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AN INTRODUCTION TO AN ANALYTICAL WAY TO COMPUTE THE VOLUME OF BLOBS

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Abstract: Using implicit surfaces is of interest in the field of interactive modelling. A first interest lies in automatic changes of topology. Volumic representation of objects is another interest. However, modelling non-compressible material with implicit surfaces remains difficult. The volume of implicit objects changes with the position of their centres. We can compute volume using numerical method, but these kinds of methods bring a quick but inaccurate result. In this paper, we propose an analytical method to obtain exact results. We present this method for two blobs. Afterward, we prove we can extend this method to several blobs.

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1. Introduction

These days, we are using computer graphics in several ways, e.g. mechanical tools building, physical phenomena simulations, or virtual ob-

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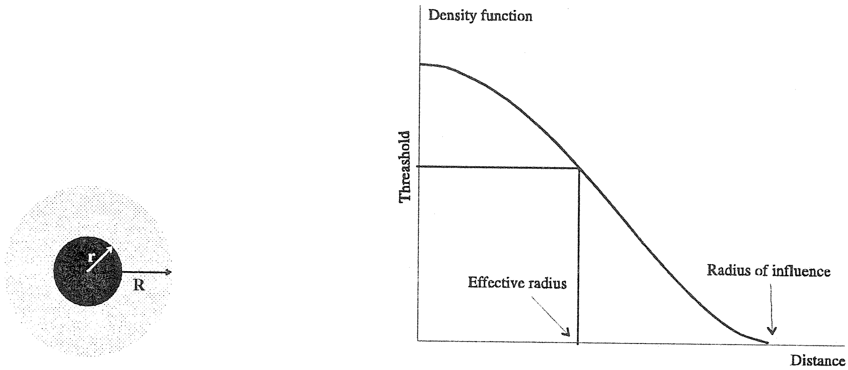


Figure 1: A blob

function F is the sum of n functions

$$F(M) = \sum_{i=1}^{n} F_i(M) .$$

A function $F : R^3 \to R$ and the threshold $T > 0$ define an equipotential surface S satisfying:

$$S = \{M \in R^3 / F(M) = T\} .$$

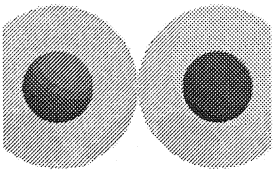
The surface S encloses the equipotential volume V that we now define by:

$$V = \{M \in R^3 / F(M) \geq T\} ,$$

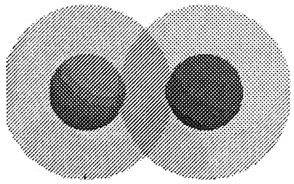
T regulates the size of the object composed of several blobs. With each blob B_i we associate a skeleton[2] Sk_i, a radius of influence R_i and an equipotential function $F : R^3 \to R$ monotonically decreasing with the distance to Sk_i. A skeleton may be a point, a segment, a curve or a surface. The equipotential function F_i is traditionally defined as the composition of two functions - the potential function $f_i : R^+ \to R$ and the distance function $d_i : R^3 \to R^+$ normalized by the radius of influence R_i

$$F_i(M) = f_i\left(\frac{d_i(M)}{R_i}\right) .$$

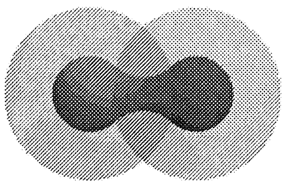
[2]A skeleton $Sk(X)$ of an object X is the set containing the centres of maximal balls.



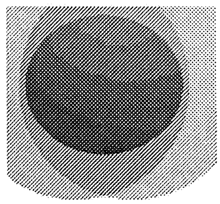
(a) The two blobs are merging. There is no influence



(b) Two separated parts with a deformation due to a mutual influence



(c) Only one object appears



(d) Only one object visually appears without any bumps

Figure 2: Fusion of two blobs

We will study these four different cases in this paper. We will use the items listed above to recognize the case we are in.

3. Volume of Two Blobs

Simply let us assume, volume computation of blobs is just an integral computation whose bounds are unknown. Extracting bounds requires first studying the simple case, where two blobs are on one of the axes of the reference triad. If the two blobs are localized everywhere in space, we simply change the triad using rotation and translation to return to the reference triad.

In the following, we will briefly study volume computation of a sphere, then volume of two blobs on an axis, and lastly we will extend the method to a generalized case.

