
a_{00} &= (2(h_2 + h_4 + h_6 + h_8) - (h_1 + h_3 + h_7 + h_9) + 5h_5)/9
\end{aligned}$$



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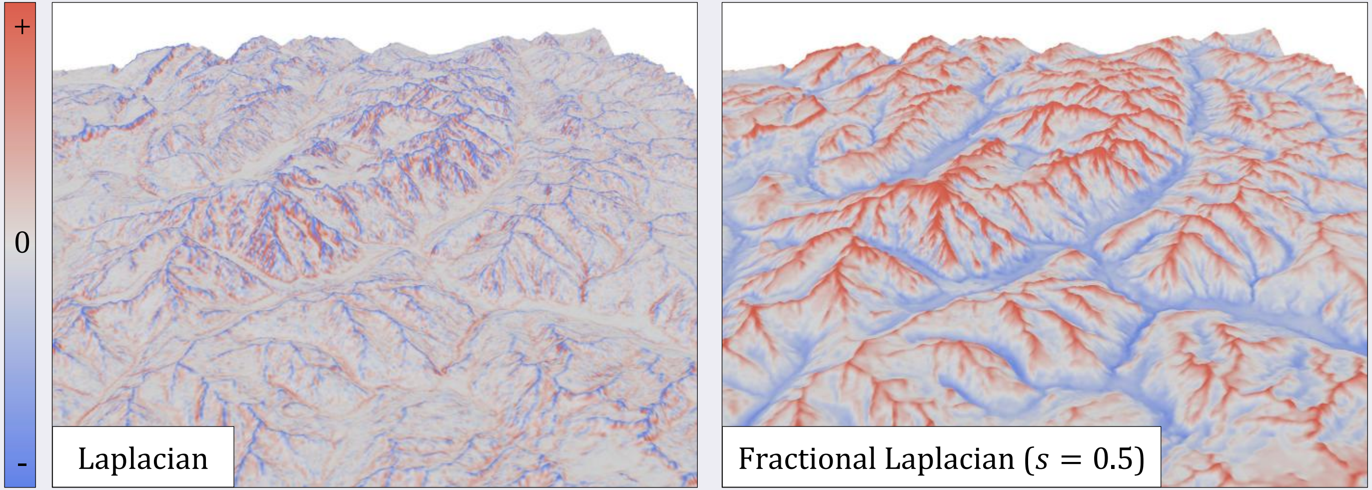
Laplacian and Fractional Laplacian

Mathematical definition

Several definitions exist, we used the well-defined integral on $\mathbf{R}^2$
Implementation as a shader is straightforward

$$\Delta h = \nabla^2 h = \frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2}$$

$$(-\Delta)^s h(\mathbf{p}) = -\frac{1}{2} C(2,s) \int_{\mathbf{R}^2} \frac{h(\mathbf{p}+\mathbf{y}) - 2h(\mathbf{p}) + h(\mathbf{p}-\mathbf{y})}{\|\mathbf{y}\|^{2+2s}} \mathrm{d}\mathbf{y}$$



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Local descriptors

Hydrology

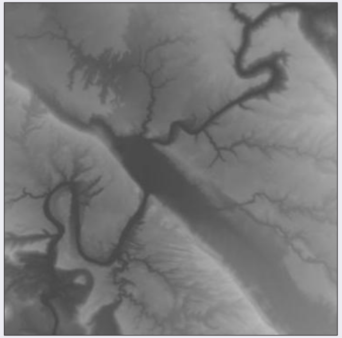
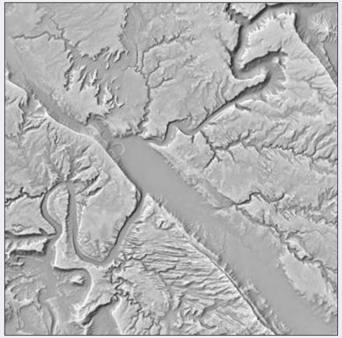
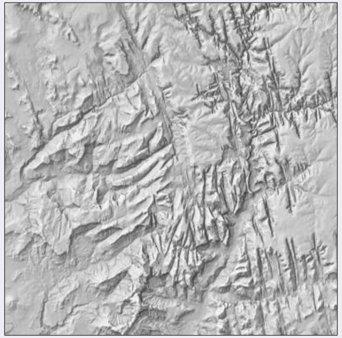
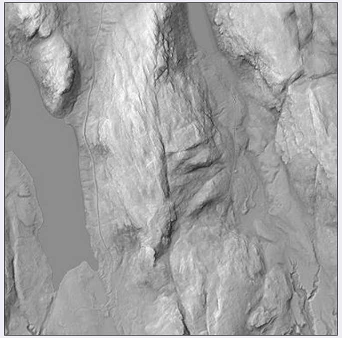
Cartography

