Wavejets: A Local Frequency Framework for Shape Details Amplification
Supplementary Material

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This supplementary material gives the mathematical proofs for the various theorems and corollaries.

1 Proof of the wavejets decomposition

Equation 1 of the paper contains terms such as $x^{k-j}y^j$, which can be rewritten as linear combinations of powers of $e^{i\theta}$.

$$\begin{aligned}
x^{k-j}y^j &= r^k \cos^{k-j}\theta \sin^j \theta \\
&= r^k \left(\frac{e^{i\theta}+e^{-i\theta}}{2}\right)^{k-j}\left(\frac{e^{i\theta}-e^{-i\theta}}{2i}\right)^{j} \\
&= \frac{r^k}{2^k i^j}\left(\sum_{l=0}^{k-j}\binom{k-j}{l} e^{(k-j-2l)i\theta}\right)\left(\sum_{l=0}^{j}\binom{j}{l}(-1)^l e^{(j-2l)i\theta}\right) \\
&= \frac{r^k}{2^k i^j}\sum_{l_1=0}^{k-j}\sum_{l_2=0}^{j}(-1)^{l_2}\binom{k-j}{l_1}\binom{j}{l_2} e^{(k-2l_1-2l_2)i\theta} \\
&= \frac{r^k}{2^k i^j}\sum_{l=0}^{k}\sum_{h=0}^{l}(-1)^{h}\binom{k-j}{h}\binom{j}{l-h} e^{(k-2l)i\theta} \\
&= \frac{r^k}{2^k i^j}\sum_{\substack{n=-k\\ n \text{ and } k \text{ have}\\ \text{same parity}}}^{k}\;\sum_{h=0}^{\frac{n-k}{2}}\binom{k-j}{h}\binom{j}{\frac{n-k}{2}-h}(-1)^h e^{in\theta} \\
&= r^k \sum_{n=-k}^{k} b(k,j,n) e^{ni\theta}
\end{aligned} \tag{1}$$

with $b(k,j,n)=0$ if k and n do not have the same parity and $b(k,j,n)=\frac{1}{2^k i^j}\sum_{h=0}^{\frac{n-k}{2}}\binom{k-j}{h}\binom{j}{\frac{n-k}{2}-h}(-1)^h$ otherwise.

Using Equations 2 of the paper we get:

$$\phi_{k,n}=\sum_{j=0}^{k}\frac{b(k,j,n)}{j!(k-j)!}f_{x^{k-j}y^j}(0,0). \tag{2}$$

2 Proof of the stability theorem (theorem 1)

Let us first recall the setting of this theorem. Let us call $\mathcal{T}(p)$ the true tangent plane and $\mathcal{P}(p)$ the chosen parameterization plane, also passing through p. One can find an axis (p,u) and angle γ such that the rotation of axis (p,u) and angle γ transforms $\mathcal{P}(p)$ into $\mathcal{T}(p)$. Since p belongs to both planes, (p,u) is aligned with line $\mathcal{T}(p)\cap\mathcal{P}(p)$. Let us parameterize $\mathcal{T}(p)$ and $\mathcal{P}(p)$ so that a point of the surface has coordinates ($x = r\cos\theta, y = r\sin\theta, h$) over $\mathcal{T}(p)$ and ($x = R\cos\Theta, y = R\sin\Theta, H$) over $\mathcal{P}(p)$. Let us first assume that θ (resp. Θ) corresponds to the angular coordinate of point q with respect an origin vector aligned with u in $\mathcal{T}(p)$ (resp. with u in $\mathcal{P}(p)$). We will state our main theorem in this setting and the generalization will follow naturally. In this setting the surface wavejets decomposition at point q writes $\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\phi_{k,n}r^k e^{\boldsymbol{i}n\theta}$ over the tangent plane $\mathcal{T}(p)$ and as $\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\Phi_{k,n}r^k e^{\boldsymbol{i}n\Theta}$ over $\mathcal{P}(p)$. We can express the $\Phi_{k,n}$ coefficients with respect to $\phi_{k,n}$ and the rotation angle γ. To generalize the theorem to arbitrary origin vectors for the angular coordinate in $\mathcal{T}(p)$ and $\mathcal{P}(p)$ for θ and Θ, recall that a change of reference vector in $\mathcal{T}(p)$ amongs to a phase shift μ, one can always change the origin vector, compute the wavejets coefficients $\phi_{k,n}$ and recover the real wavejets coefficients as $\phi_{k,n}e^{\boldsymbol{i}n\mu}$ (similar formulas hold for $\Phi_{k,n}$).

