Local certification of graph decompositions and applications to minor-free classes

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This talk

In this paper, we make progress on the following question:

Question: Does every graph class characterized by forbidden minors have a compact local certification?

In this talk, I will:

- Explain local certification.
- Explain graph minors.
- Give a quick overview of our techniques.

Local checking of the network

A scenario:

- The nodes of a network want to compute a X.
  ($X =$ matching, spanner, coloring, minimum spanning tree ...)
- They are supposed to be in a tree, and they have a fast local algorithm that works only on trees.
- But there are faults, and the network might not be a tree.

$\rightarrow$ How to check locally ($=$ efficiently) that the network is a tree?

More generally: We are interested in checking locally that the graph belongs to some given graph class $\mathcal{C}$.

Local checking of the network

Problem: Local checking is impossible for trees.

More generally: Except for a few graph classes (e.g. regular graphs), it is impossible to locally check the structure.

Local certification

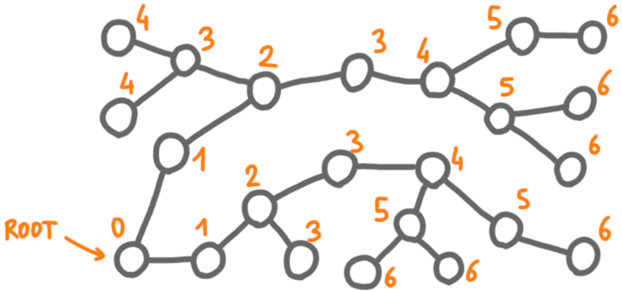
Idea: Have a labeling of the nodes that certifies that the network is in the class. This labeling can be checked locally.

Requirements: Given the local verification algorithm:

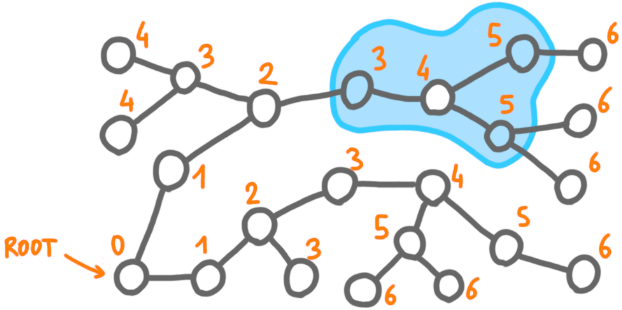
- For every graph in the class, there exists a labeling such that the algorithm accepts.
- For every graph *not* in the class, for every labeling, the algorithm rejects on at least one node.

Measure of performance: The size of the certificates. It is the memory/message complexity, but also a measure of the locality of the class.

Certification of trees



Certification of trees



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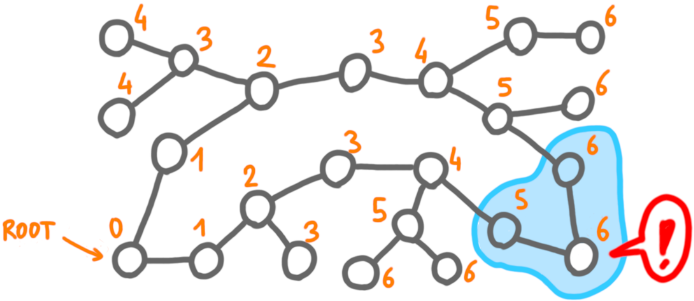
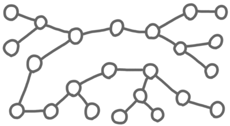
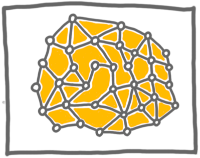
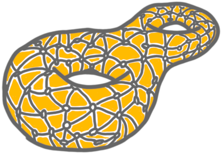


Compact certification

Known results:

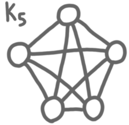
- Every class can be certified with $O(n^2)$-bit certificates.
- Trees can be certified with $O(\log n)$-bit certificates.
- Planar graphs can be certified with $O(\log n)$-bit certificates.
- Bounded-genus graphs can be certified with $O(\log n)$-bit certificates.

