What can be certified compactly?

Compact local certification of MSO properties in tree-like graphs

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Unofficial subsubtitle:

Distributed computing on graphs
meets model checking
meets parameterized complexity

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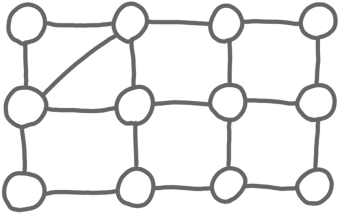
Disclaimer: citations are at the end, send me emails if you want pointers.

Distributed computing motivation

A **model of distributed computing**:

- Network represented by a graph.
- Computation step: a vertex accesses the state of its neighbors and updates its state.



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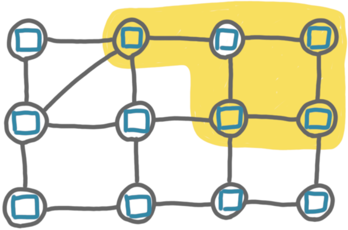

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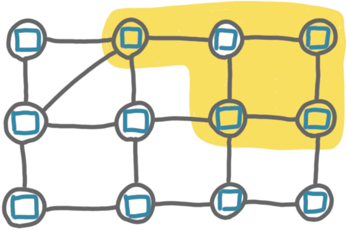
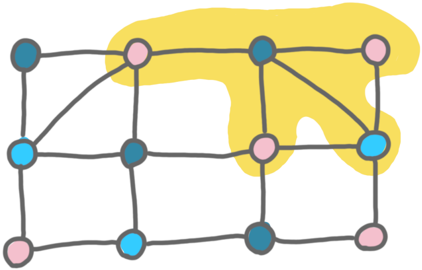


- There are faults: some set of states can be modified arbitrarily.
- Goal: converge to a correct solution (eg a spanning tree).

Distributed computing motivation

A simpler problem: Check correctness.
If not correct: a node should raise an alarm.

Examples: 3-coloring and spanning tree



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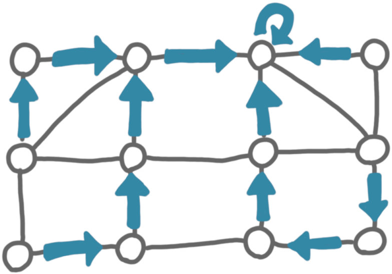

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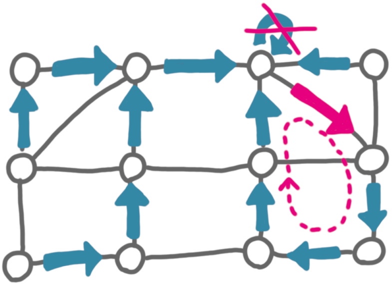

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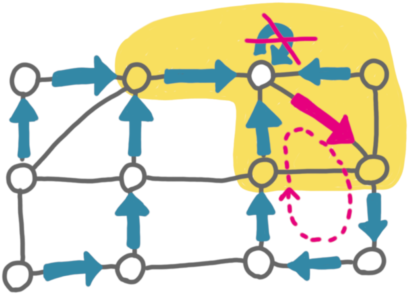

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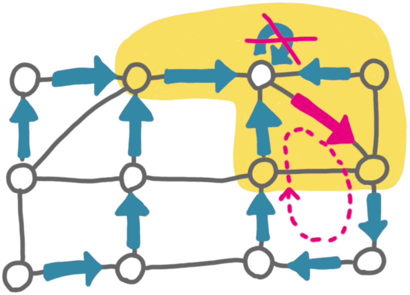

Problem: Acyclicity is a global property that cannot be checked locally.

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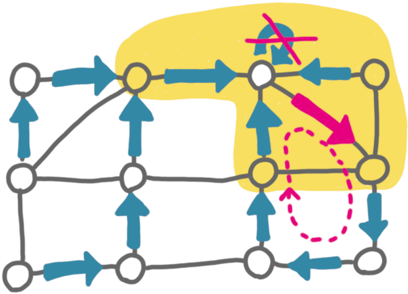


Problem: Acyclicity is a global property that cannot be checked locally.
Solution: Keep additional information to certify the correctness.

Moving towards graphs

Now: checking that the graph belongs to a given class.

Let $\mathcal{C}$ be a class of connected graphs.

A local decision algorithm: a mapping from the neighborhoods at distance d ($d = 1$ here) to *accept* or *reject*.

Local recognition of $\mathcal{C}$: A local decision algorithm such that:

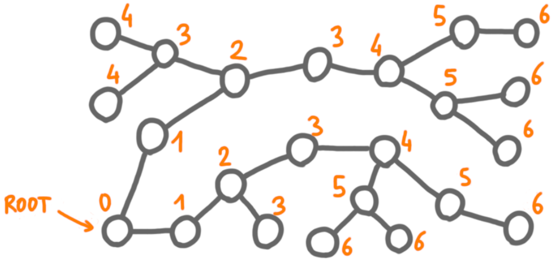
- If $G \in \mathcal{C}$ then all the vertices accept.
- If $G \notin \mathcal{C}$ then at least one vertex rejects.