The volume of the implicit object is computed in three steps:
- Evaluating radii variation as a function of z (with function $h(z)$),
- Evaluating values on the axis for which $g(z) = 0$,
- Integration of the function found in the first step over the range computed in the previous step.

3.3. Computation of Radius

Let T be a common threshold between B_1 and B_2. Functions of density for blobs B_1 and B_2 are $f\left(\frac{r_1(z)}{R_1}\right)$ and $f\left(\frac{r_2(z)}{R_2}\right)$ in which r_1 and r_2 are computed with Pythagoras Theorem (see Figure 3). That is, $r_1(z)^2 = r^2(z) + (z - z_1)^2$ and $r_2(z)^2 = r^2(z) + (z - z_2)^2$. The distance $r(z)$ from a point of the blob to its projection on the z-axis depends on the localisation of the blob centres. Thus the equation describing an implicit surface is:

$$f\left(\frac{r_1(z)}{R_1}\right) + f\left(\frac{r_2(z)}{R_2}\right) = f\left(\frac{\left(r^2(z) + (z - z_1)^2\right)^{1/2}}{R_1}\right) + f\left(\frac{\left(r^2(z) + (z - z_2)^2\right)^{1/2}}{R_2}\right) = T\,.$$

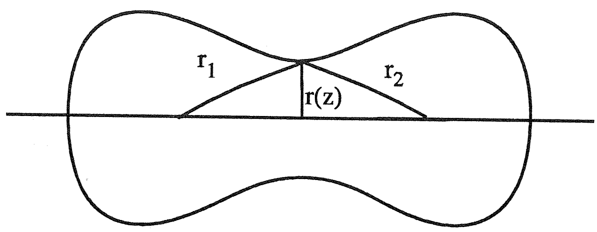


Figure 3: r_1 and r_2 are computed with Pythagoras Theorem

Let $h(z) = f\left(\frac{r_1(z)}{R_1}\right) + f\left(\frac{r_2(z)}{R_2}\right) - T$. We write the surface of the object as a function of radii and displacement along the z-axis. $h(z) = 0$ represents the radius variation as a function of displacement. The intersection of the surface with the axis passing through the centres (here z-axis) is the locus of the point for which $r(z) = 0$. If $r(z)$ is 0 in

To identify the case we are in, let $h(z) = f\left(\frac{r_1(z)}{R_1}\right) + f\left(\frac{r_2(z)}{R_2}\right) - T$. This function gives variations of radii between the two blobs. With this, we can isolate the different cases enumerated before. Thanks to $g(z) = 0$ we find intersections of implicit object with centres-axis. Let α_1 and α_2 be the two real roots of $g(z) = 0$ if they exist. If $h\left(\frac{r_1(z)+r_2(z)}{2}\right) > 0$ then the object is ovoid and mutual influence is complete. In the other case, there are two separate, influenced, implicit components. If roots of $g(z) = 0$ are complex (not real) then $h(z)$ does not intersect centres-axis. Intersections α_1 and α_2 of implicit object with centres-axis are then found thanks to $g_1(z) = 0$ and $g_2(z) = 0$.

If $h\left(\frac{r_1(z)+r_2(z)}{2}\right) > 0$ then radius of object is given by $h_1(z) = 0$, $h(z) = 0$ and $h_2(z) = 0$. In the other case, there is no influence between blobs.

Case 1. The two blobs are not influenced. Roots are then computed independently for B_1 and B_2. We must solve $g_1(z) = 0$ and $g_2(z) = 0$. We have two spherical objects. Therefore, volume is just the sum of volumes of two blobs considered as spheres.

Case 2. (see Figure 5) The two blobs are in mutual influence. Solving $g(z) = 0$ gives two roots and $g_1(z) = 0$ and $g_2(z) = 0$ give two other roots. The object is divided in four parts by the intersection of $h(z)$ with at first $h_1(z)$. Then $h_2(z)$ gives α_1 and α_2 as seen previously, and $g(z) = 0$ gives α_5 and α_6. These points correspond to the disappearance and appearance of the object. $g_1(z) = 0$ gives α_3, and $g_2(z) = 0$ gives α_4. It follows:

$$V = \int_{\alpha_3}^{\alpha_1} \pi\left(r_1^2(z)\right) dz + \int_{\alpha_1}^{\alpha_5} \pi\left(r^2(z)\right) dz + \int_{\alpha_6}^{\alpha_2} \pi\left(r^2(z)\right) dz + \int_{\alpha_2}^{\alpha_4} \pi\left(r_2^2(z)\right) dz \,.$$

Case 3. The two blobs are in fusion. Just one object appears. However, neither blob totally influences the other. Two roots are computed from $g_1(z) = 0$ and $g_2(z) = 0$. The other roots are found with $g_1(z) = 0$, $h_1(z) = h(z)$ and $h_2(z) = h(z)$.

