Distributed algorithms for networks – Exam 2025

- **Duration: 2h.**
- **Lecture notes, calculators, and dictionaries allowed.**
- **Exercises are independent. Any unjustified answer will bear no point, but presenting intuitions and partial answers is encouraged. You may answer in English or French.**
- **In all exercises, the graph is connected, n is the number of nodes, and the nodes are equipped with unique identifiers on $O(\log n)$ bits, unless stated otherwise.**
- **The distribution of points is only informative. The total is 21 points.**

1 LOCAL versus CONGEST [1 point]

Cite a problem with complexity $O(1)$ in the (deterministic) LOCAL model and $\Omega(f(n))$ in the (deterministic) CONGEST model, with f as large as possible. [1 point]

2 Average complexity of 3-coloring in paths [4 points]

Consider a variant of the LOCAL model where every node can commit to an output. After the commitment, the computation can continue in the graph, but the node cannot change its output. Note that different nodes can commit at different steps. The complexity of a node is the time when it commits to an output. The *average complexity* is the sum of the complexities of all nodes, divided by n.

1. Design a randomized algorithm for 3-coloring a path with expected average complexity $O(1)$, and give an intuitive explanation of why it has constant average complexity. [1 point]
2. Prove formally correctness and expected average complexity. [2 points]
3. Give an upper bound on the time at which all nodes have stopped with high probability in your algorithm. Give an intuitive justification. [1 point]

3 Complexity classification in directed cycles [9 points]

Remember that a 1-locally verifiable problem P on a directed cycle is described by a finite set of output labels $\{\ell_1, \ell_2, ..., \ell_r\}$ and set of correct directed balls of radius 1: (ℓ_i, ℓ_j, ℓ_k). An output assignment is correct for P if and only if all the output-labeled neighborhoods of radius 1 in the graph are in L.

We will prove that for 1-locally verifiable problems, the complexity in the deterministic LOCAL model in directed cycles is either in $O(1)$, $\Theta(\log^* n)$ or $\Theta(n)$.

1. Give an example of a problem in each class, and justify that there is no problem of complexity strictly larger than n. [1 point]

We associate to every 1-verifiable problem P, a directed graph D_P, define the following way (using the notation of the first paragraph):

- The nodes are all the couples (ℓ_i, ℓ_j)

- There is an edge $(\ell_i, \ell_j) \to (\ell_k, \ell_p)$ if $\ell_j = \ell_k$ and (ℓ_i, ℓ_j, ℓ_p) is a correct neighborhood.

A node u of the directed graph D_P is called *flexible* if there exists an integer k such that for all $k' \geq k$ there exists a walk of length k' that starts and ends in u.

5. Consider the problems of c-coloring for all integer $c \geq 1$. For which values of c does the directed graph have a flexible state? [1 point]
6. State and prove a sufficient and necessary condition on the structure of D_P, for a node to be flexible. [1 point]
7. Prove that a problem whose graph has a flexible state can be solved in $O(\log^* n)$ rounds. [3 points]
8. Suppose that for all node u of D_P, there exists two integers a_u and m_u, such that there is no walk of length $a_u \mod m_u$ that starts and ends in u. Prove that in that case P requires $\Omega(n)$ rounds. Deduce that a problem with no flexible state requires $\Omega(n)$ rounds. [1.5 points]
9. State a sufficient condition on D_P to have a complexity $O(1)$, and show that it is necessary in the special case where there is no unique identifiers. [1.5 points]

4 Maximal matchings and sloppy matchings (4 points)

In this exercise, we consider the deterministic LOCAL model. A matching is a set of edges that do not share endpoints. A *maximal matching* is a matching that is maximal by inclusion.

1. Prove that one can compute a maximal matching in $O(\text{polylog } n)$ rounds in general graphs. [1 point]

A *sloppy matching* in a tree is a matching where all non-leaf nodes are matched (leaves are unconstrained).

2 Prove that computing a sloppy matching in a d-regular tree takes $\Omega(\log n)$ rounds. [1.5 points]

3 Prove that computing a sloppy matching in a d-regular tree can be done in $O(\log n)$ rounds. [1.5 points]

5 Local certification of tree properties (3 points)

Consider an arbitrary infinite set of trees S.

1. Prove that for any S, we can locally certify that a graph is in S with $O(n \log n)$-bit certificates. [1.5 points]
2. Prove that there are some S for which one needs $\Omega(n)$ bits. [1.5 points]