How local constraints influence network diameter

and applications to LCL generalizations

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Warm-up

Caterpillar example: A path, where vertex *i* gets *i* leaves.

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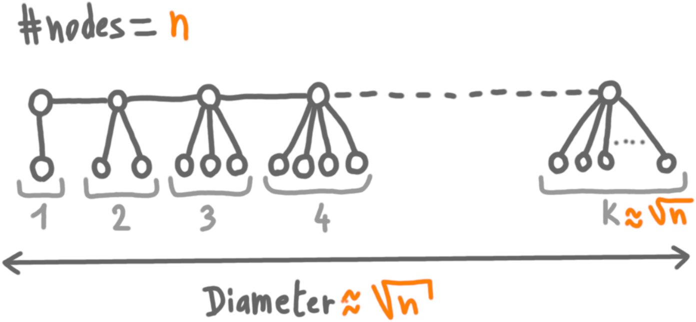

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Local verification algorithm:
For every v: Look at distance 2.
If degree(v)=1, accept.
Otherwise, check:$\leq$ 2 non-leaf neighbors and degrees are consistent.

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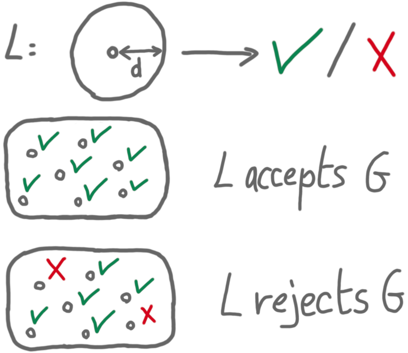
Local checkers

Definition: A *local checker* at distance d is a distributed graph algorithm that:

- Takes a snapshot of its distance-d neighborhood.
- Outputs a binary decision: Accept or reject.

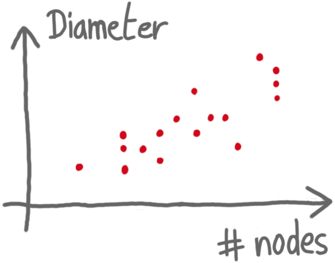
Definition: A local checker L *accepts* a graph G if all nodes output accept.



Focus and question

Definition: $\mathcal{G}(L)$: graphs accepted by L.

Focus: The diameter of $\mathcal{G}(L)$ as a function of n for **trees**.



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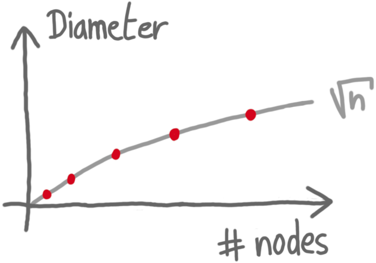

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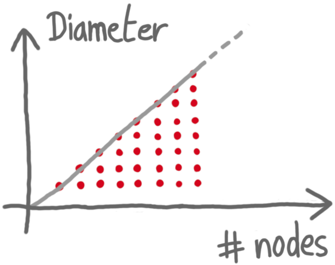

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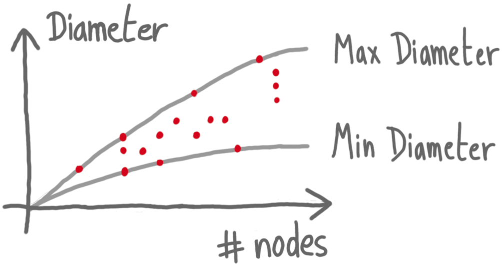

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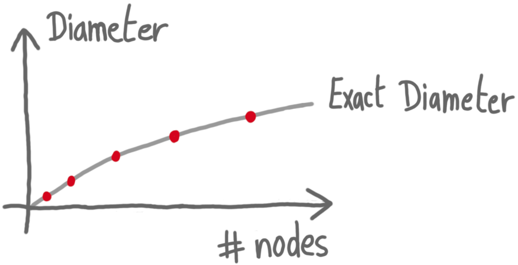

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Definition: $\mathcal{G}(L)$: graphs accepted by L.

