1 Proof of the wavejets decomposition

Equation 1 of the paper contains terms such as $x^{k-j}y^j$, which can be rewritten as linear combinations of powers of $e^{i\theta}$.

\[
x^{k-j}y^j = r^k \cos^{k-j} \theta \sin^j \theta
= r^k \left( \frac{e^{i\theta} + e^{-i\theta}}{2} \right)^{k-j} \left( \frac{e^{i\theta} - e^{-i\theta}}{2i} \right)^j
= r^k \frac{k-j}{2^k i^j} \sum_{l=0}^{k-j} \binom{k-j}{l} e^{(k-j-2l)i\theta} \left( \sum_{l=0}^{j} \binom{j}{l} (-1)^l e^{(j-2l)i\theta} \right)
= r^k \frac{k-j}{2^k i^j} \sum_{l=0}^{k-j} \sum_{h=0}^{j} (-1)^h \binom{k-j}{h} \binom{j}{l} e^{(k-2l-2h)i\theta}
= r^k \frac{k-j}{2^k i^j} \sum_{l=0}^{k-j} \sum_{h=0}^{j} \binom{n-k}{h} \binom{j}{l} \binom{l}{h} (-1)^h e^{n i\theta}
= r^k \sum_{n=-k}^{k} b(k, j, n) e^{n i\theta}
\]

with $b(k, j, n) = 0$ if $k$ and $n$ do not have the same parity and $b(k, j, n) = \frac{1}{2^{k+j} j! (k-j)!} f_{x^{k-j}y^j}(0, 0)$ otherwise.

Using Equations 2 of the paper we get:

\[
\phi_{k,n} = \sum_{j=0}^{k} b(k, j, n) j!(k-j)! f_{x^{k-j}y^j}(0, 0).
\]
coordinates $(x = r \cos \theta, y = r \sin \theta, h)$ over $T(p)$ and $(x = R \cos \Theta, y = R \sin \Theta, H)$ over $\mathcal{P}(p)$. Let us first assume that $\theta$ (resp. $\Theta$) corresponds to the angular coordinate of point $q$ with respect an origin vector aligned with $u$ in $T(p)$ (resp. with $u$ in $\mathcal{P}(p)$). We will state our main theorem in this setting and the generalization will follow naturally. In this setting the surface wavejets decomposition at point $q$ in $T(p)$ can be expressed with respect to the new coefficients $\Phi_{k,n}$.

**Theorem 1.** The new coefficients $\Phi_{k,n}$ can be expressed with respect to the $\phi_{k,n}$ as follows:

\[
\begin{align*}
\Phi_{0,0} &= 0 \\
\Phi_{1,1} &= \Phi_{1,-1} = \frac{\gamma}{2} e^{-i\frac{\pi}{2}} + o(\gamma) \\
\Phi_{k,n} &= \phi_{k,n} + \gamma F(k,n) + o(\gamma)
\end{align*}
\] (3)

**Proof.** The rotation matrix $R$ of axis $u = (1,0,0)_p$ and angle $\gamma$ transforms the coordinates $(X,Y,H)$ of a surface point $p$ in the parameterization of $\mathcal{P}(p)$ into coordinates $(x,y,h)$ in the parameterization of $\mathcal{P}(p)$. Let us assume that $\gamma^2$ is small enough. Then the rotation has the following expression:

\[
R = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & -\gamma \\
0 & u_\gamma & 1
\end{pmatrix} + o(\gamma)
\] (4)

Thus, relation between $(x,y,f(x,y) = h)$ and $(X,Y,F(X,Y) = H)$ is the following:

\[
\begin{align*}
x &= X + o(\gamma) \\
y &= Y - \gamma H + o(\gamma) \\
h &= \gamma Y + H + o(\gamma)
\end{align*}
\] (5)

Let us switch to polar coordinates $(r,\theta)$ (resp. $(R,\Theta)$) such that $x = r \cos \theta$ and $y = r \sin \theta$ (resp. $X = R \cos \Theta$ and $Y = \sin \Theta$). Let $z = x + iy$ and $Z = X + iY$. Equation (5) yields:

\[
h = H + \gamma RT(\Theta) + o(\gamma)
\] (6)

With $T(\Theta) = \frac{1}{2} \left( e^{i(\Theta - \frac{\pi}{2})} + e^{-i(\Theta - \frac{\pi}{2})} \right)$.

