Introduction to local certification

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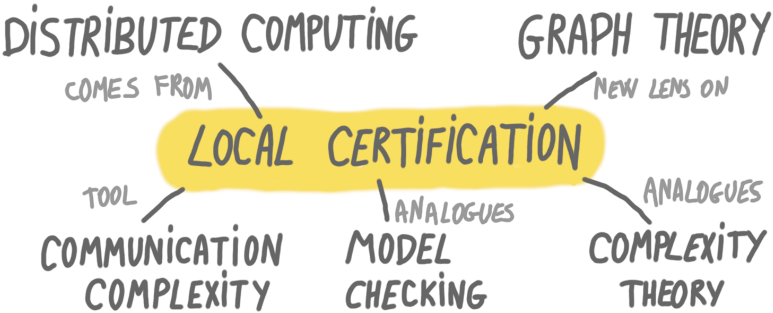
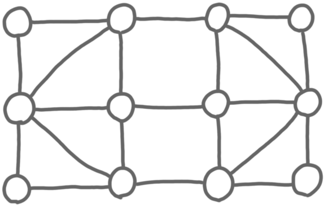
CNRS and University of Lyon · GT Graphes

JNIFM · Grenoble · March 2024

Landscape

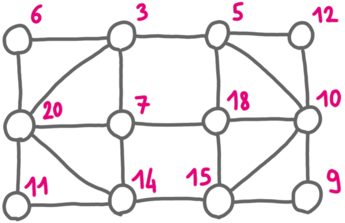

DISTRIBUTED COMPUTING
GRAPH THEORY
COMES FROM
NEW LENS ON
LOCAL CERTIFICATION
TOOL
ANALOGUES
ANALOGUES
COMMUNICATION COMPLEXITY
MODEL CHECKING
COMPLEXITY THEORY

Distributed perspective on graphs



- The graph represents a network.
- Nodes are machines.
- Edges are communication channels.

Distributed perspective on graphs



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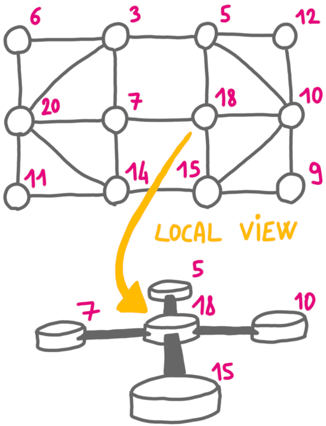


- The graph represents a network.
- Nodes are machines.
- Edges are communication channels.
- Unique identifiers.
- In this talk:
  - Every node sees its neighbors,
  - runs the algorithm,
  - outputs a binary decision.

Local recognition of graph classes

Let $\mathcal{C}$ be a class of connected graphs (*e.g.* planar graphs)

Local recognition of $\mathcal{C}$: A local decision algorithm such that:

- If $G \in \mathcal{C}$ then all the vertices accept.
- If $G \notin \mathcal{C}$ then at least one vertex rejects.

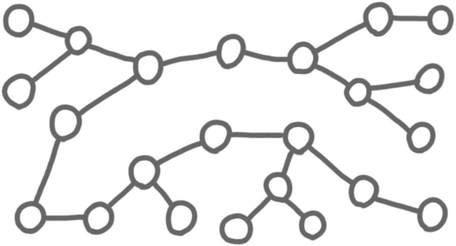
Examples:

- Graphs of degree 3 can be recognized locally.
- For any set $S \subseteq \mathbb{N}$, graph with all degrees in S can be recognized locally.
- Trees cannot be recognized locally

Quick proof for trees

- Consider a graph with a unique cycle that is too large to fit in the view of a node.
- For correctness, a least one node v rejects.
- Now, remove an edge far from v, to cut the cycle. Rerun the checking. Node v still rejects. Contradiction.

