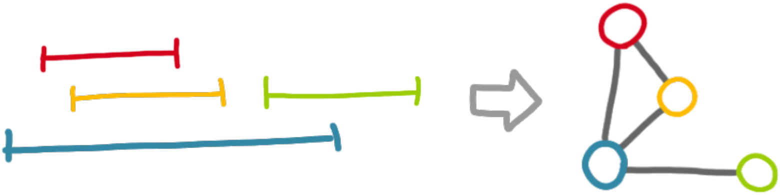
Graph classes and forbidden patterns on three and four vertices

Laurent Feuilloley and Michel Habib

Based on
Graph classes and forbidden patterns on three vertices
(to appear in SIDMA)
and on on-going work.

IRIF Graph Seminar · 24th November 2020

Interval graphs : geometry



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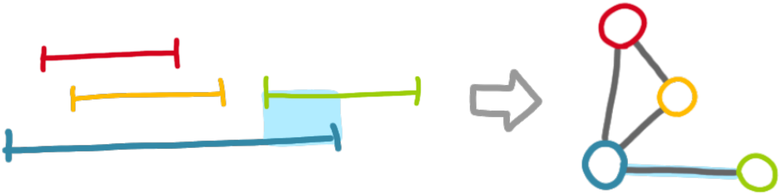

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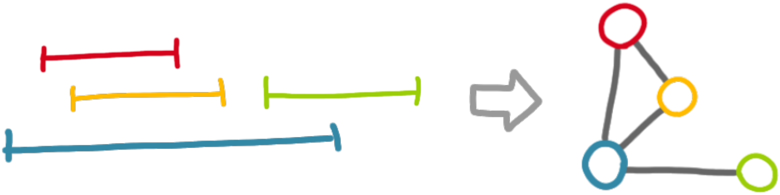

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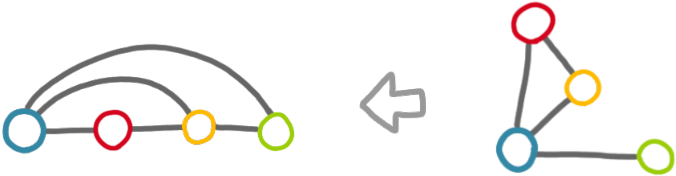

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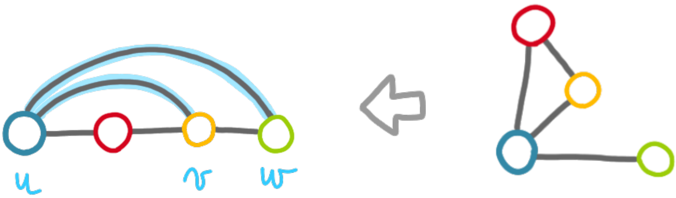

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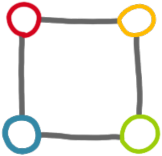

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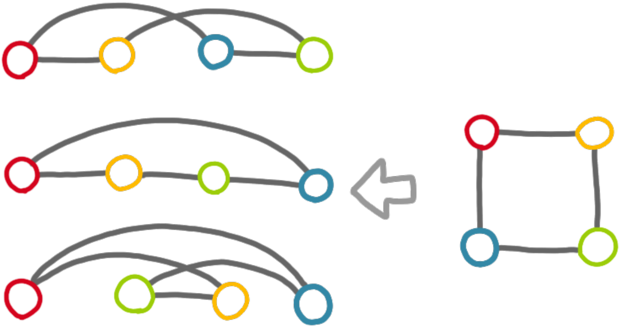

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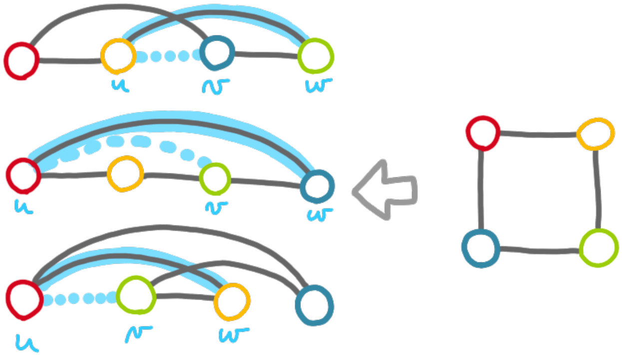