Theorem 1. *The new coefficients $\Phi_{k,n}$ can be expressed with respect to the $\phi_{k,n}$ as follows:*

$$\begin{aligned}\Phi_{0,0} &= 0\\ \Phi_{1,1} &= \Phi^*_{1,-1} = \frac{\gamma}{2}e^{-\boldsymbol{i}\frac{\pi}{2}} + o(\gamma)\\ \Phi_{k,n} &= \phi_{k,n} + \gamma F(k,n) + o(\gamma)\end{aligned} \tag{3}$$

Proof. The rotation matrix $\mathbf{R}$ of axis $\boldsymbol{u} = (1,0,0)_{\mathcal{P}}$ and angle γ transforms the coordinates (X,Y,H) of a surface point p in the parameterization of $\mathcal{P}(p)$ into coordinates (x,y,h) in the parameterization of $\mathcal{P}(p)$. Let us assume that γ^2 is small enough. Then the rotation has the following expression:

$$\mathbf{R} = \begin{pmatrix}1 & 0 & 0\\ 0 & 1 & -\gamma\\ 0 & u_\gamma & 1\end{pmatrix} + o(\gamma) \tag{4}$$

Thus, relation between $(x, y, f(x,y) = h)$ and $(X, Y, F(X,Y) = H)$ is the following:

$$\begin{cases} x &= X + o(\gamma)\\ y &= Y - \gamma H + o(\gamma)\\ h &= \gamma Y + H + o(\gamma)\end{cases} \tag{5}$$

Let us switch to polar coordinates (r,θ) (resp. (R,Θ)) such that $x = r\cos\theta$ and $y = r\sin\theta$ (resp. $X = R\cos\Theta$ and $Y = \sin\Theta$). Let $z = x + \boldsymbol{i}y$ and $Z = X + \boldsymbol{i}Y$. Equation (5) yields:

$$h = H + \gamma R T(\Theta) + o(\gamma) \tag{6}$$

With $T(\Theta) = \frac{1}{2}\left(e^{\boldsymbol{i}\left(\Theta-\frac{\pi}{2}\right)} + e^{-\boldsymbol{i}\left(\Theta-\frac{\pi}{2}\right)}\right)$.

The following equation for r follows from $z = x + \boldsymbol{i}y$ and Equation 5:

$$r^k = \sqrt{|zz^*|}^k = R^k + \frac{kR^{k-1}H}{2}\gamma\left(e^{\boldsymbol{i}\left(\Theta+\frac{\pi}{2}\right)} + e^{-\boldsymbol{i}\left(\Theta+\frac{\pi}{2}\right)}\right) + o(\gamma) \tag{7}$$

Similarly, we have for all $n \in \mathbb{N}$:

$$z^n = R^n e^{\boldsymbol{i}n\Theta} + nR^{n-1}H\gamma e^{\boldsymbol{i}\left((n-1)\Theta+\mu-\frac{\pi}{2}\right)} + o(\gamma) \tag{8}$$

which yields, since $e^{\boldsymbol{i}n\theta} = (z/|z|)^n = (z/r)^n$:

$$e^{\boldsymbol{i}n\theta} = e^{\boldsymbol{i}n\Theta} + \frac{nH}{2R}\gamma\left(e^{\boldsymbol{i}\left((n-1)\Theta-\frac{\pi}{2}\right)} - e^{\boldsymbol{i}\left((n+1)\Theta+\frac{\pi}{2}\right)}\right) + o(\gamma) \tag{9}$$

Combining Equations 7 and 9, and setting $A_{k,n} = \frac{(k+n)}{2}e^{-\boldsymbol{i}\frac{\pi}{2}}$ yields:

$$r^k e^{\boldsymbol{i}n\theta} = R^k e^{\boldsymbol{i}n\Theta} + R^{k-1}e^{\boldsymbol{i}n\Theta}\gamma H\left(A_{k,n}e^{-\boldsymbol{i}\Theta} + A^*_{k,-n}e^{\boldsymbol{i}\Theta}\right) + o(\gamma) \tag{10}$$

Plugging Equation 10 in Equation 6, one has:

$$
\begin{aligned}
H &= \frac{\left(\sum_{k=0}^{\infty}\sum_{n=-k}^{k}\phi_{k,n}R^k e^{\boldsymbol{i}n\Theta}\right) - \gamma R T(\Theta)}{1-\gamma\sum_{k=0}^{\infty}\sum_{n=-k}^{k}\phi_{k,n}R^{k-1}\left(A_{k,n}e^{\boldsymbol{i}(n-1)\Theta}+A_{k,n}^{*}e^{\boldsymbol{i}(n+1)\Theta}\right)} + o(\gamma) \\
&= \left(\sum_{k=0}^{\infty}\sum_{n=-k}^{k}\phi_{k,n}R^k e^{\boldsymbol{i}n\Theta}\right) - \gamma(RT(\Theta)+F(\Theta)+G(\Theta)) + o(\gamma)
\end{aligned}
\tag{11}
$$