Introducing local certification



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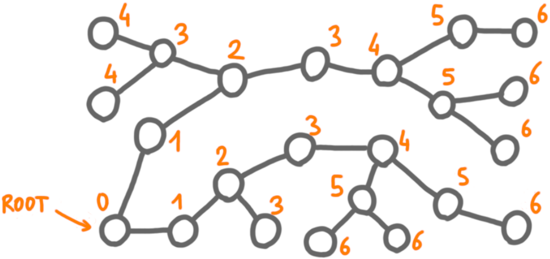
Introducing local certification



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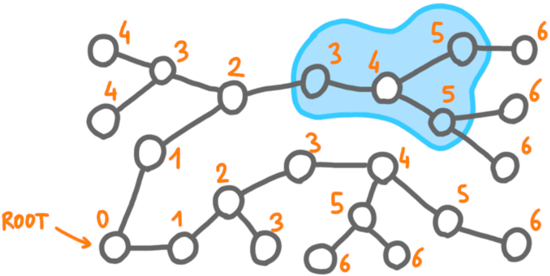

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- Sanity check at each node: the distances locally make sense.

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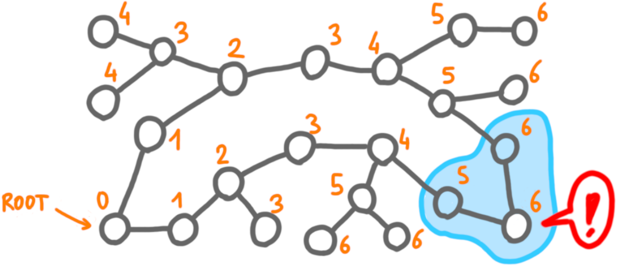


- **Idea:** Use additional information, in the form of labels.
- For trees: the distances from an arbitrarily chosen node.
- Sanity check at each node: the distances locally make sense.
- **Key property:** if the graph has a cycle, at least one node will reject.

Definition and story

Definition: A local certification of a graph class $\mathcal{C}$ is a local decision algorithm such that :

1. For $G \in \mathcal{C}$, **there exists** certificate assignment that makes **all vertices** accept.
2. For $G \notin \mathcal{C}$, **for any** certificate assignment, **at least one vertex** rejects.

Story:

- A *prover* is trying to convince all nodes that the graph is in class $\mathcal{C}$ (which might be true or false).
- If the graph is indeed in the class, it succeeds.
- If the graph is not in the class, it cannot succeed.

Any class can be certified

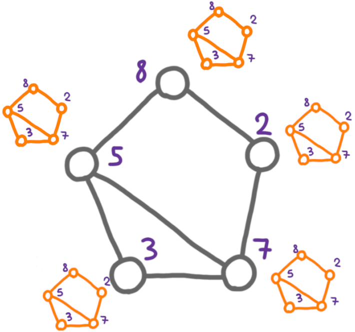
Theorem: Any graph class $\mathcal{C}$ can be certified locally.

When $G \in \mathcal{C}$ the prover gives:

- map of graph with identifiers

Nodes check:

- Same map as neighbors
- Consistent with local view
- Graph given belongs to $\mathcal{C}$.

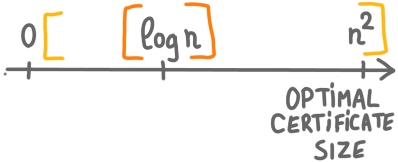


(Unique identifiers are essential here to avoid symmetry issues.)

Certificate size

Question: For a class $\mathcal{C}$, what is the minimum certificate size?

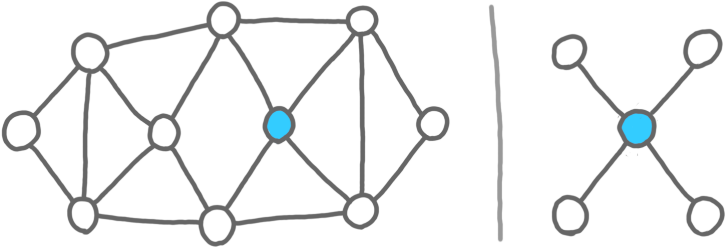
Previous slide $\rightarrow$ Upper bound of $O(n^2)$ bits.



- For some classes $O(1)$ bits suffice.
  $\rightarrow$ degree-3 graphs, 3-colorable graphs.
- For some classes $\Omega(n^2)$ is needed
  $\rightarrow$ Symmetric graphs, non-3-colorable graphs.
- A key size is $\Theta(\log n)$ (aka "compact local certifications").
  $\rightarrow$ Trees, *planar graphs*.

Certifying planarity

Theorem: We can certify planar graphs with $O(\log n)$ bits.