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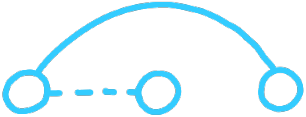

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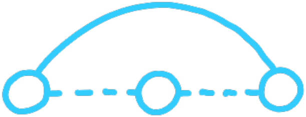

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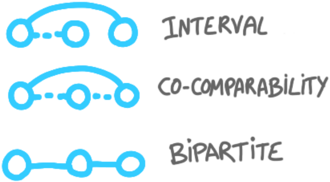

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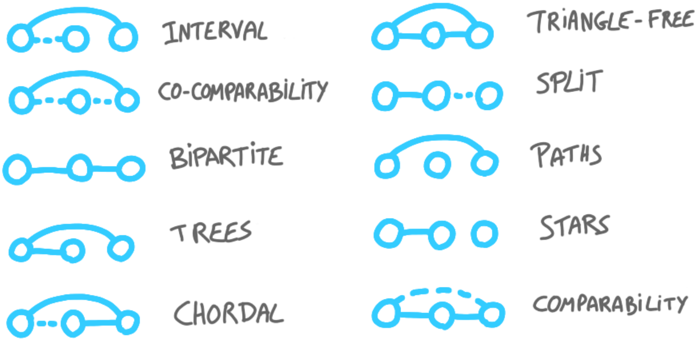

Already noted by Skrien in 82 and Damashke in 90.

Example : bipartite graphs

DEFINITION: A GRAPH IS BIPARTITE IF IT CAN BE SPLIT INTO TWO INDEPENDENT SETS

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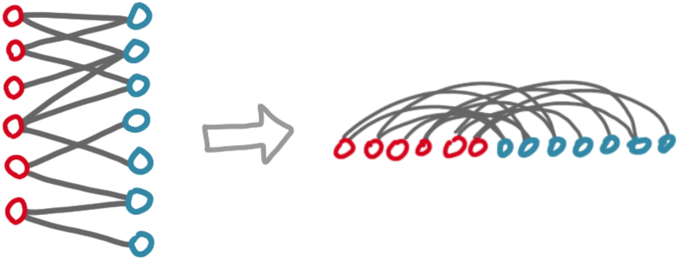

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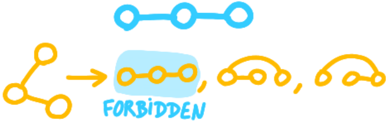

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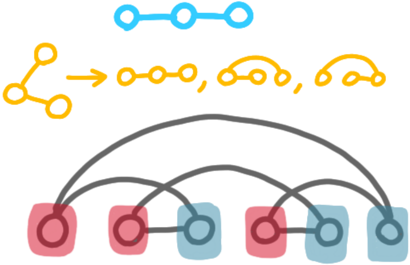

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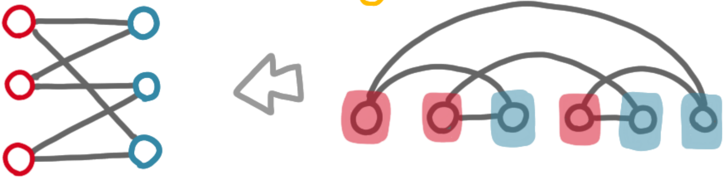
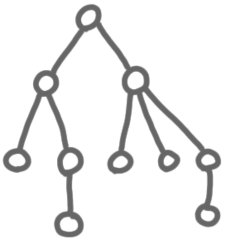


Example : trees

DEFINITION: A GRAPH IS A TREE IF IT IS ACYCLIC (*)



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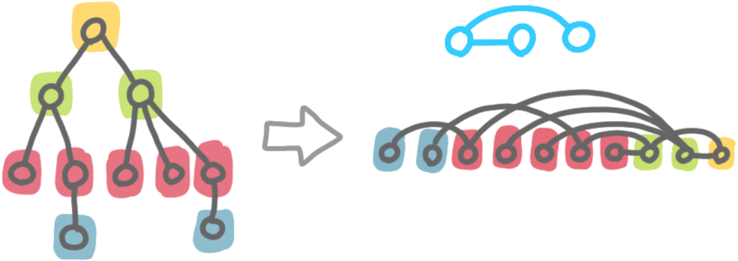