Doable: Locally recognize cycles. (Check degree=2)
Not doable: Locally recognize paths.

Local certification (intuition)

$\mathcal{C}$ = trees

Imagine the graph comes with labels, that are supposed to be the distances to a root.



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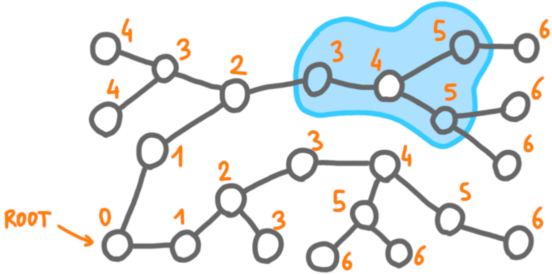

A local decision algorithm: accept iff the distances are consistent.

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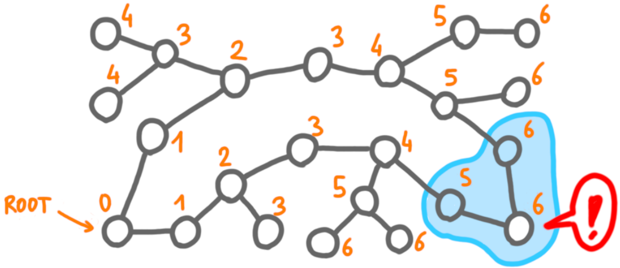


A local decision algorithm: accept iff the distances are consistent.

Fact: there exists a labeling that is consistent everywhere iff the graph is a tree.

Local certification (definition)

A local certification (of size k) for a class $\mathcal{C}$ is
a local decision algorithm such that :

1. For $G \in \mathcal{C}$, **there exists** certificate assignment (of k-bit labels) that makes **all vertices** accept.
2. For $G \notin \mathcal{C}$, **for any** certificate assignment (of k-bit labels), **at least one vertex** rejects.

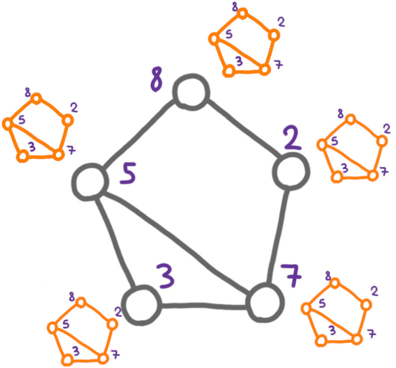
Several points of view on the notion:

- Fault-tolerance, self-stabilization.
- A distributed analogue of NP.
- Extension to the space of labeled graphs s.t.: the checking is local in this space, and the projection to unlabeled is $\mathcal{C}$.

Can we certify any graph class?

Question: Take a graph class $\mathcal{C}$ (*i.e.* an infinite set of graphs). Does there exists a local certification?

Answer: Yes (with the help of identifiers). Size: $\Theta(n^2)$.

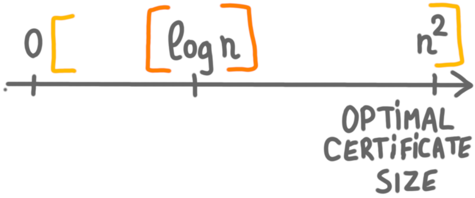


Optimal certificate size

Measure of quality: the size of the certificates.

Example: for trees,

- the optimal certificate size for trees is ≥ 1, and $\leq O(n^2)$,
- the distance labeling gives $O(\log n)$
- $O(\log n)$ is actually optimal



→ The (optimal) certificate size is a **measure of locality**.

Landscape of certificate sizes

Compact certification $= O(\log n)$ certificates (or polylog(n)):

- k-colorable graphs: $O(\log k)$
- paths, trees: $\Theta(\log n)$
- planar, bounded-genus: $\Theta(\log n)$

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Terribly non-compact certification:

- diameter ≤ 3: $\tilde{\Theta}(n)$
- non k-colorable: $\tilde{\Theta}(n^2)$
- symmetric graphs: $\Theta(n^2)$ (and $\tilde{\Theta}(n)$ for symmetric trees.)

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Intriguing open question: Does every minor-closed class admit a compact certification?

What about meta-theorems?

Courcelle's theorem: Any MSO property can be decided in linear time when the treewidth of the graph is bounded.

MSO properties: Logical formula built with:
$(u, v) \in E, \exists u, \forall u, \exists S \subset V, \forall S \subset V$, and usual connectors.

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Theorem we are looking for: Any property XXX can be certified with $O(\log n)$ labels, when the YYY of the graph is bounded.

