Local certification of graph classes

Laurent Feuilloley

Université Lyon 1

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Note: bibliographic pointers at the end of the talk.
Classic network assumptions

Classic LOCAL/CONGEST algorithms are designed for:

(a) General networks  (b) Bounded degree

Two reasons for that:

- Fundamental settings.
- Locally checkable structures (we can raise alarm if needed).
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Limitations of these two classes

These classes do not capture the properties of real-world networks.

**Example:** unit-disks are a model for wireless networks.
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**Example:** unit-disks are a model for wireless networks.

- The degree is unbounded.
- The independence number is low
  → Classic algorithms do not give a good estimate.
Most network structures cannot be checked

There exists algorithms for unit-disks, planar, small-diameter etc.  
But these network structures cannot be checked locally.

**Example:** Trees cannot be checked locally.
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Example: Trees cannot be checked locally.
New notion: Local certification

Local certification is a mechanism to allow local checking.

**Idea:** A labeling of the nodes that certifies that the network structure, and that can be checked locally. Coming from another algorithm, or from the network designer.

**Requirements:** There exists a local verification algorithm s.t.:

- For every graph in the class, there exists a labeling such that the algorithm accepts.
- For every graph *not* in the class, for every labeling, the algorithm rejects on at least one node.

**About the locality:** 1 round or $O(1)$ rounds.
Example: local certification of trees
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Can we certify any graph class?

**Question:** Take a graph class $C$ (i.e. an infinite set of graphs). Can we design a labeling and a local verification algorithm that fulfill the requirements?

**Answer:** Yes! (By abusing the assumptions and having large certificates.)
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Optimal certificate size

Measure of quality: the size of the certificates.

Example: for trees,
- the optimal certificate size for trees is $> 0$, and $\leq O(n^2)$,
- the distance labeling gives $O(\log n)$
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Research program

→ **Design optimal certification for relevant graph classes.**

List of relevant classes:

- Classes used in DC: trees, grids, planar, unit-disk, cliques, small-diameter
- Classic classes: planar, chordal, interval, cographs, bipartite
- Families of classes: \( H \)-free, \( H \)-minor-free.
A case study: 4 approaches to planar graphs

Embedding

Minors

Faces

Spanning tree
Planar graphs: via the embedding

**Embedding characterization:** Planar graphs are the graphs that can be embedded in the plane without edge crossings.
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**Certification idea:** Give the coordinates to the nodes.
Planar graphs: via the embedding

Problem: The nodes can be fooled by the coordinates.
Planar graphs: via the embedding

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Planar graphs: via minors

**Minor characterization:** Planar graphs are the graphs with no $K_5$ or $K_{3,3}$ minor.
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Planar graphs: via faces

- Given a planar embedding, we can define faces.
- But this is not enough the surface can be more complicated.
- Euler formula: \(|V| - |F| + |E| = 2\), only in planar graphs.
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Planar graphs: via faces

Certification idea:

- Use rotation systems to encode faces.
- Use a spanning tree to gather $|V|$, $|F|$ and $|E|$ at one node.
Planar graphs: via a spanning tree

Spanning tree characterization: For any spanning tree, there is no crossing of the outer edges.
Planar graphs: via a spanning tree

**Spanning tree characterization:** For any spanning tree, there is no crossing of the outer edges.
Planar graphs: summary and theorem

Theorem: Planar graphs can be certified with $O(\log n)$ bits.*
*: Taming high degrees

Problem: In the certifications given, the certificate size can be of size $\delta \log n$, where $\delta$ is the vertex degree.

Solution: In every planar graph there exists a vertex of degree at most 6. $\rightarrow$ Degeneracy ordering $\rightarrow$ Certificate load balancing.
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A class that requires large certificates

Class: Graph of diameter at most 3.
Model: Look at distance 1.
Theorem: Optimal certificate size in $\tilde{\Omega}(n)$. 
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**Theorem:** Optimal certificate size in $\tilde{\Omega}(n)$. 
Status and research directions

**Direction 1:** Aim for a generalization of the log $n$ region.
**Direction 2:** Target other relevant graph classes and parameters.
Generalization to $H$-minor-free

- $H$-minor-free is a natural generalization of the "good classes".
- They are hereditary, which is good for compact certification.
- But we don’t know how to certify that something is not there.
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Generalization to $H$-minor-free

Open question: Does every graph class characterized by forbidden minors have a compact local certification?

Theorem: When the minors are small ($|H| \leq 4$, or $|H| = 5$ with a special structure), the answer is positive.

A key tool: Certification of 2(and 3)-connectivity.
Other classes and parameters

List of relevant classes:

- Classes used in DC: trees, grids, planar, unit-disk, cliques, small-diameter.
- Classic classes: planar, chordal, interval, cographs, bipartite.

Open questions:

- Do unit-disk graphs have a compact certification?
- Can we certify treewidth $k$ efficiently?
- What about $k$-connectivity?
Bibliographic pointers

Local certification papers mentioned:

▶ Proof-labeling schemes (Korman, Kutten, Peleg - 2010). doi:10.1007/s00446-010-0095-3

▶ Memory-efficient self stabilizing protocols for general networks (Afek, Kutten, Young - 1990). doi:10.1007/3-540-54099-7_2

▶ Locally checkable proofs in distributed computing (Göös, Suomela - 2016). doi:10.4086/toc.2016.v012a019

Tutorial on local certification

▶ Introduction to local certification (Feuilloley - 2021). doi:10.46298/dmtcs.6280 + Gem talk at PODC (on youtube).
Bibliographic pointers

Certification of planar and bounded-genus graphs

- Compact distributed certification of planar graphs (Feuilloley, Fraigniaud, Montealegre, Rapaport, Rémila, Todonca, 2021) doi:10.1007/s00453-021-00823-w + Talks at PODC by Montealegre

- Local Certification of Graphs with Bounded Genus (Same as above.) arxiv:2007.08084

- Local certification of graphs on surfaces (Esperet, Leveque - 2021) arxiv:2102.04133

Small diameter lower bound

Bibliographic pointers

Certification of \( H \)-minor-free graphs

- Local certification of graph decompositions and applications to minor-free classes (Bousquet, Feuilloley, Pierron - 2021)  
  `arxiv:2108.00059 + BA at DISC`

Other specific classes

- Compact Distributed Interactive Proofs for the Recognition of Cographs and Distance-Hereditary Graphs (Montealegre, Ramírez-Romero, and Rapaport - 2021)  
  `arxiv:2012.03185 (+ personal communication)`