Reconfiguration and Combinatorial Games

Marc Heinrich

under the supervision of:

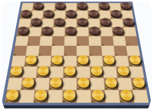
Eric Duchêne	Director
Sylvain Gravier	Co-director
Nicolas Bousquet	Co-supervisor

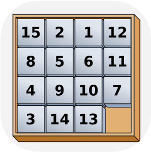
July 9th, 2019

Games

- No hidden information.
- No randomness.
- 1 or 2 players.



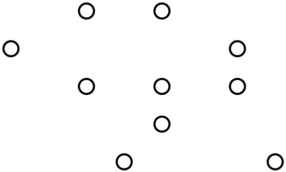




Definition

Graph:

- A set of vertices.



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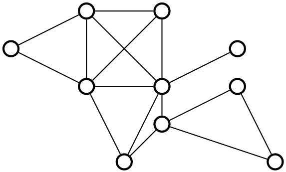

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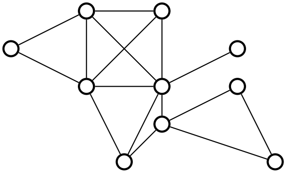

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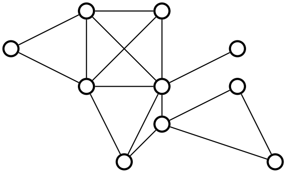

- Some games are played on a graph.

- Games can be also be represented by a graph.

General Organisation

Reconfiguration problems

(1-player games)

Reconfiguration of graph colourings

Reconfiguration of perfect matchings

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Algorithmic Applications

Sampling Colourings

Online colouring

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Combinatorial Games

(2-player games)

- Partizan Subtraction Games
- Rules composition

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Colouring reconfiguration

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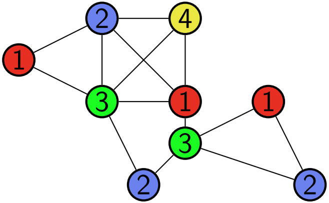
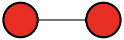
k-colouring: assignment of colours between 1 and k to the vertices of a graph such that there is no monochromatic edge.



forbidden: 

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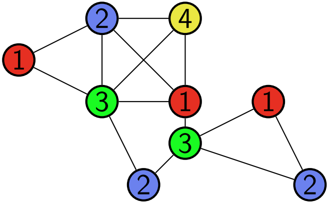
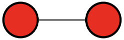


forbidden: 

Applications:

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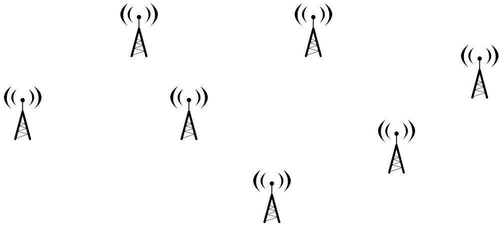

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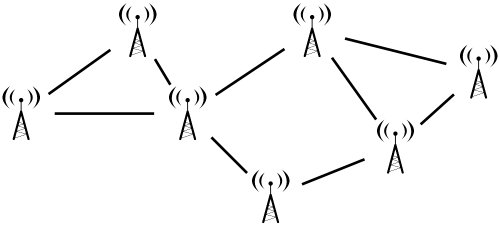

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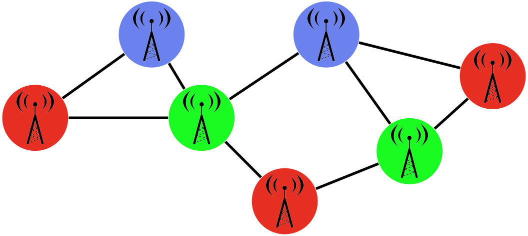

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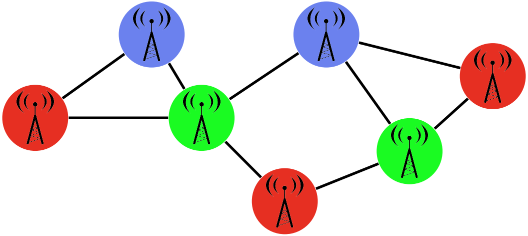

Applications:

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- Scheduling
- Statistical Physics.