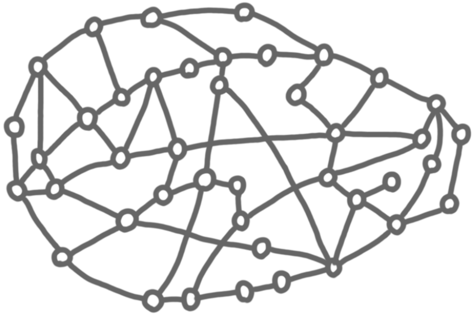




What these classes have in common:
They can be characterized by forbidden minors.

Graph minors

Informal definition: A (large) graph G has a (small) graph H as a minor, if H is hidden in G.





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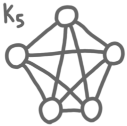

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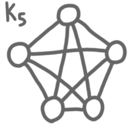


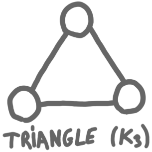
Formal definition: G has H as a minor, if one can transform G into H by vertex and edge deletions, and edge contractions.

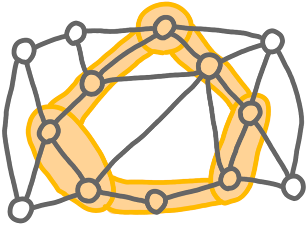
Minor-free graphs

Given a graph H, one can define the class of the graphs that do not have H as a minor.

Examples:

- Trees are exactly the triangle-minor-free graphs.
- Planar graphs are exactly the $(K_5, K_{3,3})$-minor-free graphs.
- Graphs of genus x are exactly the (...)-minor-free graphs.





Question and partial answer

$\rightarrow$ **Can we certify all minor-free classes with** $O(\log n)$ **bits ?**

Weaker version:
Can we certify all minor-free classes with $O(\mathit{poly} \log n)$ bits ?

A reason to hope:
Properties that we know require large labels are non-hereditary.

A reason to doubt:
Such result would imply that these graphs can be constructively described.

Our result:
For small minors, the answer (to the strong version) is positive.
("Small" means $|V| \leq 4$, or $|V| \leq 6$ with additional assumptions.)

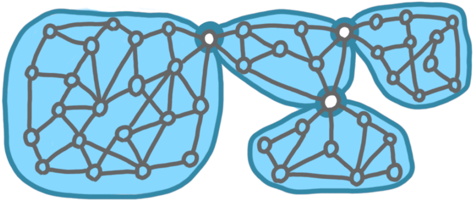
Certification framework

For concreteness: focus on K_4-minor-free graphs.

Theorem: A graph is K_4-minor-free if and only if all its 2-connected components are series-parallel graphs.

Natural approach:

- Certify a decomposition in 2-connected components
- Certify that each component is series-parallel.



Key tool: ear decompositions

(Plain) ear decomposition

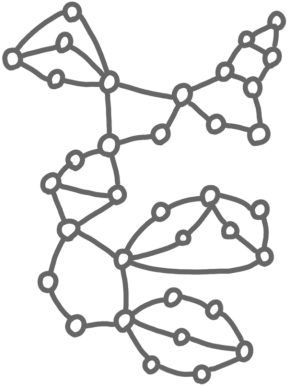
Characterizes 2-connectivity.

Nested ear decomposition.

Characterizes 2-connected series-parallel graphs.

(The dark blue paths are actually closed into cycles.)

Key tool: ear decompositions



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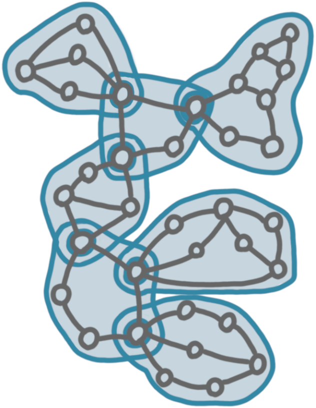