The following equation for $r$ follows from $z = x + iy$ and Equation 5:

\[
r^k = \sqrt{|z|^2} = R^k + k R^{k-1} \frac{H}{2} \gamma \left( e^{i(\Theta + \frac{\pi}{2})} + e^{-i(\Theta + \frac{\pi}{2})} \right) + o(\gamma)
\] (7)

Similarly, we have for all $n \in \mathbb{N}$:

\[
z^n = R^n e^{i\Theta} + n R^{n-1} H \gamma e^{i((n-1)\Theta + \frac{\pi}{2})} + o(\gamma)
\] (8)

which yields, since $e^{i\Theta} = (z/|z|)^n = (z/r)^n$:

\[
e^{i\Theta} = e^{i\Theta} + \frac{n H}{2R} \gamma \left( e^{i((n-1)\Theta + \frac{\pi}{2})} - e^{i((n+1)\Theta + \frac{\pi}{2})} \right) + o(\gamma)
\] (9)

Combining Equations 7 and 9, and setting $A_{k,n} = \frac{(k+n)}{2} e^{-i\frac{\pi}{2}}$ yields:

\[
r^k e^{i\Theta} = R^k e^{i\Theta} + R^{k-1} e^{i\Theta} H \left( A_{k,n} e^{-i\Theta} + A_{k,n-1} e^{i\Theta} \right) + o(\gamma)
\] (10)

Plugging Equation 10 in Equation 6, one has:
Furthermore by definition of $A$ and $\Phi$, 

$$
H = \frac{\sum_{k=0}^{\infty} \sum_{n=-k}^{k} \phi_{k,n} R^k e^{i\theta}}{1 - \gamma \sum_{k=0}^{\infty} \sum_{n=-k}^{k} \phi_{k,n} R^{k-1} \left( A_{k,n} e^{i(n-1)\theta} + A^*_{k,n} e^{i(n+1)\theta} \right)} + o(\gamma)
$$

(11)

With:

$$
F(\Theta) = \left( \sum_{k=0}^{\infty} \sum_{n=-k}^{k} \phi_{k,n} R^k e^{i\theta} \right) \left( \sum_{j=1}^{\infty} \sum_{m=-j}^{j} \phi_{j,m} A_{j,m} R^{j-1} e^{i(m-1)\theta} \right)
$$

(12)

$$
G(\Theta) = \left( \sum_{k=0}^{\infty} \sum_{n=-k}^{k} \phi_{k,n} R^k e^{i\theta} \right) \left( \sum_{j=1}^{\infty} \sum_{m=-j}^{j} \phi_{j,m} A^*_{j,-m} R^{j-1} e^{i(m+1)\theta} \right)
$$

(13)

Recall that if $k$ and $n$ do not share the same parity, $\phi_{k,n} = 0$, then if $m = -j - 1$, $\phi_{j+1,m+1} = 0$. Furthermore by definition of $A$, if $m = -j - 2$ then $A_{j+1,m+1} = 0$. Thus we can write:

$$
F(\Theta) = \left( \sum_{k=0}^{\infty} \sum_{n=-k}^{k} \phi_{k,n} R^k e^{i\theta} \right) \left( \sum_{j=0}^{\infty} \sum_{m=-j}^{j} \phi_{j+1,m} A_{j+1,m} R^{j} e^{i(m-1)\theta} \right)
$$

(14)

Finally:

$$
F(\Theta) = \sum_{k=0}^{\infty} \sum_{n=-k}^{k} \sum_{j=0}^{\infty} \sum_{m=-j}^{j} \phi_{k-n+p,j+1,m+1} A_{j+1,m+1} R^k e^{i\theta}
$$

(15)

A similar computation yields:

$$
G(\Theta) = \sum_{k=0}^{\infty} \sum_{n=-k}^{k} \sum_{j=0}^{\infty} \sum_{m=-j}^{j} \phi_{k-n+p,j+1,m-1} A^*_{j+1,m-1} R^k e^{i\theta}
$$

(16)

