Details of the Wardetzky proof for *Manifold Harmonics* lecture

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We expand the proof of [1] that there is no Laplace Beltrami discretization satisfying (SYM) + (LOC) + (LIN) + (POS). Nothing new is added to [1], the idea is just to add some details in relation to the dual graph with negative weights.

We proceed by showing:

1. (SYM)+(LOC)+(LIN) ⇔ there exists an orthogonal dual graph

2. There exist an orthogonal dual graph with positive primal weights ⇔ the triangulation is regular

Assume that a vertex $i$ and its 1-ring neighborhood lie on a plane. The (LOC) and (LIN) conditions are equivalent to:

$$\sum_{j \in N_1(i)} w_{ij} (x_i - x_j) = 0$$

This can be further interpreted as the equilibrium of a system of pulling and pushing forces applied on particles on a plane where $w_{ij} > 0$ implies a pulling force attracting $x_i$ towards $x_j$ and $w_{ij} < 0$ implies a pushing force repulsing $x_i$ from $x_j$.

According to the Maxwell-Cremona theorem, the system is at the equilibrium if and only if there exists an orthogonal dual framework.

Assume we have a graph $\Gamma = (V, E, F)$, its orthogonal dual is a graph $\Gamma^* = (V^*, E^*, F^*) = (F, E^*, V)$ such that an edge of $\Gamma^*$ is orthogonal to the corresponding edge $\Gamma$. Notice that we authorize edges to be swapped inducing self-crossings in the dual graph (yet the edges should still form cycles).

1. **(SYM)+(LOC)+(LIN) ⇔ there exists an orthogonal dual graph**

Assume the graph $\Gamma$ admits a Laplacian satisfying (SYM)+(LOC)+(LIN). For each primal edge $e_{ij}$, let us build $\ast e_{ij}$ an edge corresponding to the 90° rotated vector $u_{ij}$ with $u_{ij} = w_{ij} e_{ij}$.

Since $\sum_i w_{ij} (x_i - x_j) = \sum_j u_{ij} = 0$, we can apply the rotation yielding: $\sum_j R u_{ij} = 0$, thus the $\ast e_{ij}$ form a cycle.

Hence the dual graph $\Gamma^*$ of $\Gamma$ exists and is a valid graph.

Exercise: Draw the graph containing 5 vertices $v_0, v_1, v_2, v_3, v_4$ and edges $(v_0, v_1), (v_0, v_2), (v_0, v_3), (v_0, v_4), (v_1, v_2)$ compute the dual graph around $v_0$ for $w_{01} = 1, w_{02} = 2/9, w_{03} = -5/9, w_{04} = 1$. There should be a self crossing of the graph.
Assume the graph $\Gamma$ admits an orthogonal dual graph $\Gamma^*$. An edge $e_{ij}$ in $\Gamma$ corresponds to an edge $*e_{ij}$ of $\Gamma^*$. Let $\|\|$ denote the euclidean length then let us define:

$$w_{ij} = \varepsilon_{ij} \frac{\|*e_{ij}\|}{\|e_{ij}\|}$$

where $\varepsilon_{ij} \pm 1$. To come up with a sign convention, let us notice that edge $*e_{ij}$ connects two vertices $*f_1$ and $*f_2$ corresponding to two facets $f_1$ and $f_2$ that were connected by edge $e_{ij}$ in the primal graph $\Gamma$. We shoot a ray from $*f_1$ to $*f_2$ and set the sign as follow. If in this ray direction the facet $f_1$ lies before $f_2$ $\varepsilon_{ij} = 1$ and $-1$ otherwise. The negative sign occurs typically when there is a self-crossing in the dual graph (example built above).

The dual graph exists so that $\sum_j *e_{ij} = 0$ thus $\sum_j w_{ij} e_{ij} = 0$, since the self-crossing case is handled with the sign choice.

Bringing together these two results we get (SYM)+(LOC)+(LIN) $\iff$ there exists an orthogonal dual graph

2 There exists an orthogonal dual graph with positive primal weights $\iff$ the triangulation is regular

Result from Aurenhammer (1987) Straight-line triangulation is regular iff it allow for a positive orthogonal dual (no self-crossing)

If the triangulation is regular a positive orthogonal dual exists so that the derived laplacian satisfies (SYM)+(LOC)+(LIN)+(POS).

Assume a laplacian satisfies (SYM)+(LOC)+(LIN)+(POS) then by construction $\Gamma^*$ exists and has positive primal weights (since (POS)) so that by Aurenhammer the triangulation is regular.

References