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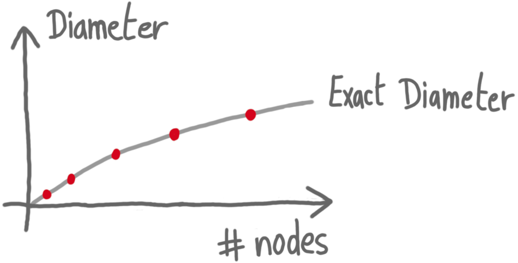
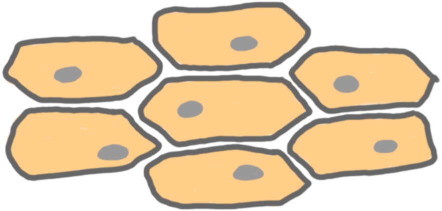
Question: For which functions f is there a local checker L_f, such that $\mathcal{G}(L_f)$ has maximum (resp. minimum, exact) diameter f?



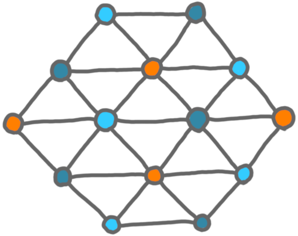
Motivation 1: Modeling living beings

Simple model for how living beings maintain a given shape, without centralized information.



Motivation 2: LCLs beyond bounded degree

Definition: Locally checkable labelings (LCL): problems on bounded degree graphs whose output can be checked locally



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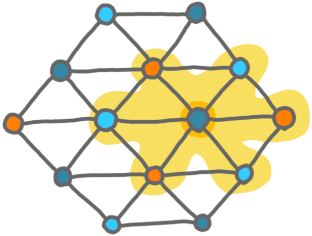

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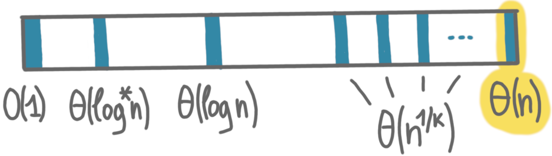

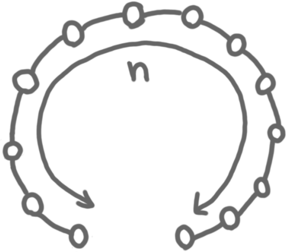

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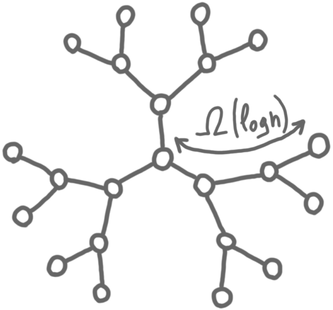

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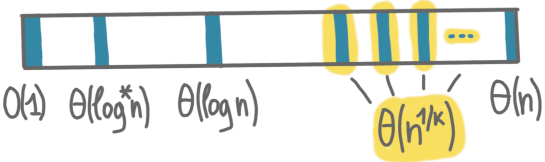
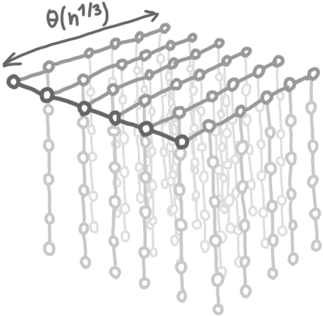
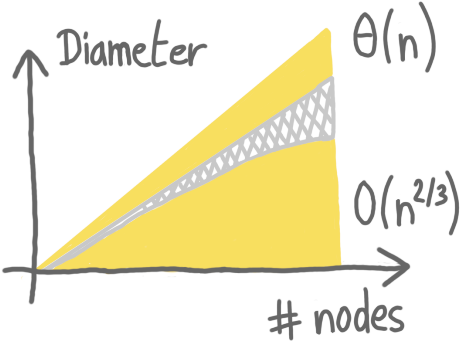




Results on exact diameter

Theorem: For all f in $\Theta(n)$ or $\leq n^{2/3}$, there exists a local checker L at distance 2, such that $\mathcal{G}(L)$ has exact diameter is f.



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- Refining the $\sqrt{n}$ construction.
- Smaller diameter via padding
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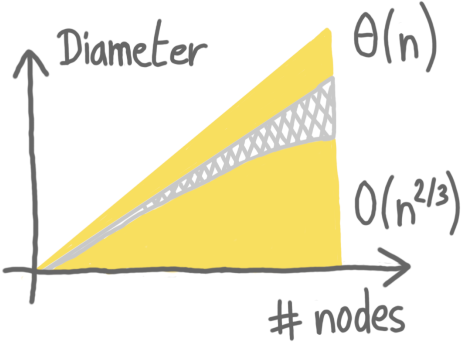