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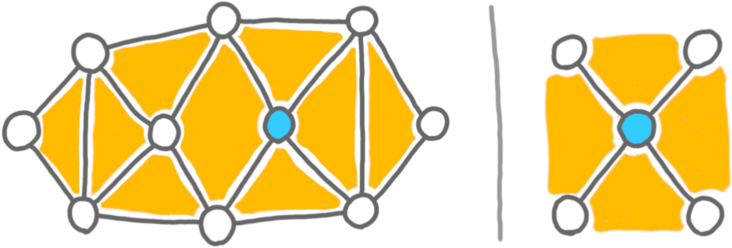
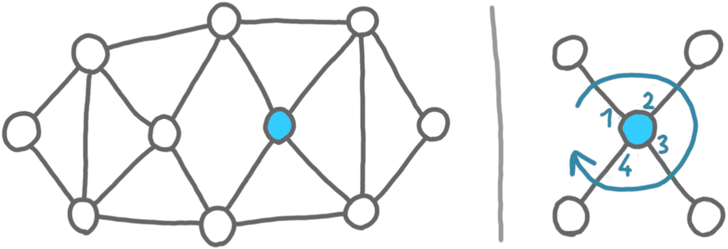


Part 1: Rotational system



- First component: orientations of the edges around each node.

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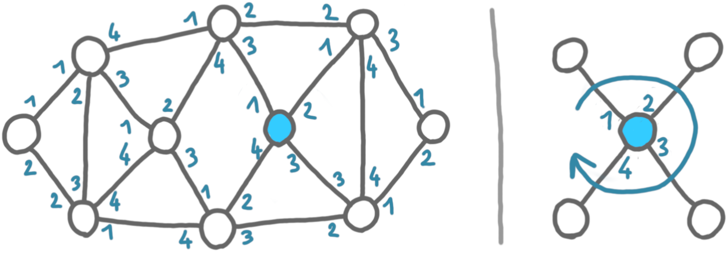

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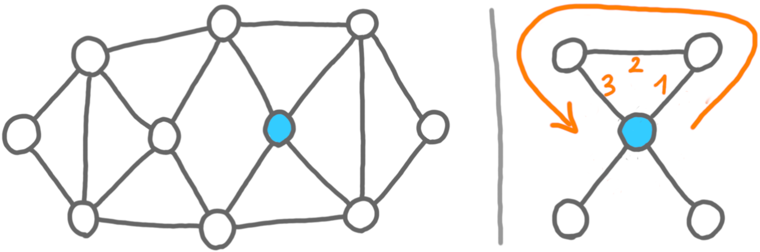

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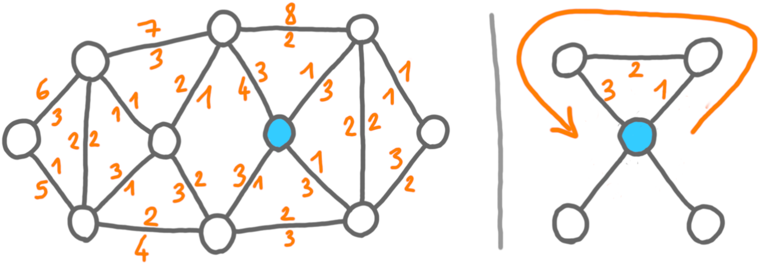

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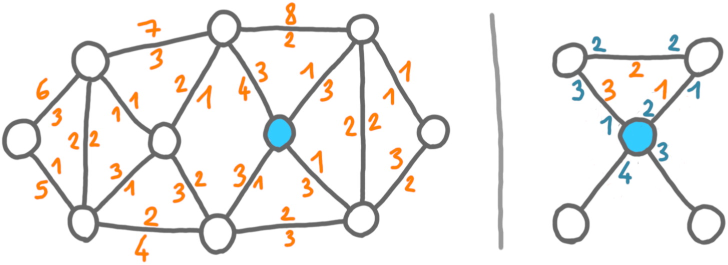
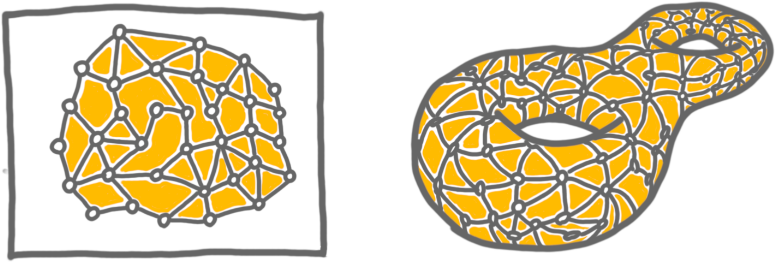


- First component: orientations of the edges around each node.
- Second component: orientations of the edges around each *face*.
- The consistency a system can be checked locally
  $\rightarrow$ Good for certification, if we allow edge certificates.