Note: restricting both the logic and the structure is also necessary for compact certification:

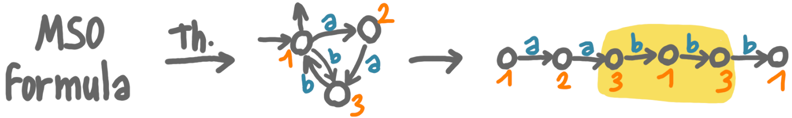
- Diameter > 3 is in FO, but certificate size is $\tilde{\Omega}(n)$
- Symmetric trees are... trees, but certificate size is $\tilde{\Omega}(n)$

On trees

Theorem : Any MSO property can be certified with $O(1)$ bits in trees.

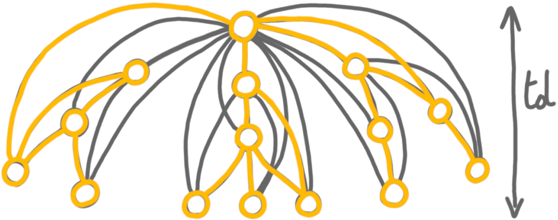
Technique on edge-labeled oriented paths (= words):

- Known theorem: MSO on path = regular languages
- Use the automaton states as labels.



On bounded-treedepth

Theorem : Any MSO property can be certified with $O(\log n)$ bits in graphs of bounded treedepth.



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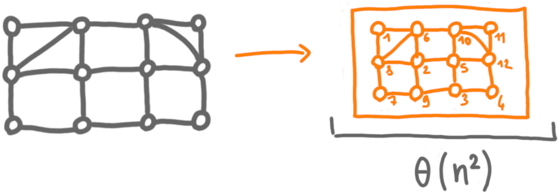

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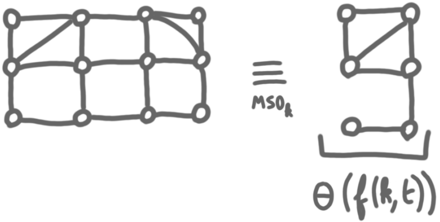

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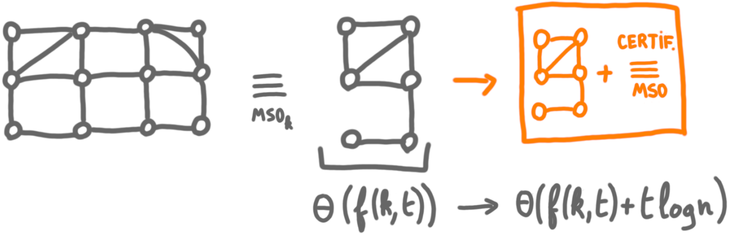

Other results and open questions

- An analogue for treewidth has been proved with certificates of size $O(\log^2 n)$ bits. Is it optimal?
- Other trade-offs between expressivity/structure/certification?
- Certifying all minor-closed classes in $O(\log n)$?
- What about completely different type of classes, like unit-disks?

Bibliographic pointers

Local certification papers mentioned:

- Proof-labeling schemes (Korman, Kutten, Peleg - 2010). doi:10.1007/s00446-010-0095-3
- Memory-efficient self stabilizing protocols for general networks (Afek, Kutten, Young - 1990). doi:10.1007/3-540-54099-7_2
- Locally checkable proofs in distributed computing (Göös, Suomela - 2016). doi:10.4086/toc.2016.v012a019

Tutorial on local certification

- Introduction to local certification (Feuilloley - 2021). doi:10.46298/dmtcs.6280 + Gem talk at PODC (on youtube).

Bibliographic pointers

Certification of planar and bounded-genus graphs

- Compact distributed certification of planar graphs (Feuilloley, Fraigniaud, Montealegre, Rapaport, Rémila, Todinca, 2021) doi:10.1007/s00453-021-00823-w + Talks at PODC by Montealegre
- Local Certification of Graphs with Bounded Genus (Same as above.) arxiv:2007.08084
- Local certification of graphs on surfaces (Esperet, Leveque - 2021) arxiv:2102.04133

Small diameter lower bound

- Approximate proof-labeling schemes (Censor-Hillel, Paz, Perry - 2020) doi:10.1016/j.tcs.2018.08.020

Bibliographic pointers

Certification of H-minor-free graphs

- Local certification of graph decompositions and applications to minor-free classes (Bousquet, Feuilloley, Pierron - 2021) arxiv:2108.00059 + BA at DISC.

Other specific classes

- Compact Distributed Interactive Proofs for the Recognition of Cographs and Distance-Hereditary Graphs (Montealegre, Ramírez-Romero, and Rapaport - 2021) arxiv:2012.03185 (+ personal communication)

MSO on bounded treewidth

- A Meta-Theorem for Distributed Certification (Fraigniaud, Montealegre, Rapaport, Ioan Todinca - 2022)