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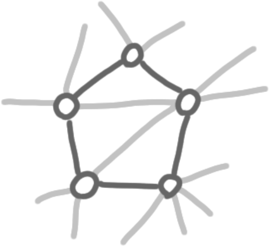

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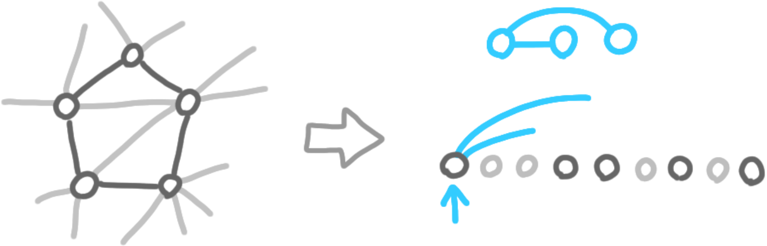

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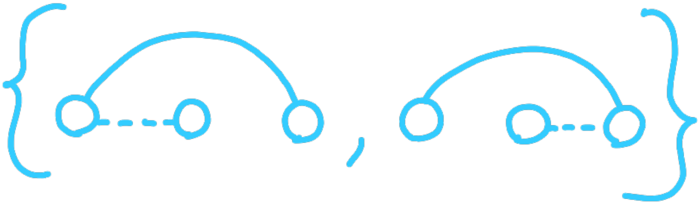

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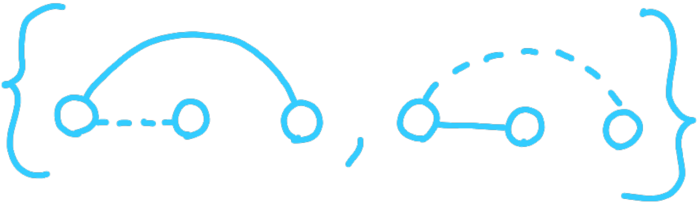


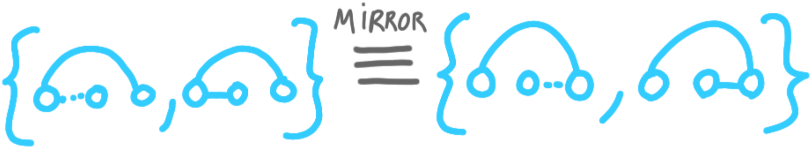
Theorem

Theorem : Up to a few simple operations, the non-trivial classes defined by a set of pattern (on three nodes) are :

1. forests
2. linear forests
3. stars
4. interval
5. split
6. bipartite
7. chordal
8. comparability
9. triangle-free
10. permutation
11. threshold
12. proper interval
13. caterpillar
14. trivially perfect
15. bipartite chain
16. 2-star
17. 1-split
18. augmented clique
19. bipartite permutation
20. triangle-free $\cap$ co-chordal
21. clique
22. complete bipartite

Structure

- Mirror patterns



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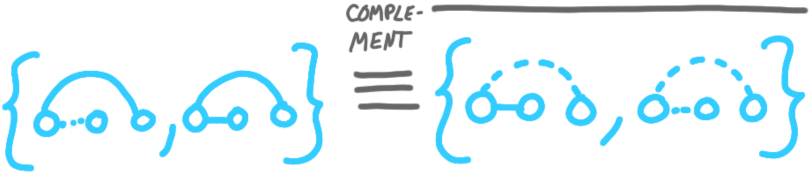

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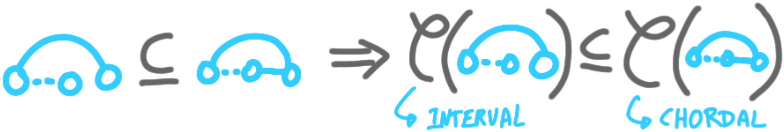