The radius function is in three pieces. It is necessary to know where each piece is changing. That means where to compute radius of the first blob, where to compute radius of the sum, and where to compute radius

Extracting roots is equivalent to considering $r(z) = 0$ in $h(z)$. Thus $g(z) = f\left(\frac{z-z_1}{R_1}\right) + f\left(\frac{z-z_2}{R_2}\right) - T$. At first, we must solve $g(z) = 0$ (for instance, for Murakami function we have to solve a 4-th degree polynomial). We obtain two real roots r_1 and r_2. Volume is then computed using displacement of a disk whose radius is $r(z)$ and integration over the interval (α_1, α_2). The disk displacement $r(z)$ is found by solving $h(z) = 0$. Therefore $V = \int_{\alpha_1}^{\alpha_2} \pi r^2(z) dz$.

4. Generalisation to n Blobs

4.1. Minimizing a System of Equations

Before generalising for n blobs, we must study minimization of a system of equations.

Let f be a function of density defining blobs B_1 and B_2. Functions of density may be written

$$\begin{cases} f_1(x,y,z) = f\left(\dfrac{x-x_1}{R_1}, \dfrac{y-y_1}{R_1}, \dfrac{z-z_1}{R_1}\right), \\ f_2(x,y,z) = f\left(\dfrac{x-x_2}{R_2}, \dfrac{y-y_2}{R_2}, \dfrac{z-z_2}{R_2}\right). \end{cases}$$

Finding the isosurface is equivalent to finding the blob boundary hence

$$\{(x,y,z) \in B_1 \cup B_2 / f_1(x,y,z) + f_2(x,y,z) = T\}.$$

Let $f_2(x,y,z) = p$, $p \in]0,T[$. We are searching for each (x,y,z) verifying the last equality[4]. Finding the isosurface and solving the following parametric system are equivalent:

$$Sp_1 = \begin{cases} f_1(x,y,z) = T - p, \quad p \in]0,T[\,, \\ f_2(x,y,z) = p. \end{cases}$$

The problem is now a maxima-minima problem. Extremes may be obtained by squaring members of the system. We obtain

$$Sp_2 = \begin{cases} \min(f_1(x,y,z) - T + p)^2, \\ \min(f_2(x,y,z) - p)^2. \end{cases}$$

[4]Cases with $p = 0$ and $p = T$ are trivial.

the inequalities[6]

$$\left(\frac{x-x_1}{R_1}\right)^2+\left(\frac{y-y_1}{R_1}\right)^2+\left(\frac{z-z_1}{R_1}\right)^2<1\,, \quad \text{and}$$
$$\left(\frac{x-x_2}{R_2}\right)^2+\left(\frac{y-y_2}{R_2}\right)^2+\left(\frac{z-z_2}{R_2}\right)^2<1\,,$$

$$Sp_2 \Leftrightarrow \begin{cases} Fx * Gy - Fy * Gx = 0\,, & (\mathrm{eq}_1)\\ Fx * Gz - Fz * Gx = 0\,, & (\mathrm{eq}_2)\\ Fy * Gz - Fz * Gy = 0\,, & (\mathrm{eq}_3)\\ \left(\frac{x-x_1}{R_1}\right)^2+\left(\frac{y-y_1}{R_1}\right)^2+\left(\frac{z-z_1}{R_1}\right)^2 - m = 0\,, & \\ \qquad m \in [0,1]\,, & (\mathrm{eq}_4)\\ \left(\frac{x-x_2}{R_2}\right)^2+\left(\frac{y-y_2}{R_2}\right)^2+\left(\frac{z-z_2}{R_2}\right)^2 - m = 0\,, & \\ \qquad m \in [0,1]\,. & (\mathrm{eq}_5) \end{cases}$$

Simplifying equations (eq_1), (eq_2), and (eq_3) with $f_1(x,y,z)-T+p$ and $f_2(x,y,z)-p$ gives (eq_{1a}), (eq_{2a}), and (eq_{3a}).

