Introduction to local certification

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Landscape

DISTRIBUTED COMPUTING

COMES FROM

TOOL

COMMUNICATION COMPLEXITY

GRAPH THEORY

NEW LENS ON

ANALOGUES

MODEL CHECKING

ANALOGUES

COMPLEXITY THEORY

LOCAL CERTIFICATION
Distributed perspective on graphs

- The graph represents a network.
- Nodes are machines.
- Edges are communication channels.
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- Unique identifiers.
- In this talk:
  - Every node sees its neighbors,
  - runs the algorithm,
  - outputs a binary decision.
Local recognition of graph classes

Let $C$ be a class of connected graphs (e.g. planar graphs)

**Local recognition of $C$:** A local decision algorithm such that:

- If $G \in C$ then all the vertices accept.
- If $G \notin C$ then at least one vertex rejects.

**Examples:**

- Graphs of degree 3 can be recognized locally.
- For any set $S \subseteq \mathbb{N}$, graph with all degrees in $S$ can be recognized locally.
- Trees cannot be recognized locally
Quick proof for trees

- Consider a graph with a unique cycle that is too large to fit in the view of a node.
- For correctness, at least one node $v$ rejects.
- Now, remove an edge far from $v$, to cut the cycle. Rerun the checking. Node $v$ still rejects. Contradiction.
Introducing local certification

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▶ **Idea:** Use additional information, in the form of labels.
▶ For trees: the distances from an arbitrarily chosen node.
▶ Sanity check at each node: the distances locally make sense.
▶ **Key property:** if the graph has a cycle, at least one node will reject.
Definition: A local certification of a graph class $\mathcal{C}$ is a local decision algorithm such that:

1. For $G \in \mathcal{C}$, there exists certificate assignment that makes all vertices accept.
2. For $G \notin \mathcal{C}$, for any certificate assignment, at least one vertex rejects.

Story:

- A prover is trying to convince all nodes that the graph is in class $\mathcal{C}$ (which might be true or false).
- If the graph is indeed in the class, it succeeds.
- If the graph is not in the class, it cannot succeed.
Any class can be certified

**Theorem:** Any graph class $C$ can be certified locally.

When $G \in C$ the prover gives:
- map of graph with identifiers

Nodes check:
- Same map as neighbors
- Consistent with local view
- Graph given belongs to $C$.

(Unique identifiers are essential here to avoid symmetry issues.)
Certificate size

Question: For a class $C$, what is the minimum certificate size?

Previous slide $\rightarrow$ Upper bound of $O(n^2)$ bits.

- For some classes $O(1)$ bits suffice.
  $\rightarrow$ degree-3 graphs, 3-colorable graphs.
- For some classes $\Omega(n^2)$ is needed
  $\rightarrow$ Symmetric graphs, non-3-colorable graphs.
- A key size is $\Theta(\log n)$ (aka ”compact local certifications”).
  $\rightarrow$ Trees, planar graphs.
Certifying planarity

**Theorem:** We can certify planar graphs with $O(\log n)$ bits.
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Part 1: Rotational system

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- Second component: orientations of the edges around each face.
- The consistency a system can be checked locally → Good for certification, if we allow edge certificates.
Part 2: Checking Euler formula

- A rotational system $\rightarrow$ local embedding. Maybe not planar.
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- Euler formula characterizes planar embeddings.

\[ |V| - |E| + |F| = 2 \]

\( G \) is planar iff
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Each edge certificate goes to the smallest-index vertex. $\rightarrow$ Ok for the verification phase, and $O(\log n)$ certificates!
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- Each edge certificate goes to the smallest-index vertex. → Ok for the verification phase, and \( O(\log n) \) certificates!
Forbidden minors

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- **Open question:** Is it true that every class defined by excluded minors can be certified with $O(\log n)$ bits?
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- **Open question:** Is it true that every class defined by excluded minors can be certified with $O(\log n)$ bits?
- Known true for bounded-genus, small minors, planar minors.
Origin: distributed computing

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- Fault-tolerance: be able to detect locally if the tree is broken.
- Impossible if only the pointers are kept in memory, but possible if one also keeps ID of and distance to the root.
- Algorithms that can cope with such faults are called self-stabilizing.
Tool: Communication complexity

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- Tailored instances for reductions.
- Each player gets one part of the graph. Argue about certificate size at boundary.
**Analogy: Complexity theory**

**Class NP:**
There exists a **polytime** verification algorithm $A$ such that:

Input is correct $\iff$ Exists $c$ such that $A(c, \text{input})$ accepts.

**Local certification:**
There exists a **local** sanity check $A$ such that:

Input is correct $\iff$ Exists $c : V \rightarrow \text{labels}$ such that $A(c, \text{input})$ accepts at every node.
Analogy: Complexity theory

One can define many other analogues of the classic complexity classes: probabilistic classes, interactive proofs, zero-knowledge, polynomial hierarchy etc.
Analogy: Model checking

General model checking approach: Check efficiently that some restricted properties on restricted structures.

Courcelle theorem: Any MSO formula can be checked in polynomial-time in graphs of bounded treewidth.

Recent analogues: Any MSO formula can be certified with $O((\text{poly}) \log n)$ bits in graphs on graphs of bounded treedepth/treewidth/cliquewidth.

Techniques: Kernelization, automata theory.
Wrapping up

- Local certification is about checking locally a graph property (or data structure), thanks to certificates.
- It originates from the study of fault-tolerance in distributed computing.
- It is connected to several other areas of TCS.
- There are still exciting open questions!
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- It is connected to several other areas of TCS.
- There are still exciting open questions!

Thanks for your attention!

(I cannot be around for the rest of the week, do not hesitate to send me an email for additional questions!)