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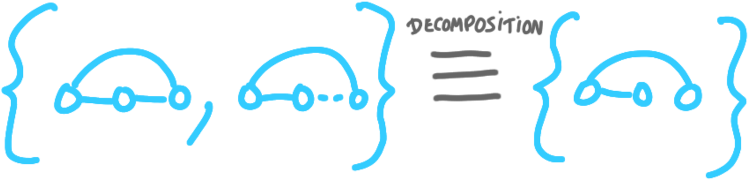
- "Splitting patterns"



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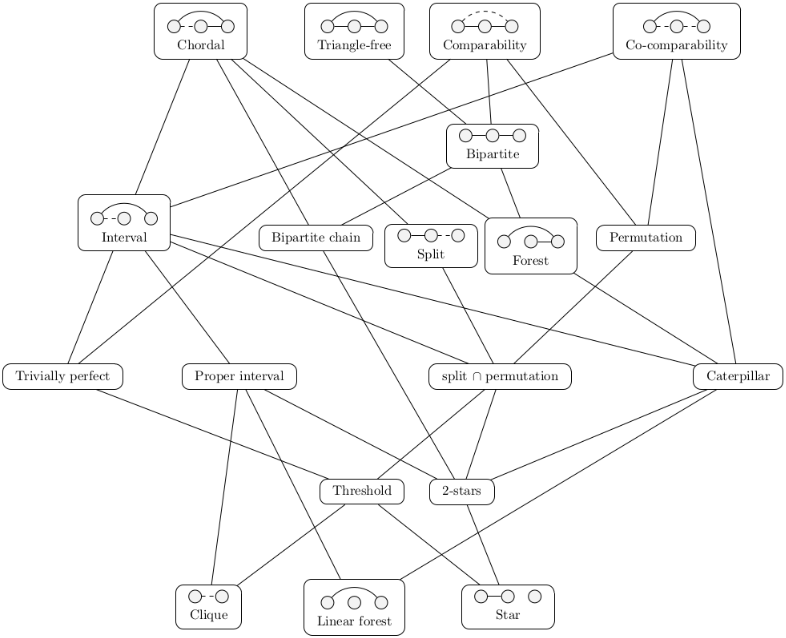

Chordal
Triangle-free
Comparability
Co-comparability
Bipartite
Interval
Bipartite chain
Split
Forest
Permutation
Trivially perfect
Proper interval
split ∩ permutation
Caterpillar
Threshold
2-stars
Clique
Linear forest
Star

Complexity and algorithms

An ordering can be checked in polytime $\rightarrow$ Recognition is in NP.

Theorem (Hell, Mohar and Rafiey) : Every class defined by a set of patterns on three nodes can be recognized in time $O(n^3)$.

New theorem : Every class defined by patterns on three nodes can be recognized in **linear time** except two of them (in time $O(n^{2,37})$), and mostly thanks to **graph traversals**.

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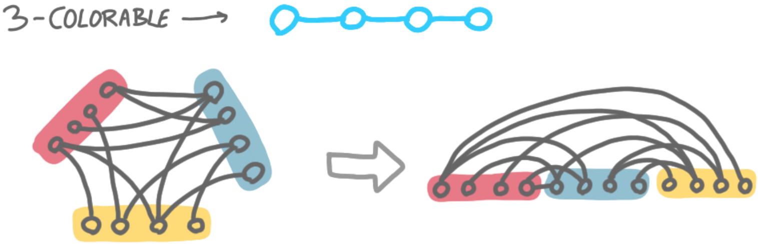

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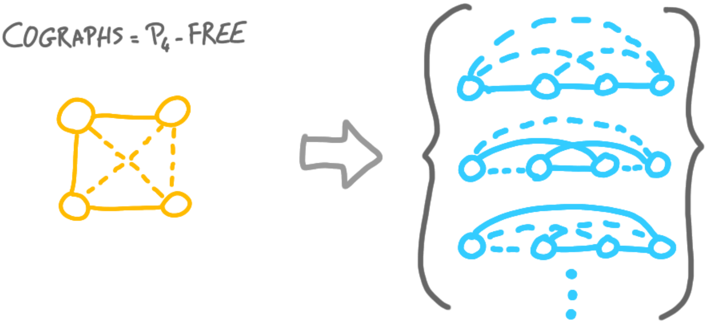