With:

$$
\begin{aligned}
F(\Theta) &= \left(\sum_{k=0}^{\infty}\sum_{n=-k}^{k}\phi_{k,n}R^k e^{\boldsymbol{i}n\Theta}\right)\left(\sum_{j=1}^{\infty}\sum_{m=-j}^{j}\phi_{j,m}A_{j,m}R^{j-1}e^{\boldsymbol{i}(m-1)\Theta}\right) \\
G(\Theta) &= \left(\sum_{k=0}^{\infty}\sum_{n=-k}^{k}\phi_{k,n}R^k e^{\boldsymbol{i}n\Theta}\right)\left(\sum_{j=1}^{\infty}\sum_{m=-j}^{j}\phi_{j,m}A_{j,-m}^{*}R^{j-1}e^{\boldsymbol{i}(m+1)\Theta}\right)
\end{aligned}
\tag{12}
$$

$$
\begin{aligned}
F(\Theta) &= \left(\sum_{k=0}^{\infty}\sum_{n=-k}^{k}\phi_{k,n}R^k e^{\boldsymbol{i}n\Theta}\right)\left(\sum_{j=0}^{\infty}\sum_{m=-j-1}^{j+1}\phi_{j+1,m}A_{j+1,m}R^{j}e^{\boldsymbol{i}(m-1)\Theta}\right) \\
&= \left(\sum_{k=0}^{\infty}\sum_{n=-k}^{k}\phi_{k,n}R^k e^{\boldsymbol{i}n\Theta}\right)\left(\sum_{j=0}^{\infty}\sum_{m=-j-2}^{j}\phi_{j+1,m+1}A_{j+1,m+1}R^{j}e^{\boldsymbol{i}m\Theta}\right)
\end{aligned}
\tag{13}
$$

Recall that if k and n do not share the same parity, $\phi_{k,n}=0$, then if $m=-j-1$, $\phi_{j+1,m+1}=0$. Furthermore by definition of A, if $m=-j-2$ then $A_{j+1,m+1}=0$. Thus we can write:

$$
\begin{aligned}
F(\Theta) &= \left(\sum_{k=0}^{\infty}\sum_{n=-k}^{k}\phi_{k,n}R^k e^{\boldsymbol{i}n\Theta}\right)\left(\sum_{j=0}^{\infty}\sum_{m=-j}^{j}\phi_{j+1,m+1}A_{j+1,m+1}R^{j}e^{\boldsymbol{i}m\Theta}\right) \\
&= \sum_{\ell=0}^{\infty}\sum_{s=0}^{\ell}R^{\ell}\left(\sum_{n=-\ell+s}^{\ell-s}\phi_{\ell-s,n}e^{\boldsymbol{i}n\Theta}\right)\left(\sum_{m=-s}^{s}\phi_{s+1,m+1}A_{s+1,m+1}e^{\boldsymbol{i}m\Theta}\right) \\
&= \sum_{\ell=0}^{\infty}\sum_{s=0}^{\ell}R^{\ell}\left(\sum_{n=-\ell+s}^{\ell-s}\phi_{\ell-s,n}e^{\boldsymbol{i}n\Theta}\right)\left(\sum_{m=-s}^{s}\phi_{s+1,m+1}A_{s+1,m+1}e^{\boldsymbol{i}m\Theta}\right)
\end{aligned}
\tag{14}
$$

Finally:

$$
F(\Theta)=\sum_{k=0}^{\infty}\sum_{n=-k}^{k}\Big(\sum_{j=0}^{k}\sum_{\substack{p+m=n\\|p|\le k-j\\|m|\le j}}\phi_{k-j,p}\phi_{j+1,m+1}A_{j+1,m+1}\Big)R^k e^{\boldsymbol{i}n\Theta}
\tag{15}
$$

A similar computation yields:

$$
G(\Theta)=\sum_{k=0}^{\infty}\sum_{n=-k}^{k}\Big(\sum_{j=0}^{k}\sum_{\substack{p+m=n\\|p|\le k-j\\|m|\le j}}\phi_{k-j,p}\phi_{j+1,m-1}A_{j+1,-m+1}^{*}\Big)R^k e^{\boldsymbol{i}n\Theta}
\tag{16}
$$

Since $H=\sum_{k=0}^{\infty}\sum_{n=-k}^{k}R^k e^{\boldsymbol{i}n\Theta}$, by coefficient identification one has $\Phi_{0,0}=\phi_{0,0}+o(\gamma)$ and $\Phi_{1,1}=\phi_{1,1}+\frac{\gamma}{2}e^{-\boldsymbol{i}\frac{\pi}{2}}+o(\gamma)$, however since $\phi_{0,0}=\phi_{1,1}=0$ (since $\mathcal{T}(p)$) is the tangent plane, we have: $\Phi_{0,0}=o(\gamma)$ and $\Phi_{1,1}=\frac{\gamma}{2}e^{-\boldsymbol{i}\frac{\pi}{2}}+o(\gamma)$.