Part 2: Checking Euler formula



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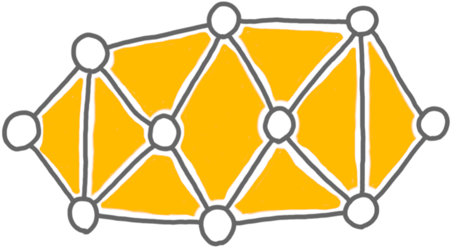

G is planar iff

$|V| - |E| + |F| = 2$

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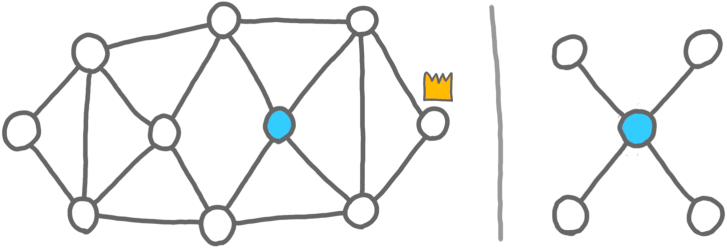

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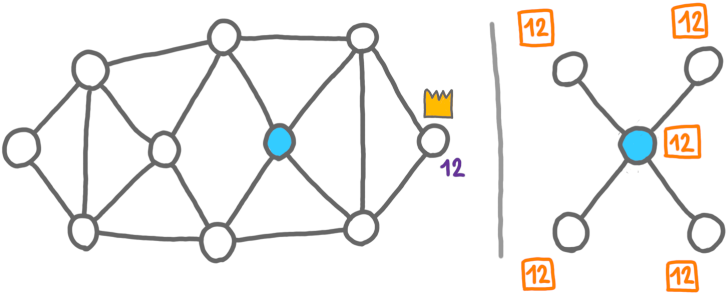

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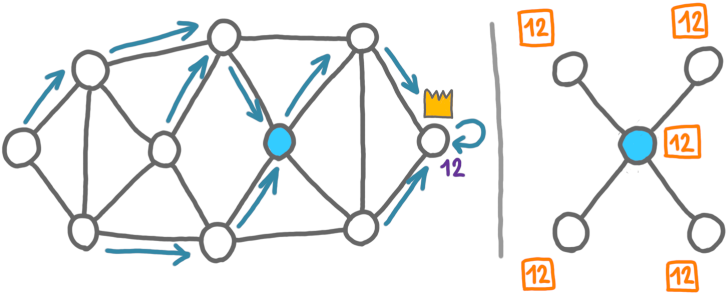

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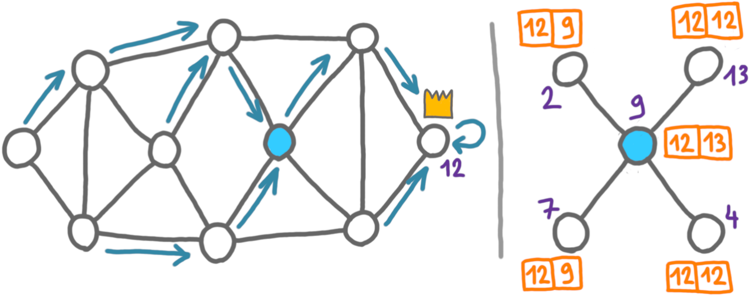

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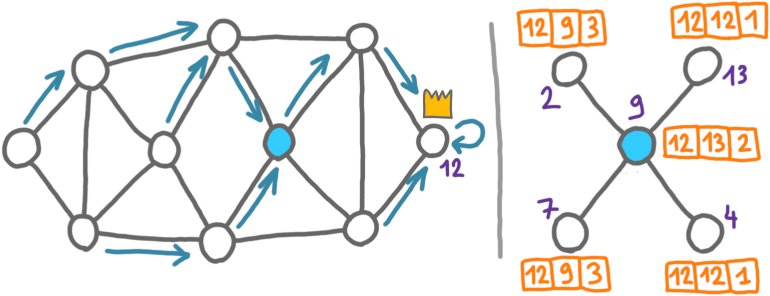