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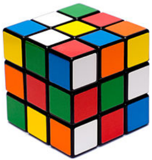

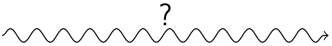

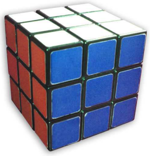

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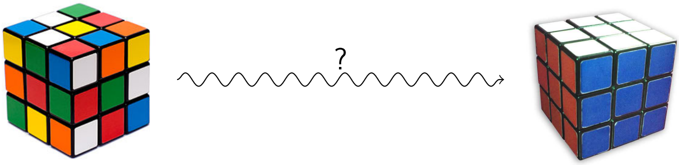

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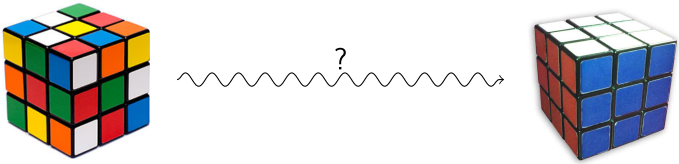

  - Colouring reconfiguration.

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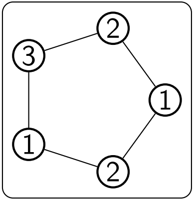

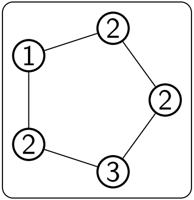

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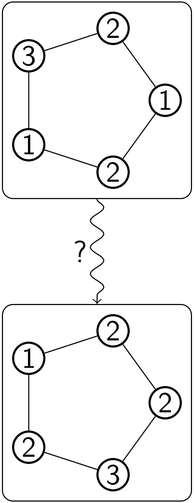

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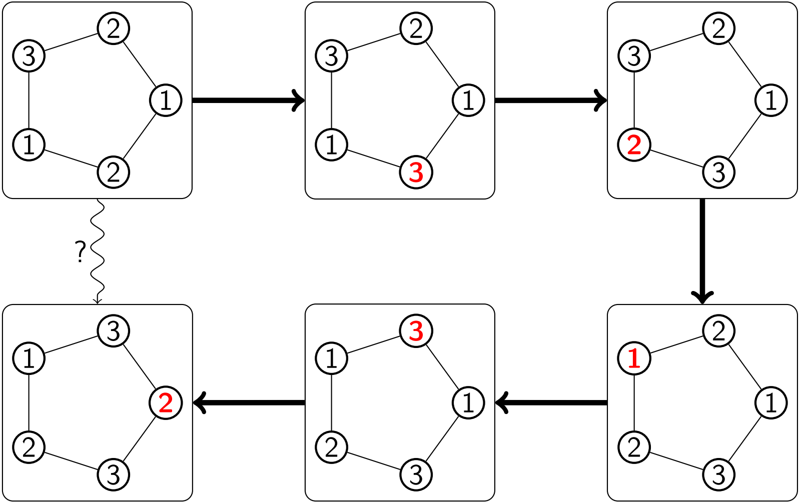

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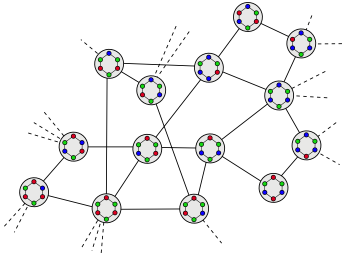
$\mathcal{G}(k, G)$: reconfiguration graph of k-colourings of G.



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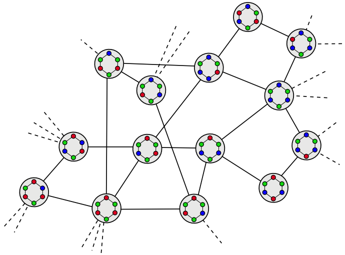

Combinatorial puzzles (1-player games) can also be defined with this formalism.

Motivations

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  $\rightarrow$ Operational research

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- Generating random colourings
  $\rightarrow$ Statistical physics

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- How many steps are required (in the worst case)?
  $\longrightarrow$ What is the diameter of $\mathcal{G}(k, G)$?

