

# The moirés of circles

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When we draw on a screen of computer or when we print the family of all the concentric circles of integer radius less or equal to  $R$  with  $R$  large enough (see the figure 1), we can observe “moiré effects” that is to say regular white patterns. The goal of this paper is to give an explanation for such phenomena.

In fact, the moiré effects that are generally dealt in the literature ([1],[2] for example) are made by two (or more) periodic structures, which contain black regions (like lines or points) and transparent ones, that overlap. For example, two figures which contains parallel black lines and which made a small angle, or two figures that contain concentric circles. Contrary to these studies, in this paper, we just consider one figure which contains concentric circles. So, our goal is to explain the moiré effects that we can observe on the figure made by the following sequence of instructions:

```
For i:=1 to rmax do
  DrawCircle(0,0,i);          /*Draw the circle centered on (0,0)
                              with an integer radius equal to i*/
```

with a large `rmax` (equals to 270 for example, in the figure 1).

As the circles drawn on a screen are not reals but discrete ones, we will try to explain these “moiré effects” examining the structure of discrete circles. In [3], we have studied the discrete circles and we have seen that there exists various definitions. But, we have proved that a discrete circle can be seen as a set of horizontal, vertical and diagonal lines that lean on bundles of parabolas. As a matter of fact, we have the following equivalence: for all integers  $x \geq 0$ ,  $y \geq 0$ ,  $k \geq 1$ ,

$$\begin{cases} x^2 + y^2 = R^2 \\ x = R - k \end{cases} \Leftrightarrow \begin{cases} y^2 = 2kx + k^2 \\ x = R - k \end{cases}$$

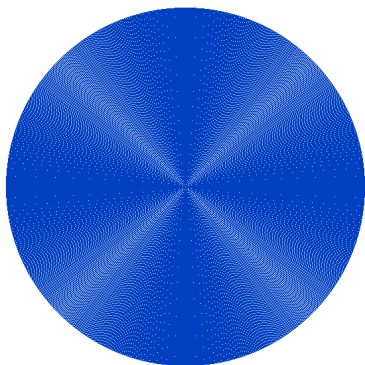


Figure 1: Family of all concentric circles with integer radii less or equal to 270.

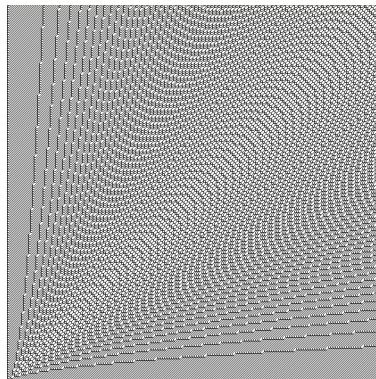


Figure 2: Discrete parabolas  $H_k$  and  $V_k$  with  $1 \leq k \leq 200$ .

and so, for a fixed first coordinate  $x = R - k$ , the points that belong to a digitization of the circle a radius  $R$  are the integer points that belong between respective digitizations of  $y = \sqrt{2kx + k^2}$  and  $y = \sqrt{2(k+1)x + (k+1)^2}$ . Moreover, we have seen that all the well-known digitizations can be deduced from the digitization that consists in “taking the floor”. Consequently, instead of studying the moiré effects made by concentric circles, we will focus us on the moirés that are produced by the family of “floor parabolas”.

A floor parabola, denoted  $H_k$  with  $k$  an integer ( $k \geq 1$ ), is defined as follows:

$$H_k = \{(x, y) \in [(i, j), (i+1, j')]; i, j, j' \in \mathbb{Z}, j = \lfloor \sqrt{2ki + k^2} \rfloor \text{ and } j' = \lfloor \sqrt{2k(i+1) + k^2} \rfloor\}.$$

The moiré effects we want to explain are made by the intersections of the floor parabolas  $H_k$  (with  $k \geq 1$ ) and the parabolas symmetric to these ones according to the first diagonal: the parabolas denoted by  $V_{k'}$  (with  $k' \geq 1$ ). The figure 2 give a graphical representation of these parabolas with  $0 \leq x, y \leq 200$ .

When we observe this figure, we can distinguish three parts: a central part that we call the “heart” of the figure and two symmetric lateral parts denoted  $DH_0$  and  $DV_0$  as it is indicated on the figure 3. In the heart, we can see regular curves that have the first diagonal as axis of symmetry (see the figure 4) and, in  $DH_0$  and  $DV_0$ , it seems that there is an infinity of moirés as the figure 5 shows it. We successively explain all these elements.