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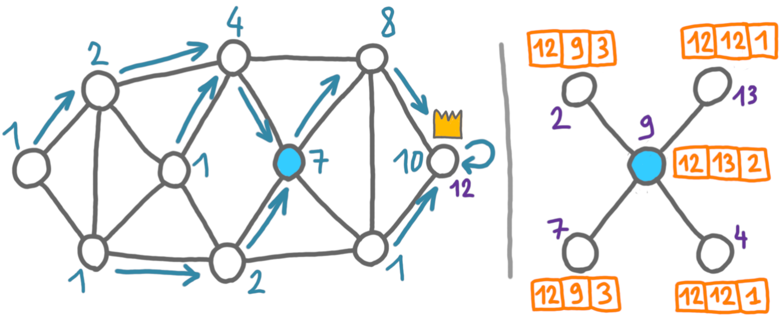

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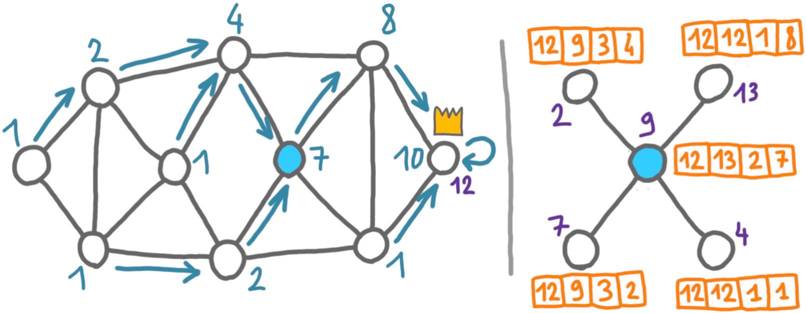

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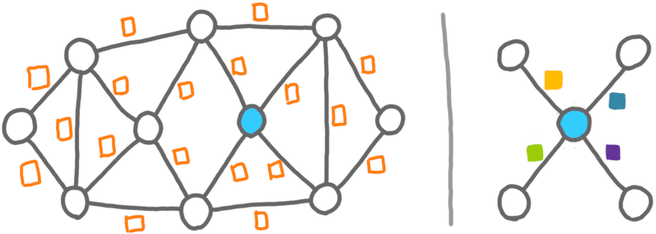

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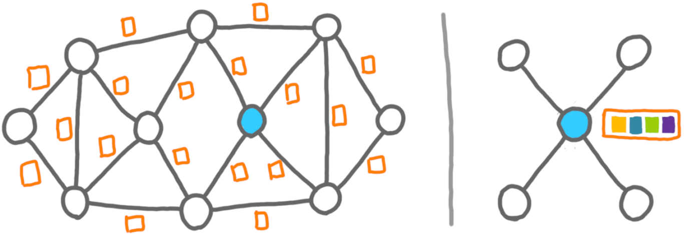

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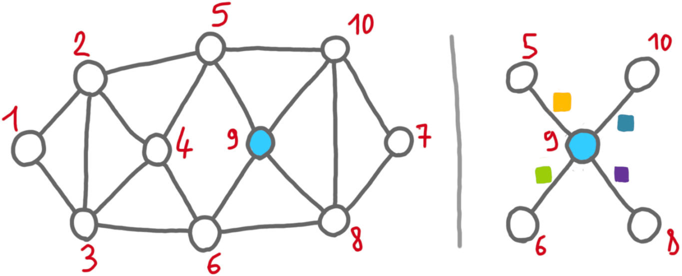