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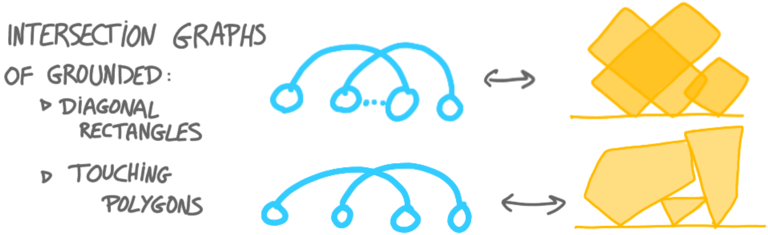
- Grounded intersection graphs



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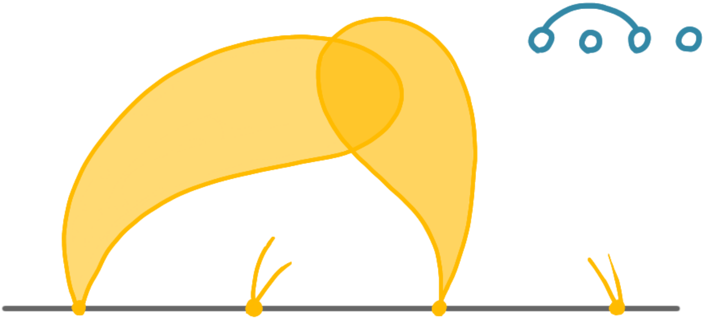

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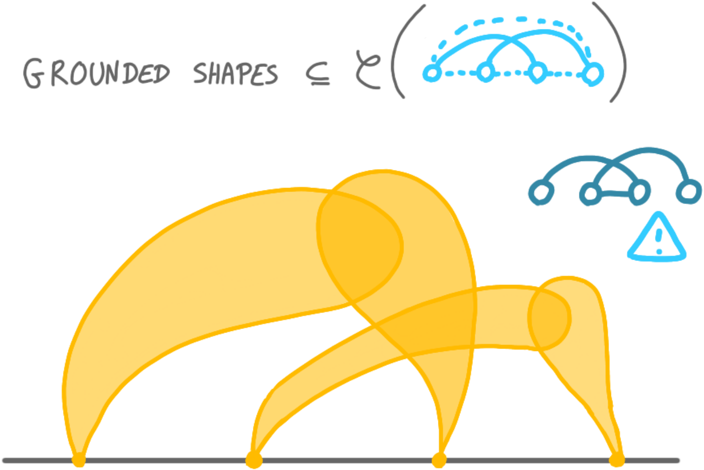

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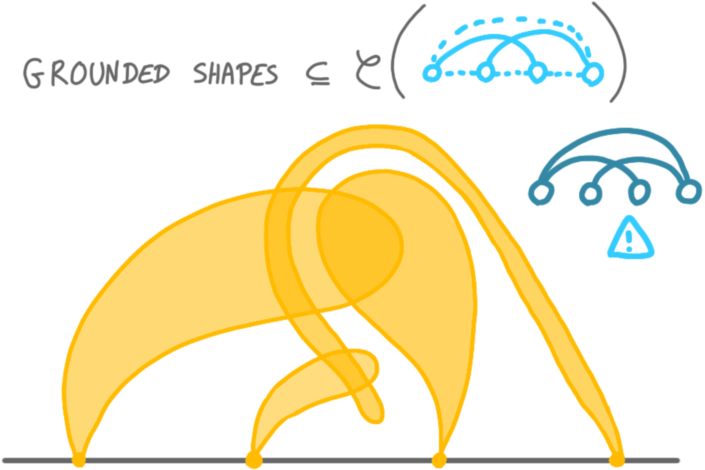

Open problems

A few concrete open problems :

- Clarify the interplay between grounded intersection model and patterns.
- Complexity of recognition of grounded rectangle graphs : P or NP ?
- Find a criterion for deciding the complexity of the class based on its patterns.