Generate maps that convey information for analysis, visualization, interpretation [Kenelly2014, Brown2014]
Comparison of geological features

Drainage area — Fractional Laplacian — Convolution for smoothing at different resolutions

$$c \approx t(h, \nabla h, A)\, s\big(h, \nabla h, \Delta h, \Delta^s h, \alpha(h)\big) * k$$

Tint, mostly elevation gradient and wetness — Shade accounting for illumination

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J.P. Kennelly & A.J. Stewart. General sky models for illuminating terrains. International Journal of Geographical Information Science **28**, 2 (2014).

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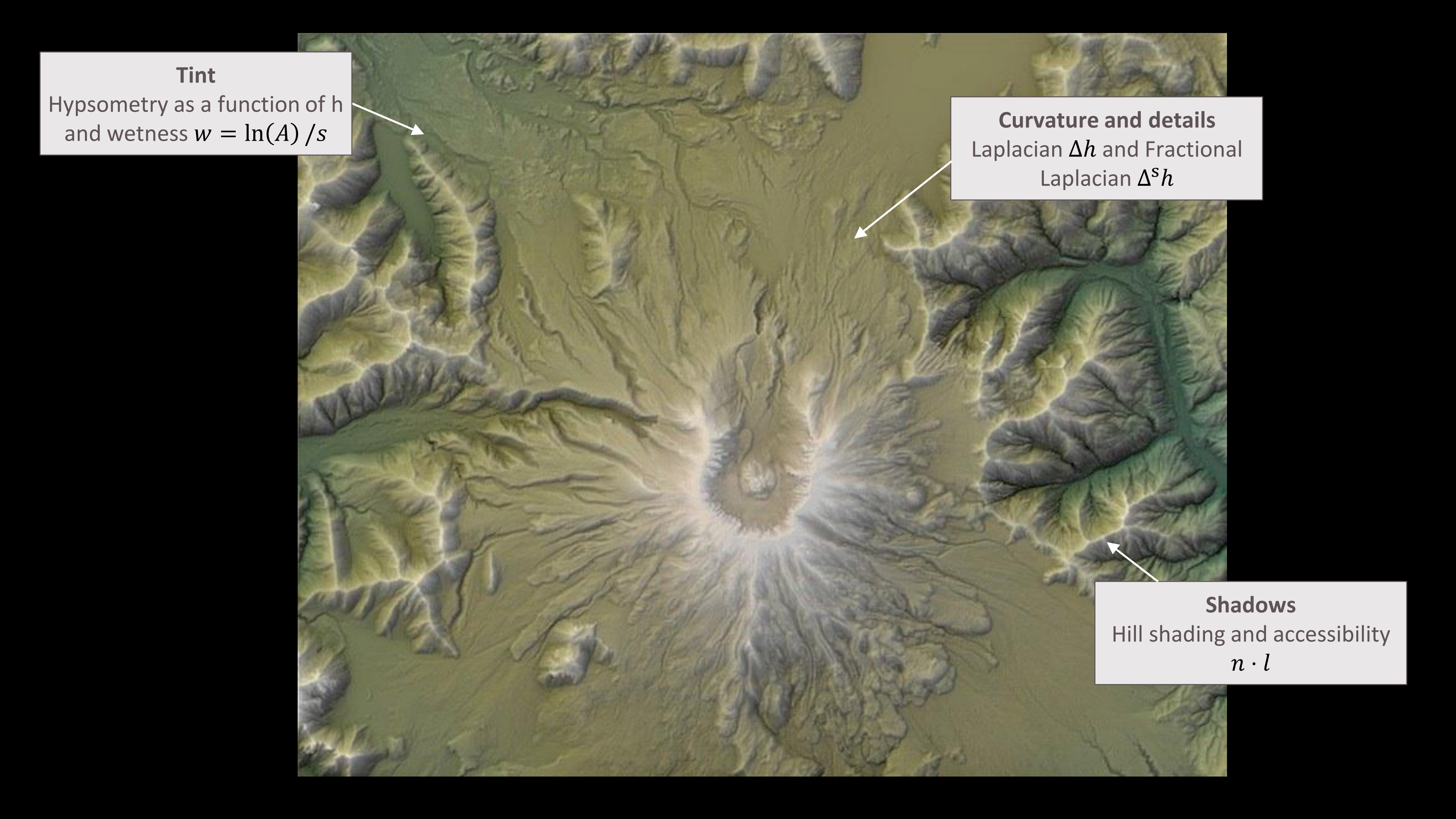
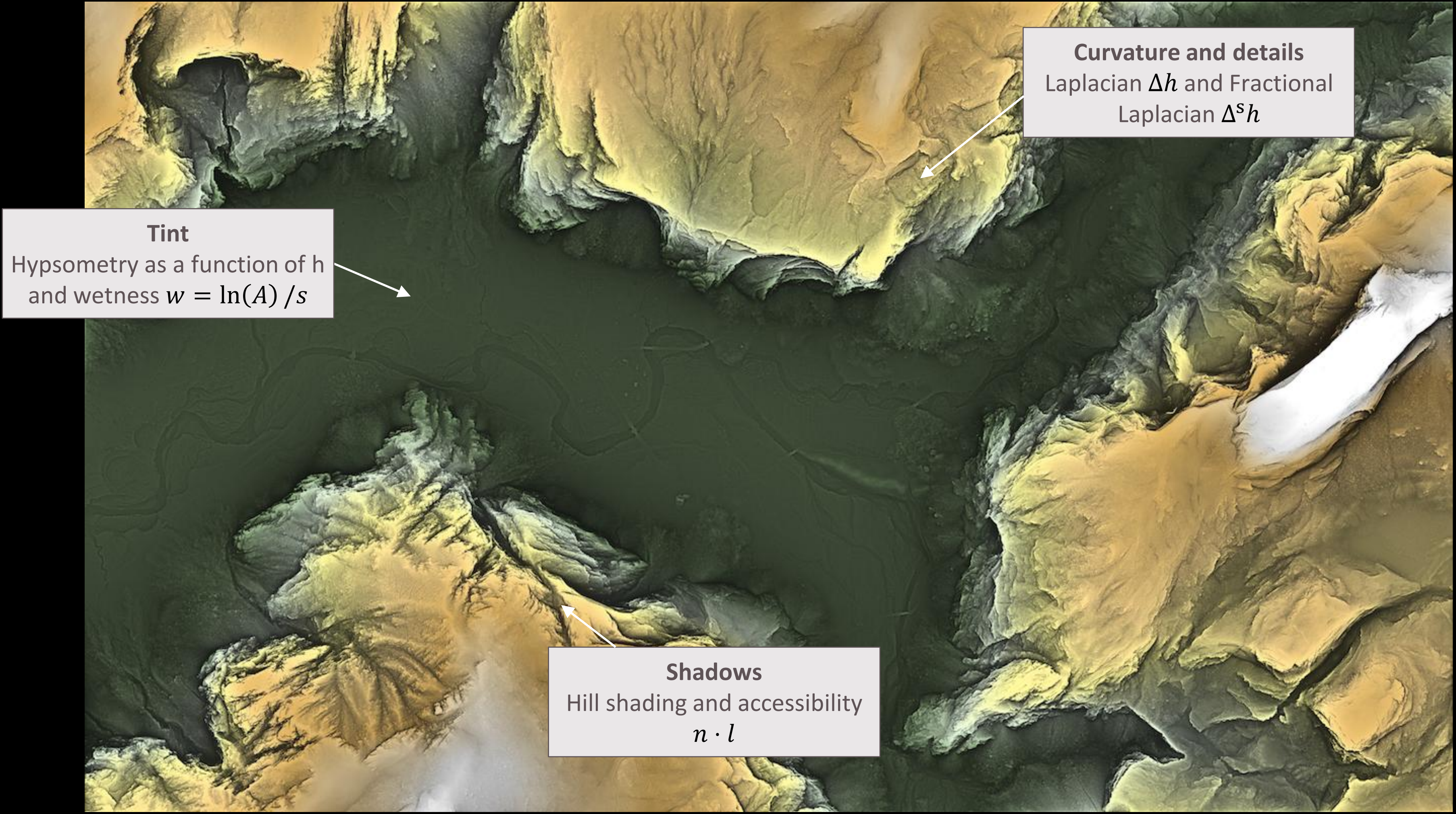