For $k > 1$, one has the following relationship:

$$\begin{aligned}\Phi_{k,n} &= \phi_{k,n} + \gamma \sum_{j=0}^{k} \sum_{\substack{p+m=n \\ |p|\leq k-j \\ |m|\leq j}} \phi_{k-j,p}(\phi_{j+1,m+1}A_{j+1,m+1} + \phi_{j+1,m-1}A^*_{j+1,-m+1}) + o(\gamma) \\ &= \phi_{k,n} + \gamma F(k,n) + o(\gamma)\end{aligned} \tag{17}$$

□

3 Proof of Corollary 1

Corollary 1. *It follows from Theorem 1 that $\gamma = 2|\Phi_{1,1}| + o(\gamma)$ and $arg(\Phi_{1,1}) = \frac{\pi}{2} + o(\gamma)$. Thus if the rotation is small enough, it is possible to correct the parameterization by performing a rotation along axis $(1,0,0)$ with rotation angle $2|\Phi_{1,1}|$.*

Proof. From Theorem 1, we have $\Phi_{1,1} = \frac{\gamma}{2}e^{-\boldsymbol{i}\frac{\pi}{2}} + o(\gamma)$. Then $|\Phi_{1,1}| = \gamma/2 + o(\gamma)$ and $arg\Phi_{1,1} = -\frac{\pi}{2} + o(\gamma)$. To recover the tangent plane, one has thus to perform a rotation of angle $2|\Phi_{1,1}|$ around the rotation axis (p,u). □

4 Proof of Corollary 2

Corollary 2. *One can recover the* true *coefficients $\phi_{k,n}$ iteratively by the following formula:*

$$\phi_{k,n} = \Phi_{k,n} - \gamma \sum_{j=1}^{k-2} \sum_{\substack{p+m=n \\ |p|\leq k-j \\ |m|\leq j}} \phi_{k-j,p}(\phi_{j+1,m+1}A_{j+1,m+1} + \phi_{j+1,m-1}A^*_{j+1,-m+1}) + o(\gamma) \tag{18}$$

In particular, $\phi_{2,0} = \Phi_{2,0} + o(\gamma)$, $\phi_{2,2} = \Phi_{2,2} + o(\gamma)$ and $\phi_{2,-2} = \Phi_{2,-2} + o(\gamma)$ which means that the mean curvature is also stable in $o(\gamma)$.

Proof. Let us rewrite Equation 17 as:

$$\phi_{k,n} = \Phi_{k,n} - \gamma \sum_{j=1}^{k} s_{j,k,n} + o(\gamma) \tag{19}$$

- For $j = 0$, $s_{0,k,n} = \phi_{k,n}(\phi_{1,1}A_{1,1} + \phi_{1,-1}A^*_{1,1})$ since $\phi_{1,1} = \phi_{1,-1} = 0$.
- For $j = k-1$, $s_{k-1,k,n} = \phi_{1,1}(\phi_{k,n}A_{k,n} + \phi_{k,n-2}A^*_{k,-n+2}) = 0$ since $\phi_{1,1} = 0$
- For $j = k$, $s_{k,k,n} = \phi_{0,0}(\phi_{k+1,n+1}A - k + 1, n + 1 + \phi_{k+1,n-1}A^*_{k+1,-n+1}) = 0$ since $\phi_{0,0} = 0$

Equation 17 thus yields:

$$\phi_{k,n} = \Phi_{k,n} - \gamma \sum_{j=1}^{k-2} \sum_{\substack{p+m=n \\ |p|\leq k-j \\ |m|\leq j}} \phi_{k-j,p}(\phi_{j+1,m+1}A_{j+1,m+1} + \phi_{j+1,m-1}A^*_{j+1,-m+1}) + o(\gamma) \tag{20}$$

One can notice that all $\phi_{l,p}$ coefficients appearing in the sum are such that $l < k$. The correction procedure is straightforward: assuming we have corrected all $\Phi_{l,n}$ for all $l < k$ and $-l \leq n \leq l$ and have therefore access to $\phi_{l,n}$ for all $l < k$ and $-l \leq n \leq l$, up to some error in $o(\gamma)$, one can use Equation 20 to correct coefficients $\Phi_{k,n}$ for all $-k \leq n \leq k$. □