First, we interest ourselves to the different parts we have distinguish in the figure 3. In fact, we show that we can naturally consider the figure 2 to be composed of an infinity of parts such that in each of them there is a kind of moiré .

These parts are defined as follows (see the figure 7):

- We call  $DH_{01}$  (respectively  $DV_{01}$ ) the set of integer points of coordinates  $(x, y)$  such that  $\frac{3}{4}x \leq y \leq x$  (respectively  $\frac{3}{4}y \leq x \leq y$ ).

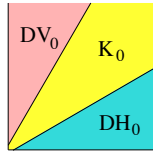


Figure 3: The three parts.

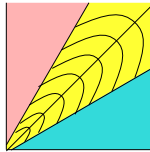


Figure 4: The regularities of the heart.

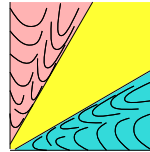


Figure 5: The regularities of  $DV_0$  and  $DH_0$ .

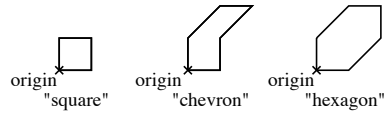


Figure 6: Patterns in the heart.

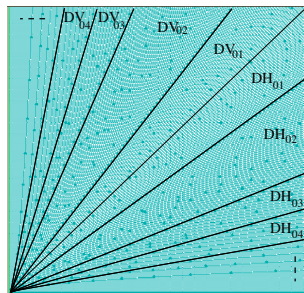


Figure 7: Different lines that separate the moirés.

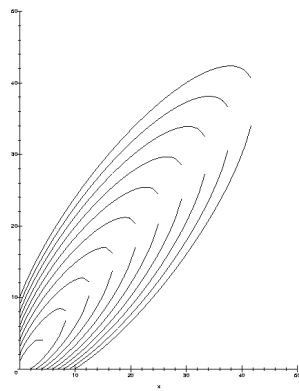


Figure 8: The ellipses on which are the origins of squares.

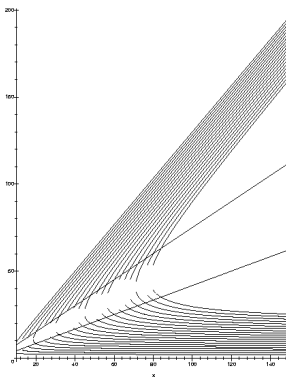


Figure 9: Graphical representations of the functions  $f(2, \alpha, x)$  and  $g(2, \alpha, x)$  for  $1 \leq \alpha \leq 20$  and  $0 \leq x \leq 150$ .

- For every integer  $i$ ,  $i > 1$ , we denote by  $DH_{0i}$  (respectively  $DV_{0i}$ ) the set of integer points of coordinates  $(x, y)$  such that  $\frac{2i+1}{2i(i+1)}x \leq y \leq \frac{2i-1}{2(i-1)i}x$  (respectively  $\frac{2i+1}{2i(i+1)}y \leq x \leq \frac{2i-1}{2(i-1)i}y$ ).

To explain the moiré effects in the heart, we study in detail the intersections between the parabolas  $H_k$  and  $V_{k'}$ . As we can prove it, the heart is only composed of four different patterns (see figure 6): a “square”, “an hexagon”, a “chevron” and its symmetrical pattern regularly disposed. In [3], we have computed the expression of the coordinates of the origin of the squares (point which is on the left and above): a point  $(x, y)$  is an origin of a square if and only if there exists two integers  $i$  and  $k$  ( $i \geq 0$ ,  $k \geq 1$ ) such that:

$$\begin{cases} x = 2k + 4i + \lfloor \sqrt{2ki + k^2} \rfloor \\ y = 2k + 3i + 2\lfloor \sqrt{2ki + k^2} \rfloor \end{cases}$$

It helps us to prove that the squares in the heart are disposed on digitizations of elliptical curves. The figure 8 represents some of them.

Finally, an important part deals with the moirés outside the heart.

## References

- [1] I. Amidror. *Analysis of moiré patterns in multi-layer superpositions*. PhD thesis, Ecole Polytechnique Federale de Lausanne, 1995.
- [2] G. Oster and Y. Nishijima. Moiré patterns. *Scientific American*, mai 1963.
- [3] L. Tougne. *Cercles discrets sur automates cellulaires*. PhD thesis, Ecole Normale Supérieure de Lyon, janvier 1997.