Tint
Hypsometry as a function of h
and wetness $w = \ln(A)/s$
Curvature and details
Laplacian Δh and Fractional
Laplacian $\Delta^s h$
Shadows
Hill shading and accessibility
$n \cdot l$


Curvature and details
Laplacian Δh and Fractional Laplacian $\Delta^s h$
Tint
Hypsometry as a function of h and wetness $w = \ln(A)\,/s$
Shadows
Hill shading and accessibility
$n \cdot l$

Topographic Position Index

Definition

A local approach to analyze the elevation range inside a square domain Ω

Local relief $\longrightarrow$ $$\Delta h_\Omega(\mathbf{p}) = \max_{\mathbf{q}\in\Omega} h(\mathbf{q}) - \min_{\mathbf{q}\in\Omega} h(\mathbf{q})$$

The topographic position index t is defined as the difference between the elevation and mean

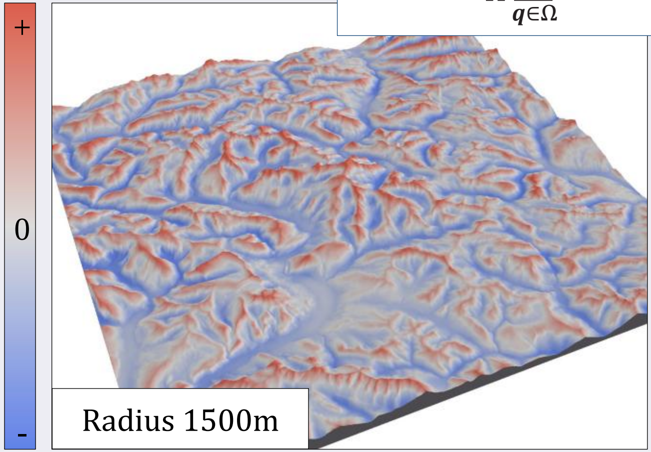
Its normalized variant in $[-1,1]$ is $\hat{t}$ is obtained by using the standard deviation

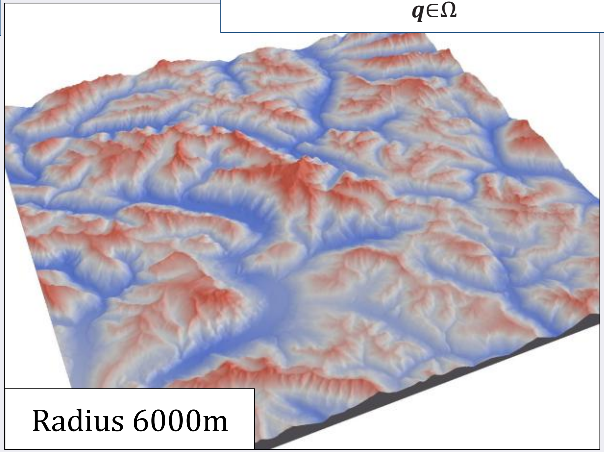
$$t(\mathbf{p}) = h(\mathbf{p}) - \bar{h}(\mathbf{p})$$

$$\hat{t}(\mathbf{p}) = t(\mathbf{p})/\sigma_z(\mathbf{p})$$

$$\bar{h}(\mathbf{p}) = \frac{1}{\Omega}\sum_{\mathbf{q}\in\Omega} h(\mathbf{q})$$

$$\sigma_z^2(\mathbf{p}) = \frac{1}{\Omega^2}\sum_{\mathbf{q}\in\Omega}\left(h(\mathbf{q}) - \bar{h}(\mathbf{p})\right)^2$$





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Terrain Ruggedness Index

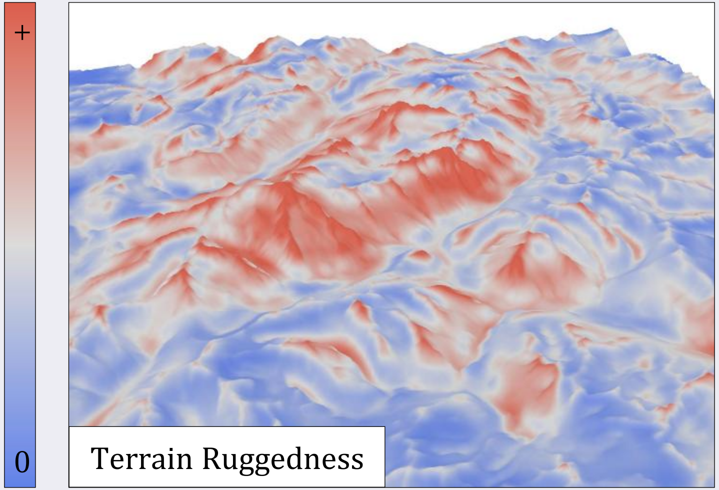
Local descriptors

Hydrology

Definition

Similar to standard deviation, but with $h(\mathbf{p})$ instead of $\bar{h}(\mathbf{p})$

$$r(\mathbf{p}) = \left(\sum_{q\in\Omega} \left(h(\mathbf{q}) - h(\mathbf{p})\right)^2\right)^{1/2}$$