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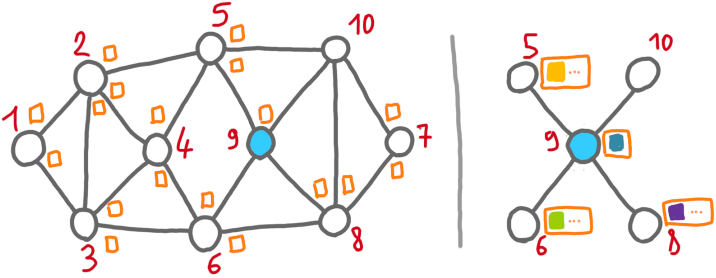
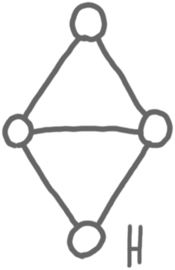
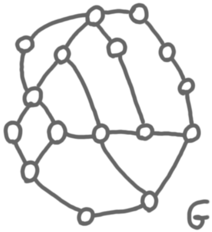


- Rotational systems naturally translate into edge certificates.
- Duplicating on both endpoints $\rightarrow$ size $=$ max-degree $\times O(\log n)$
- Planar graphs have degeneracy ≤ 6.
- Each edge certificate goes to the smallest-index vertex.
  $\rightarrow$ Ok for the verification phase, and $O(\log n)$ certificates!

Forbidden minors





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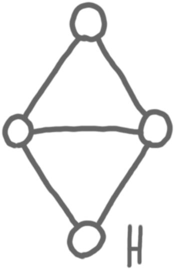

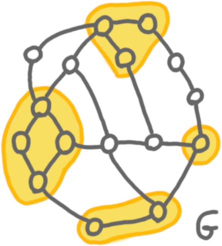

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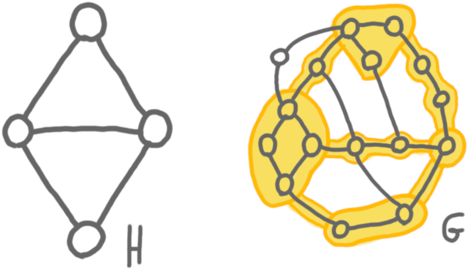

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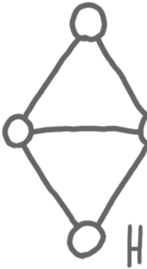

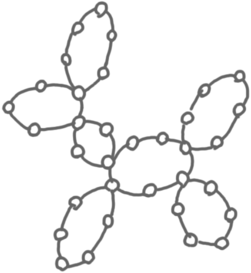

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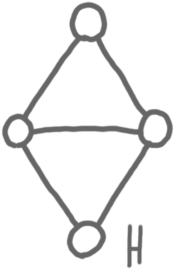

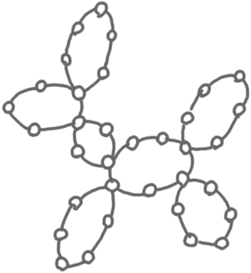

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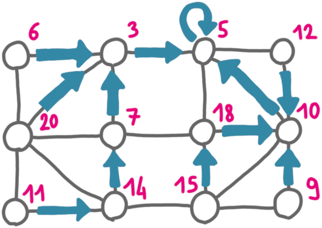
Forbidden minors

- Planar graphs = graphs that do not have $K_5, K_{3,3}$ as minors.
- **Open question:** Is it true that every class defined by excluded minors can be certified with $O(\log n)$ bits?
- Known true for bounded-genus, small minors, planar minors.

Origin: distributed computing



- A classic problem: Compute a spanning tree.

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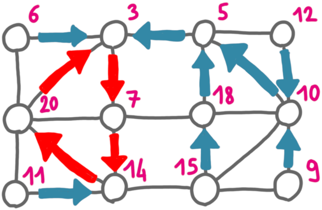

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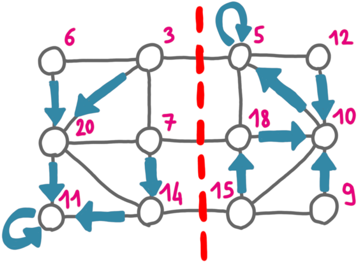

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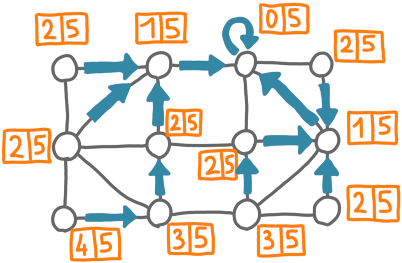