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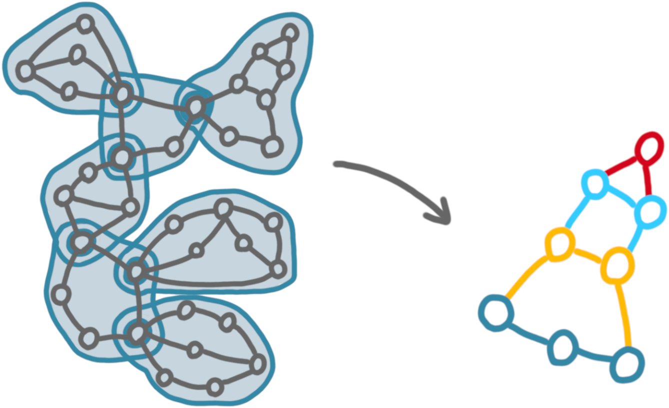

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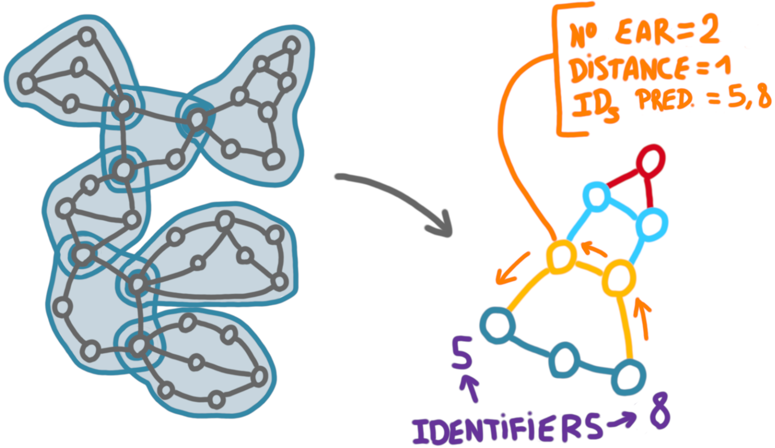
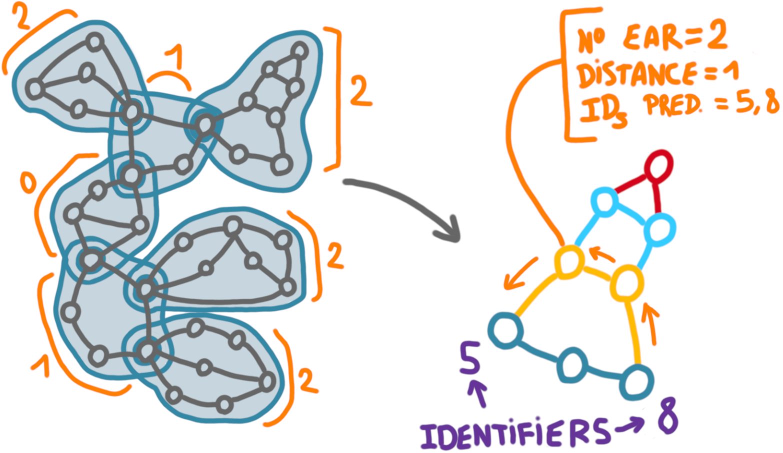


→ a challenge here: avoid congestion at connecting nodes.

Key tool: ear decompositions



Beyond K_4-minor-free graphs

- in C_4-minor-free, the 2-CC are K_2 and K_3,
- in C_5-minor-free, the 2-CC are complete bipartite and small graphs
- in diamond-minor-free, the 2-CC are induced cycles.
- in $K_{2,3}$-minor-free, the 2-CC are basically outerplanar.
- in $K_{2,4}$-minor-free, we have use 3-connectivity and tricky characterizations.
- ...

Our result:
For small minors, the answer (to the strong version) is positive.
("Small" means $|V| \leq 4$, or $|V| \leq 6$ with additional assumptions.)

Open questions

Still wide open:
Can we certify minor-free classes with $O(\log n)$ bits?

Fresh news!
For planar minors, a recent preprint proves a $O(\log^2 n)$ bound.

The simplest open problem:
What about K_5-free graphs?

About the decompositions:
Can we certify k-connectivity? What about lower bounds?