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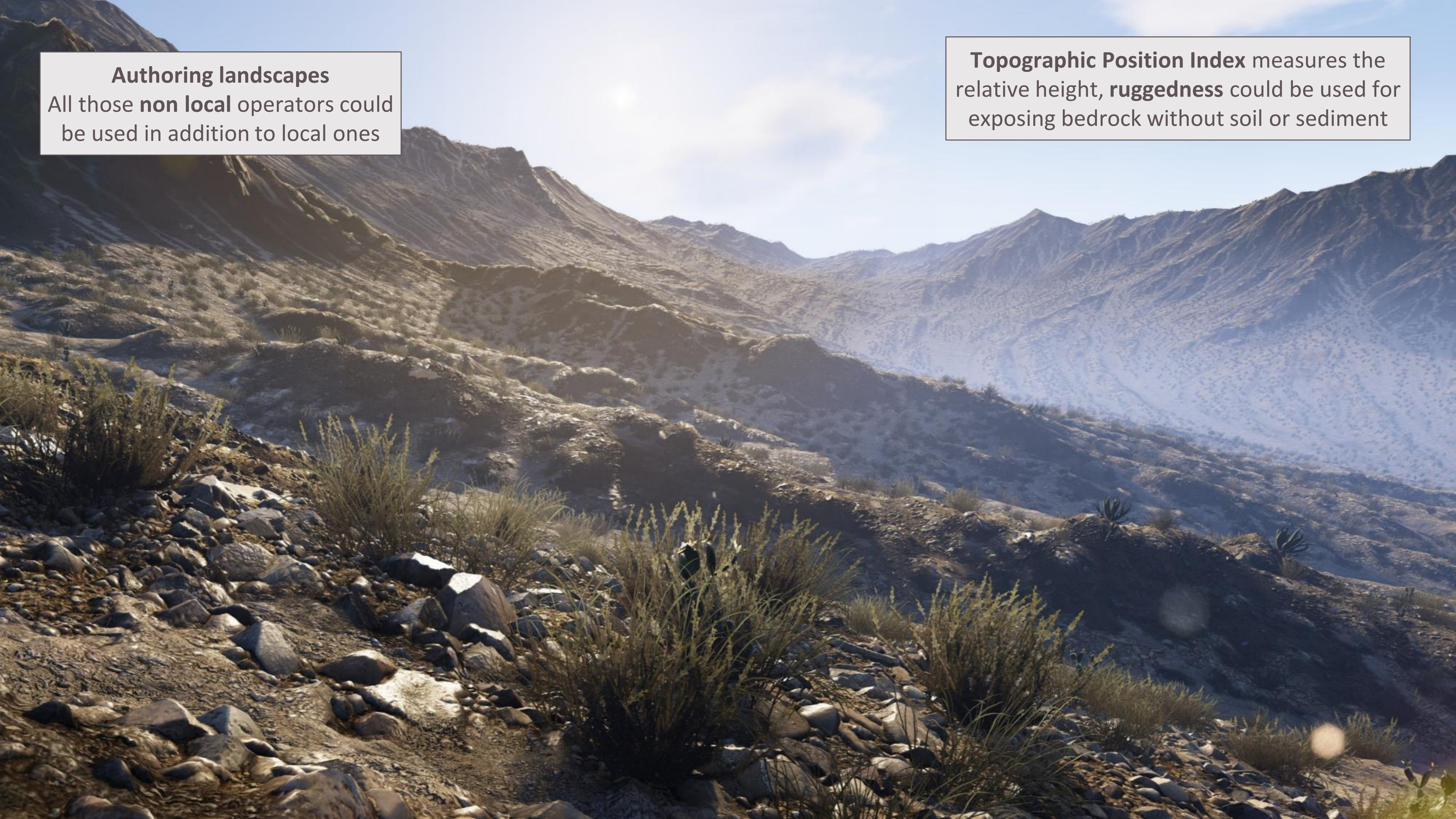
Authoring landscapes
All those **non local** operators could be used in addition to local ones

Topographic Position Index measures the relative height, **ruggedness** could be used for exposing bedrock without soil or sediment