Since $H = \sum_{k=0}^{\infty} \sum_{n=-k}^{k} R^k e^{i\theta}$, by coefficient identification one has $\Phi_{0,0} = \phi_{0,0} + o(\gamma)$ and $\Phi_{1,1} = \phi_{1,1} + \frac{1}{2} e^{-i\frac{\pi}{2}} + o(\gamma)$, however since $\phi_{0,0} = \phi_{1,1} = 0$ (since $T(p)$ is the tangent plane, we have: $\Phi_{0,0} = o(\gamma)$ and $\Phi_{1,1} = \frac{1}{2} e^{-i\frac{\pi}{2}} + o(\gamma)$.
For \( k > 1 \), one has the following relationship:

\[
\Phi_{k,n} = \phi_{k,n} + \sum_{j=0}^{k-2} \sum_{|p|\leq k-j, |m|\leq j} \phi_{k-j,p}(\phi_{j+1,m+1}A_{j+1,m+1} + \phi_{j+1,m-1}A^*_{j+1,-m+1}) + o(\gamma)
\]

\[
= \phi_{k,n} + \gamma F(k,n) + o(\gamma)
\]

(17)

3 Proof of Corollary 1

**Corollary 1.** It follows from Theorem 1 that \( \gamma = 2|\Phi_{1,1}| + o(\gamma) \) and \( \arg(\Phi_{1,1}) = \frac{\pi}{2} + o(\gamma) \). Thus if the rotation is small enough, it is possible to correct the parameterization by performing a rotation along axis \((1,0,0)\) with rotation angle \(2|\Phi_{1,1}|\).

**Proof.** From Theorem 1, we have \( \Phi_{1,1} = \frac{\pi}{2}e^{-i\frac{\pi}{2}} + o(\gamma) \). Then \( |\Phi_{1,1}| = \gamma/2 + o(\gamma) \) and \( \arg\Phi_{1,1} = -\frac{\pi}{2} + o(\gamma) \).

To recover the tangent plane, one has thus to perform a rotation of angle \(2|\Phi_{1,1}|\) around the rotation axis \((p,u)\).

\( \square \)

4 Proof of Corollary 2

**Corollary 2.** One can recover the true coefficients \( \phi_{k,n} \) iteratively by the following formula:

\[
\phi_{k,n} = \Phi_{k,n} - \gamma \sum_{j=1}^{k-2} \sum_{|p|\leq k-j} \phi_{k-j,p}(\phi_{j+1,m+1}A_{j+1,m+1} + \phi_{j+1,m-1}A^*_{j+1,-m+1}) + o(\gamma)
\]

(18)

In particular, \( \phi_{2,0} = \Phi_{2,0} + o(\gamma) \), \( \phi_{2,2} = \Phi_{2,2} + o(\gamma) \) and \( \phi_{2,-2} = \Phi_{2,-2} + o(\gamma) \) which means that the mean curvature is also stable in \( o(\gamma) \).

**Proof.** Let us rewrite Equation 17 as:

\[
\phi_{k,n} = \Phi_{k,n} - \gamma \sum_{j=1}^{k} s_{j,k,n} + o(\gamma)
\]

(19)

- For \( j = 0 \), \( s_{0,k,n} = \phi_{k,n}(\phi_{1,1}A_{1,1} + \phi_{1,-1}A^*_{1,1}) \) since \( \phi_{1,1} = \phi_{1,-1} = 0 \).
- For \( j = k-1 \), \( s_{k-1,k,n} = \phi_{k,n}(\phi_{k-1,k,n}A_{k-1,k,n} + \phi_{k,n-2}A^*_{k,n-2}) = 0 \) since \( \phi_{1,1} = 0 \)
- For \( j = k \), \( s_{k,k,n} = \phi_{0,0}(\phi_{k+1,n+1}A - k + 1, n+1 + \phi_{k+1,n-1}A^*_{k+1,-n+1}) = 0 \) since \( \phi_{0,0} = 0 \)

Equation 17 thus yields:

\[
\phi_{k,n} = \Phi_{k,n} - \gamma \sum_{j=1}^{k-2} \sum_{|p|\leq k-j} \phi_{k-j,p}(\phi_{j+1,m+1}A_{j+1,m+1} + \phi_{j+1,m-1}A^*_{j+1,-m+1}) + o(\gamma)
\]

(20)

One can notice that all \( \phi_{j,p} \) coefficients appearing in the sum are such that \( l < k \). The correction procedure is straightforward: assuming we have corrected all \( \Phi_{l,n} \) for all \( l < k \) and \( -l \leq n \leq l \) and have therefore access to \( \phi_{l,n} \) for all \( l < k \) and \( -l \leq n \leq l \), up to some error in \( o(\gamma) \), one can use Equation 20 to correct coefficients \( \Phi_{k,n} \) for all \( -k \leq n \leq k \).

\( \square \)