This implies $f_1(x,y,z)\neq 0$ and $f_2(x,y,z)\neq 0$. Roots on border of old blobs are not extracted and must be studied separately. We have to solve a system of 5 simultaneous equations $eq_{1a}=0$, $eq_{2a}=0$, $eq_{3a}=0$, $eq_4=0$, and $eq_5=0$. Roots are functions of z:

$$\text{Solutions} = \begin{cases} x_{sol} = x(z)\,,\\ y_{sol} = y(z)\,,\\ z_{sol} = z(z)\,. \end{cases}$$

We have to compute solutions of $f_1(x_{root}, y_{root}, z_{root}) + f_2(x_{root}, y_{root}, z_{root}) - T = 0$. Roots of this system are not on the border of the object, however they are known because the distance is greater than the sum of effective radius and the influence radius. Studying the density function gives number of solutions and computes roots on the border of the implicit object (as seen previously). These roots

[6]Points of blob are in ball, which origin is centre of blob and which radius is radius of influence. Dividing by Ri, $i=1,2$ gives inequalities above.

This theorem says that generalisation to n blobs is allowed. Indeed, according to this theorem g_i, $i = 1 \cdots m$ have to be free. If workspace is $\mathbb{R}^n$, then m+1 vectors are dependent. Here we are in $\mathbb{R}^3$. We have at most four free constraints. Generalising a solution to n blobs is solving the system in a vectorial normalized space whose dimension is at least $n-1$. This gap to an upper dimension is given in the next section.

Let F be a density function defining a blob. Let T be the threshold of the isosurface.

$$F : R^3 \to R\,,$$
$$(x,y,z) \to F(x,y,z), B = F^{-1}\ (\{\mathrm{T}\})\,.$$

B is a 2-surface object of $\mathbb{R}^3$ in $\mathbb{R}$, F is a submersion of $\mathbb{R}^3$ in $\mathbb{R}$. Now, we try to generalise to p blobs. For p blobs, we have $\tilde{F} : \mathbb{R}^{2+k} \to \mathbb{R}^k$.

Let $\tilde{B}$ be the implicit surface with

$$\tilde{B} = \widetilde{F^{-1}}\ (\{T, 0, 0, ..., 0\})\,.$$

We search $\tilde{F}$ with Rank$(D\tilde{F}) = k$. We choose

$$\tilde{F} : \mathbb{R}^3 \times \mathbb{R}^{k-1} \to \mathbb{R}^k\,,$$
$$(x, y, z, x_4, ..., x_{k+2}) \to (F(x,y,z), x_4, ..., x_{k+2})\,.$$

The application defines the implicit. Let I_{k-1} be the identity application of $\mathbb{R}^{k-1}$. We have $\begin{array}{ll} I : \mathbb{R}^{k-1} & \to \mathbb{R}^{k-1} \\ x & \to x \end{array}$.

The Jacobean of $\tilde{F}$ is $D\tilde{F} = \begin{pmatrix} DF & 0 \\ 0 & I_{k-1} \end{pmatrix}$.

We must check that the rank of the differential of $\tilde{F}$ is k. Since it is not a squared matrix, we cannot compute the determinant. We must extract a non-null determinant whose dimension is k. With this kind of matrix, we can extract a non-null determinant with dimensions $k-1$ (the dimension of the identity matrix). If $\frac{\partial F_3}{\partial x} = 0$, a permutation of one of the three first lines or one of the three first column of $D\tilde{F}$, we can find a non-null term because rank$(D\tilde{F})$=1. So, rank$(D\tilde{F}) = k$. Let:

$$\Pi : \mathbb{R}^3 \times \mathbb{R}^{k-1} \to \mathbb{R}^3\,,$$
$$x, y, z, x_4, ..., x_{k+2} \to (x, y, z)\,.$$

to n blobs, $n \geq 4$. The number of constraints we can solve gives the number 4. If we choose to immerse the implicit surface in an upper space dimension, this number of constraints can increase. We demonstrated it is possible to obtain volume of n blobs with $n-1$ constraints defined with functions of density. Dimension of working space must be at least $n-1$.

We can compute volume of blobs defined with different radii and different functions of density. The function must be anyone but the first derivative must be continuous. However, the use of function in pieces may be more complicated.

In future work, we would like to implement this extension, as it should be a good way to compute volume in an analytical way. We would like to use this method to compute implicit volume for physical measures or to control volume of blobs during fusion, too.

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