Hydrology

Flow routing models

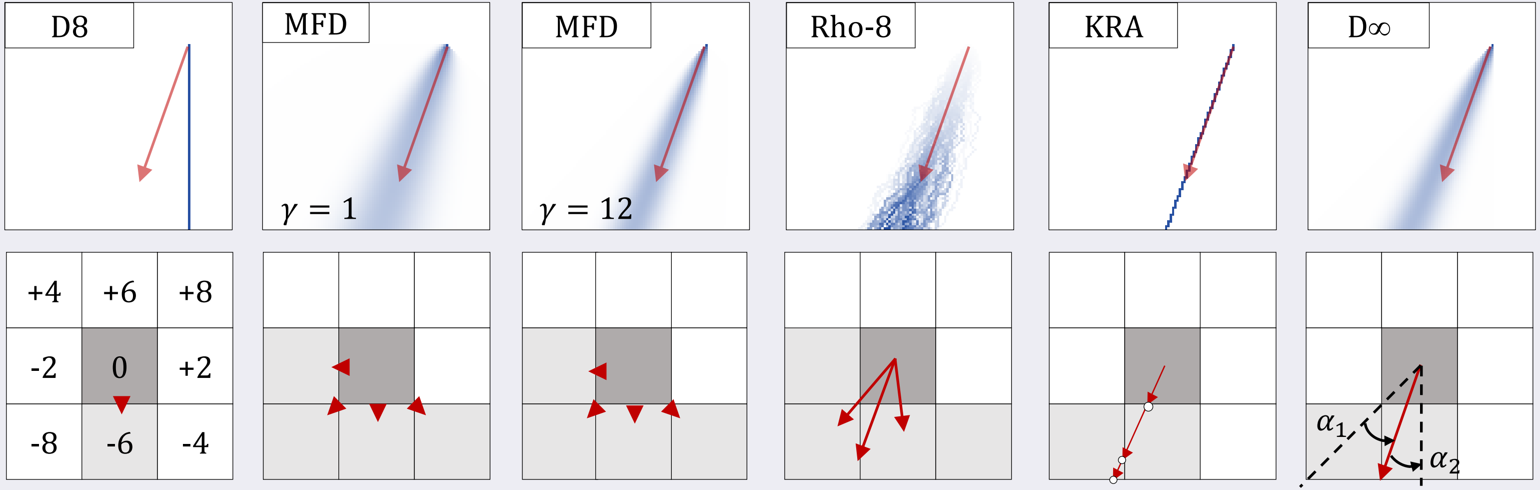
Local descriptors

Hydrology

Goal

Analyze the movement of water and the properties of the flow

Routing models describe how water flows from $\mathbf{p}$ to neighboring positions



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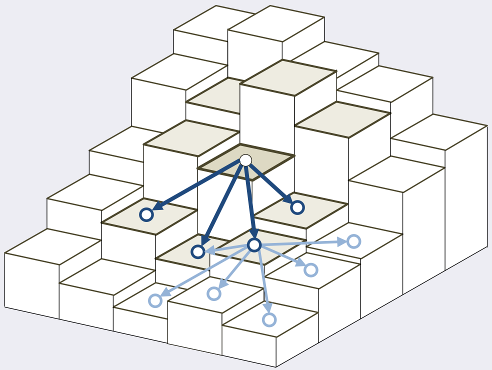
The multiple flow direction model

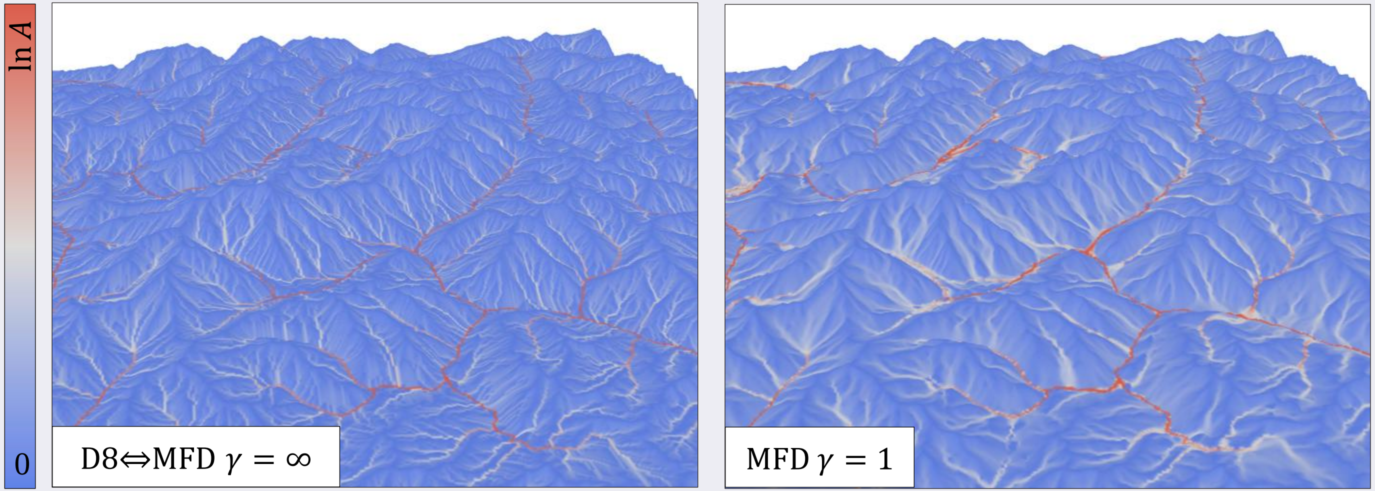
Goal

A divergent flow algorithm distribute flow to more than one cell
When $\gamma \to \infty$, it is equivalent to D8
Prescribed γ values are in $[4,6]$ [Holmgren1994]



$$f_i = \frac{s(\mathbf{p},\mathbf{q})^\gamma}{\sum_{j \in L(\mathbf{p})} s(\mathbf{p},\mathbf{q}_j)^\gamma}$$

