Distributed algorithms for networks – Final exam

- **Duration: 2h.**
- **Documents, calculators and dictionaries allowed.**
- **Exercises are independent.**
- **Any unjustified answer will bear no point, but presenting intuitions is encouraged.**
- **You may answer in english or french.**

Computing diameter in CONGEST

1. Give a $O(n)$-round procedure in the CONGEST model testing whether a graph has diameter at most 2, i.e. such that every node accepts if and only if the graph has diameter at most 2.
2. Using a reduction from Set-Disjointness, prove that this is tight (up to polylogarithmic factors), even when restricted to graphs of diameter at most 3. In other words, for every instance (S_A, S_B) of Set-Disjointness, construct a graph G of diameter at most 3 such that G has diameter 2 if and only if (S_A, S_B) is a positive instance of Set-Disjointness.

Certifying existential FO

A first order sentence is said to be *existential* if it can be written as $\exists x_1 \cdots \exists x_k \quad \varphi(x_1, \ldots, x_k)$ where $x_1, \ldots, x_k$ are vertices and φ is a quantifier-free formula. Give a $O(k \log n + k^2)$ local certification process for existential FO sentences with k quantifiers.

Edge-coloring trees

Consider the setting we had for the sinkless orientation lower bound: a 3-regular tree T where the nodes are white or black (such that these colors form a proper 2-coloring).

1. Assuming each node knows the color of its incident edges in a 4-edge-coloring of T, construct a sinkless orientation of T in 0 rounds.
2. Deduce a lower bound for computing a 4-edge-coloring.

Improved randomized coloring

Let G be an n-vertex graph of maximum degree Δ and $0 < \varepsilon < 1$. We recall the randomized procedure used to $(1+\varepsilon)\Delta$-color G during the classes in $O(\log n)$ rounds. At each iteration, each node v picks randomly a color among the colors that are not previously taken by its neighbors. If no neighbor of v chose the same color, then v gets permanently colored with it, and tells its neighbors. Otherwise, v forgets its color and waits for the next iteration.

The goal of this exercise is to improve the complexity to $O(\sqrt{\log n})$ rounds.

1. In this question, we assume that $\Delta \leqslant 2^{\sqrt{\log n}}$. We run the above procedure for $k = 4\sqrt{\log n}/\varepsilon$ iterations.
   (a) Show that, at each iteration, each node gets colored with probability at least $\varepsilon/2$.
   (b) Show that the probability that a fixed path of length k in G stays fully uncolored after k iterations is at most $n^{-8/\varepsilon}$.
   (c) As a function of n, k and Δ, how many paths of length k can G contain at most? Deduce that w.h.p., the connected components of the graph induced by the uncolored vertices have diameter at most k.
   (d) Propose an algorithm running in $O(\sqrt{\log n})$ rounds to $(1+\varepsilon)\Delta$-color G in that case.

2. We now forget the assumption $\Delta \leqslant 2^{\sqrt{\log n}}$. We recall the so-called Chernoff's bound:

   Suppose $X_1, X_2, \ldots, X_n$ are independent random variables taking values in $\{0,1\}$. Let $X = X_1 + \cdots + X_n$ and $\mu = \mathbb{E}[X]$. For every $0 < \delta < 1$, we have

$$\mathbb{P}(X \notin [\mu(1-\delta), \mu(1+\delta)]) \leqslant 2e^{-\delta^2\mu/3}$$

   (a) Show that, in one round, one can partition the vertices of G into $k = \varepsilon^2\Delta/(100\log n)$ parts such that w.h.p., each part induces a subgraph of degree at most $\Delta/k \cdot (1+\varepsilon/3)$.
   (b) Conclude.

Coloring interval graphs

The *k-th power* of G, denoted by G^k is the graph on $V(G)$ where u, v are adjacent if and only if they are at distance at most k in G. We also denote by $\omega(G)$ the size of the largest clique in G. Remind that X is a (α, β)*-ruling set* if vertices of X are pairwise at distance at least α and every vertex of G is at distance at most β from X.

1. Prove that a maximal independent set is a $(2,1)$-ruling set.

2. Deduce an algorithm (in the LOCAL model) that computes a $(k+1, k)$-ruling set. What is its complexity?

We assume that G is an interval graph, that is an intersection graph of intervals of the real line. In other words, every vertex u can be represented as an interval $[a_u, b_u]$ and two vertices are adjacent if and only if the corresponding intervals intersect. Let $k \geq 5$ and X be a $(k, k-1)$-ruling set of the interval graph G. A *box* is the neighborhood of some vertex in X.

3. Prove that, boxes are anti-complete, that is, there is no edge between $N(x)$ and $N(y)$ for every $x, y \in X$.

4. Let G' be the graph obtained from G by deleting the boxes. What can you say about diameters of connected components of G'?

5. Assume that the following holds: let G be an interval graph and u, v be two vertices at distance at least $\omega(G)$. Any proper coloring of $N(u) \cup N(v)$ can be extended into a proper $(\omega(G)+1)$-coloring of all vertices of G whose intervals contain a value in $[\min(a_u, a_v), \max(b_u, b_v)]$.

   Propose an algorithm (in the LOCAL model) to compute a $(\omega(G)+1)$-coloring of G. What is its complexity?