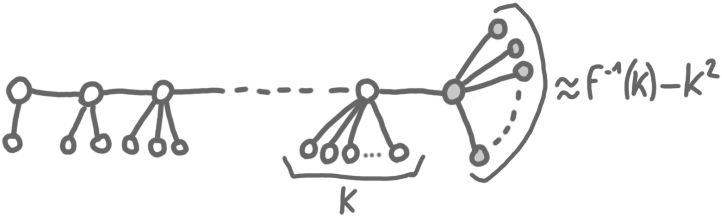

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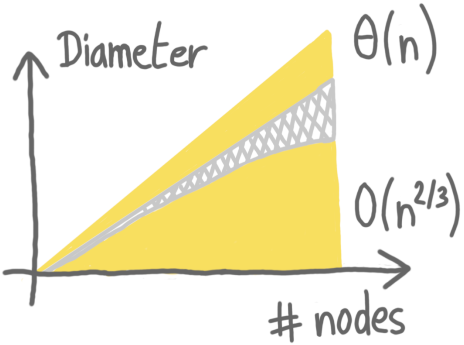

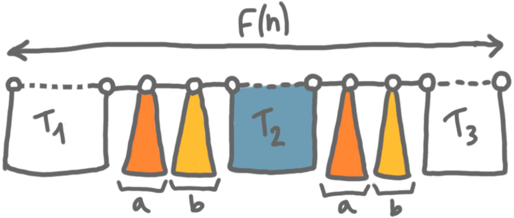

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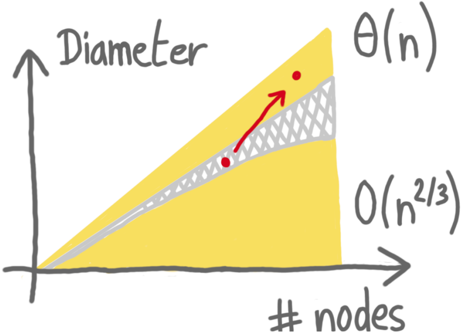

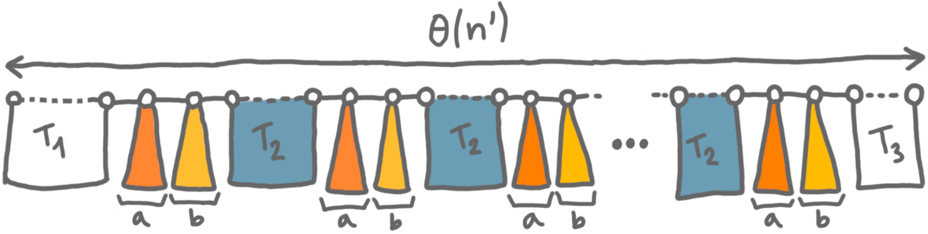

Applications to LCLs

Generalized theorem: For all f in $\Theta(n)$ or $\leq n/\log n$, there exists a local checker L at some distance d, such that $\mathcal{G}(L)$ has exact diameter is f.

$\rightarrow$ **Theorem:** For unbounded degree there exists LCLs with almost all possible complexities (ie there is no real gap in the landscape).

Proof idea: Design the LCL such that the hardest instances are the ones of $\mathcal{G}(L)$, and that on these instances one has to solve a global problem.

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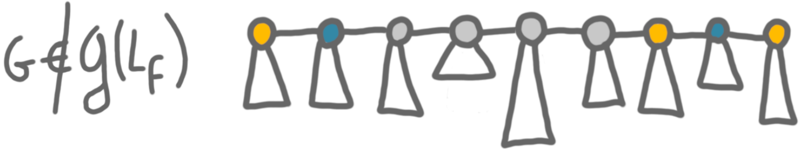

Myopic checkers

Summary: No nice complexity landscape, using arbitrarily complicated checkers.

Question: What about restricted checkers?

Definition (simplified): A local checker is myopic if it rejects as soon as one of the neighbors of the node v has degree different from 1, $deg(v)$, $deg(v) - 1$ and $deg(v) + 1$.

Theorem: For myopic checkers, the maximum diameter is in one of the following regimes: $O(1)$, $\Theta\left(\frac{\log n}{\log \log n}\right)$, $\Theta(\log n)$, $\Theta(\sqrt{n})$ and $\Theta(n)$.

Research directions

- Generalize myopic checkers to handle more cases.
- Study other restrictions with logic characterizations or restricted computational power.
- Understand the diameter of graphs with cycles.
- Global parameters different from diameter (max clique, symmetries, homogeneity...)
- Better models for living beings.