$\gamma \in [4,6]$



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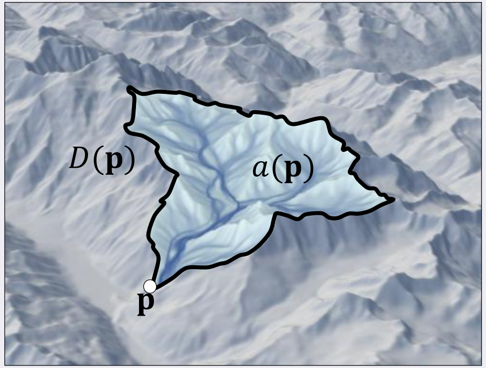
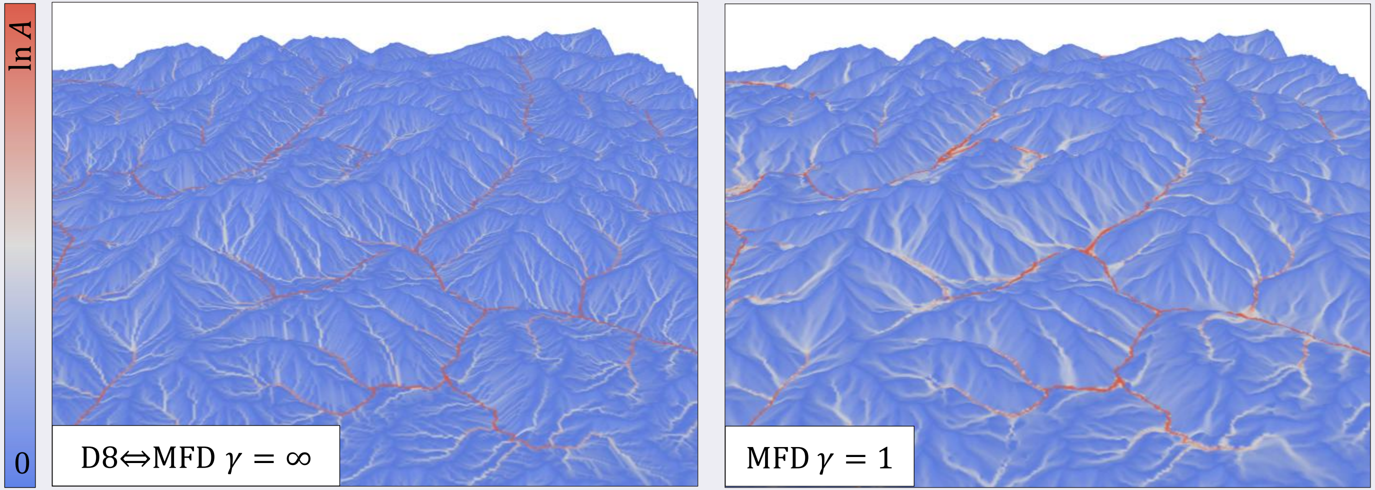
Drainage basin and drainage area

Drainage basin

It is the region $\mathrm{D}(\mathbf{p})$ of from where the surface flow can reach $\mathbf{p}$
The upslope drainage area $a(\mathbf{p})$ is the area of $\mathrm{D}(\mathbf{p})$