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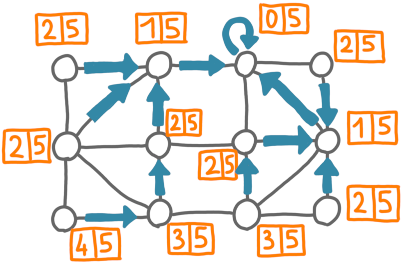
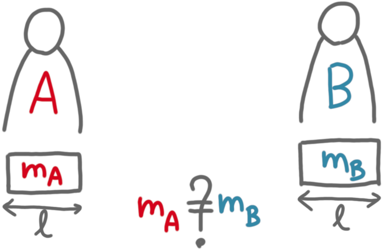


- A classic problem: Compute a spanning tree.
- Fault-tolerance: be able to detect locally if the tree is broken.
- Impossible if only the pointers are kept in memory, but possible if one also keeps ID of and distance to the root.
- Algorithms that can cope with such faults are called *self-stabilizing*.

Tool: Communication complexity



- How much communication to compute a function, when the input is distributed among several players.

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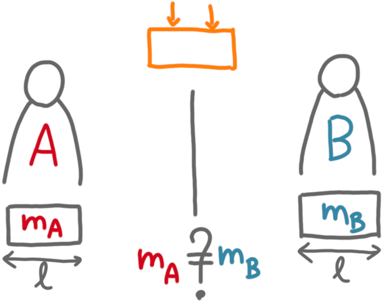

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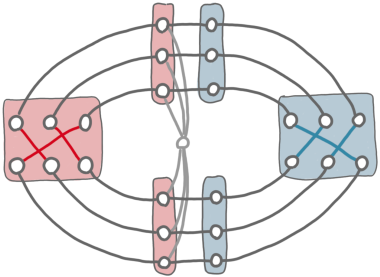

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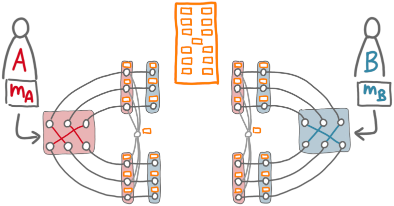


- How much communication to compute a function, when the input is distributed among several players. (Non-deterministic variant.)
- Tailored instances for reductions.
- Each player gets one part of the graph. Argue about certificate size at boundary.

Analogy: Complexity theory

Class NP:

There exists a polytime verification algorithm A such that:

Input is correct	$\Leftrightarrow$	Exists c such that $A(c, \mathit{input})$ accepts.

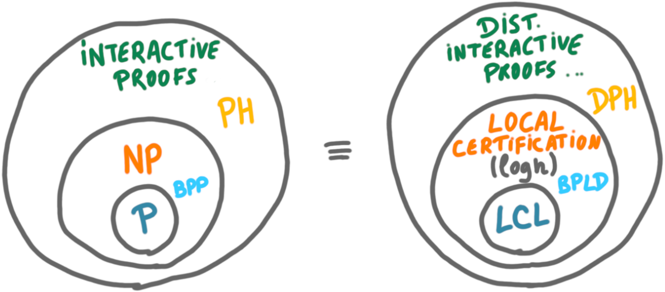
Local certification:

There exists a local sanity check A such that:

Input is correct	$\Leftrightarrow$	Exists $c : V \to$ labels such that $A(c, \mathit{input})$ accepts at every node.

Analogy: Complexity theory

One can define many other analogues of the classic complexity classes: probabilistic classes, interactive proofs, zero-knowledge, polynomial hierarchy etc.



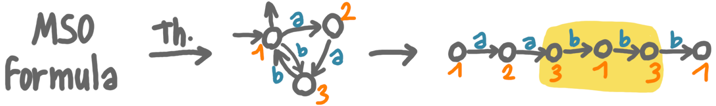
Analogy: Model checking

General model checking approach: Check efficiently that some restricted properties on restricted structures.

Courcelle theorem: Any MSO formula can be checked in polynomial-time in graphs of bounded treewidth.

Recent analogues: Any MSO formula can be certified with $O((poly)\log n)$ bits in graphs on graphs of bounded treedepth/treewidth/cliquewidth.

Techniques: Kernelization, automata theory.



Wrapping up

- Local certification is about checking locally a graph property (or data structure), thanks to certificates.
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Thanks for your attention!

(I cannot be around for the rest of the week, do not hesitate to send me an email for additional questions!)