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Drainage basin metrics

Wetness Index

Also called topographic wetness index, it is proxy to soil moisture

$$w(\mathbf{p}) = \ln\left(a(\mathbf{p})/s(\mathbf{p})\right)$$

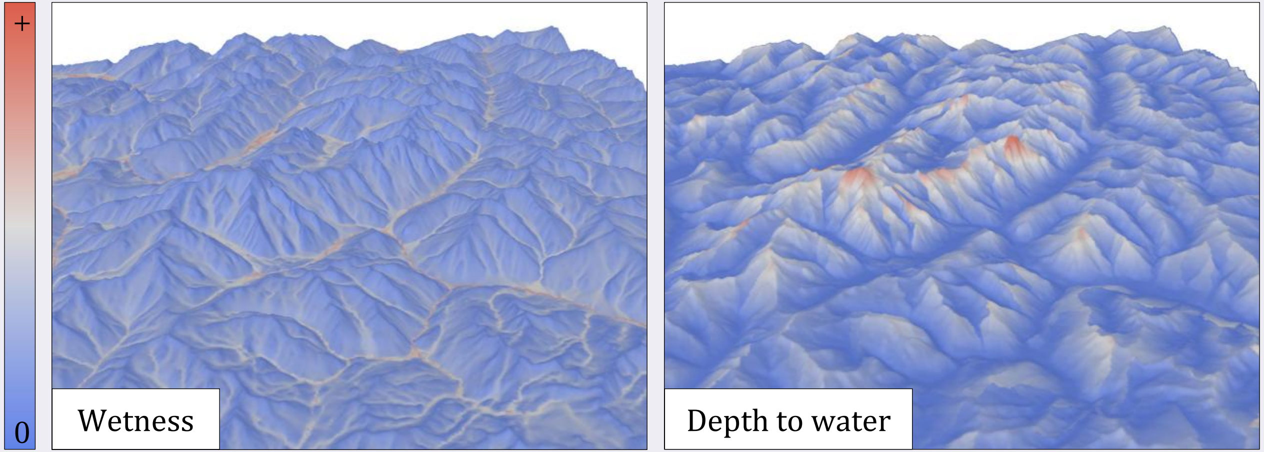
Depth to water index

Accounts for the distance and accumulated slope towards a water location

Given a path $\{\mathbf{q}_0 = \mathbf{p}, \mathbf{q}_1, \dots, \mathbf{q}_n = \mathbf{p}_w\}$ of least cumulative slope towards a water location $\mathbf{p}_w$ we have

$$d_R(\mathbf{p}) = \sum_{i=1}^{n} s(\mathbf{q}_i) \| \mathbf{q}_i - \mathbf{q}_{i-1} \| \rho(\mathbf{q}_i)$$

Coefficient of permeability



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Drainage patterns
Snow accumulation in
exaggerated gullies

Erosional indices

Stream power

The stream power equation describes the general evolution of landscape height

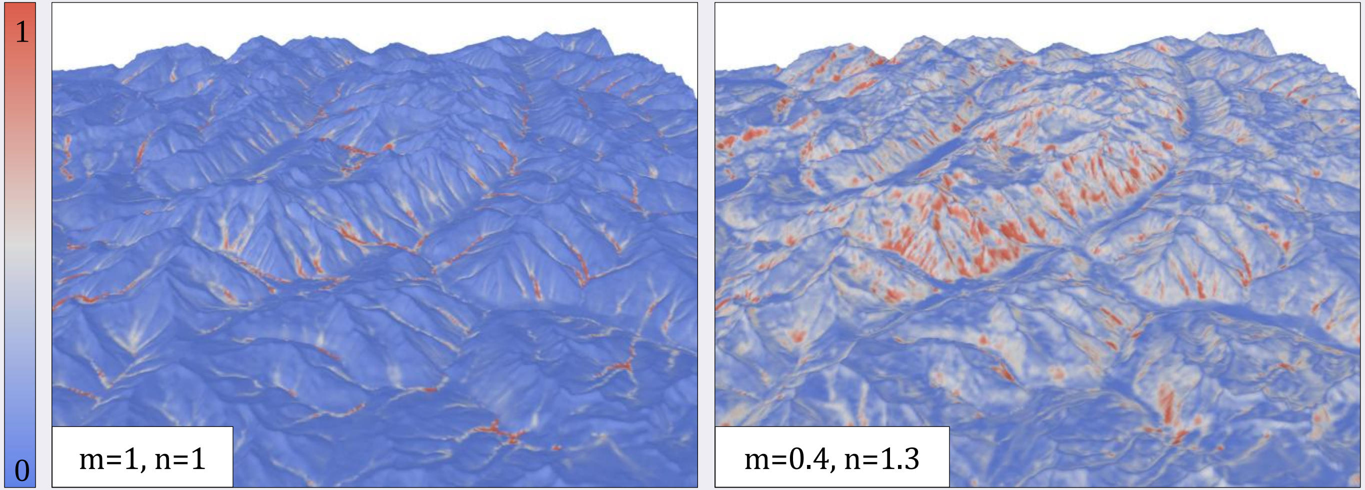
Stream power index → $\mathrm{e}(\mathbf{p}) = a^n(\mathbf{p})s^m(\mathbf{p})$

$$\frac{\partial h(\mathbf{p})}{\partial t} = u - a^n(\mathbf{p})s^m(\mathbf{p}) + \Delta h(\mathbf{p})$$

Hillslope erosion

Drainage area

Slope



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High erosion areas
Prevent the development of vegetation, adapted shading
High wetness area
Vegetation develops at the bottom of valleys