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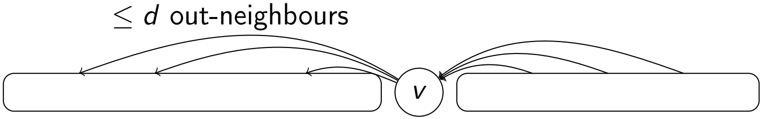

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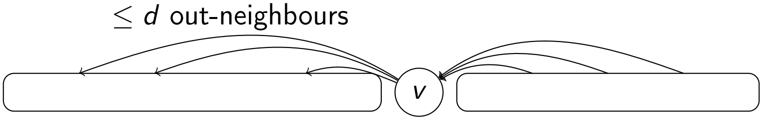
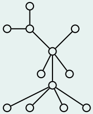
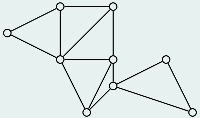


- Trees are 1-degenerate



- Planar graphs are 5-degenerate,



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Even a polynomial upper bound is open.

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The conjecture holds if we replace the degeneracy by:

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Planar graphs:

- diameter $O(n^2)$ if $k \geq 10$ [Feg19],
- diameter poly(n) if $k \geq 8$ [BP15, Feg19],
- diameter $2^{\sqrt{n}}$ if $k \geq 7$ [EF18].

Results [BH19]

number of colours	Diameter
$k \geq d+2$	$O(n^{d+1})$
$k \geq (1+\varepsilon)(d+1)$	$O(n^{\frac{1}{\varepsilon}})$
$k \geq \frac{3}{2}(d+1)$	$O(n^2)$

[Bousquet, Heinrich, 2019]

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- Remove all the vertices coloured c.

Full colours

Definition

A colour c is **full** if for all v, either:

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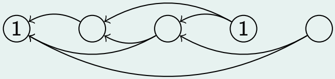

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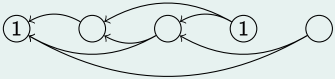

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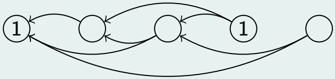

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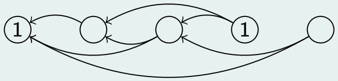

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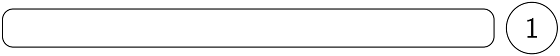

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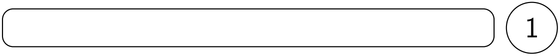

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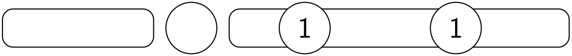

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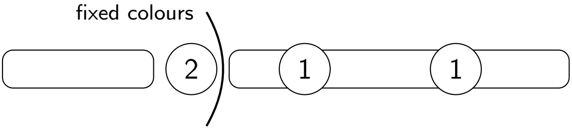

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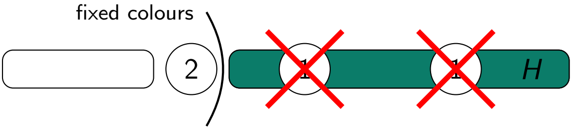

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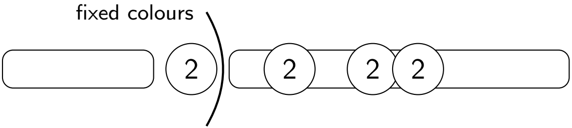

Full colour: 2.

- Remove colour 2 from H.
- Greedily recolour vertices with 2.

Reconfiguration problems

(1-player games)

Reconfiguration of graph colourings

Reconfiguration of perfect matchings

Combinatorial Games

(2-player games)

Partizan Subtraction Games

Rules composition

Algorithmic Applications

Sampling Colourings

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Definition

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- Mixing time: how long do we have to repeat this?

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Glauber dynamics for edge colourings of a tree with $\Delta + 1$ colours mixes in polynomial time.

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- Cereceda's conjecture:
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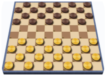

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Subtraction games, $\textsc{Sub}(S)$:

- Parametrized by a set S: the subtraction set.
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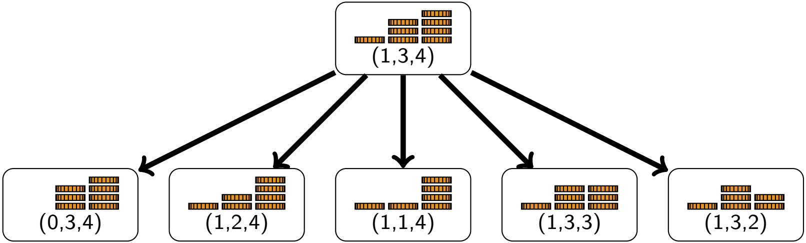

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NIM: Players can remove as many tokens as they want ($S = \mathbb{N}^+$).

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- Study of particular instances of games:

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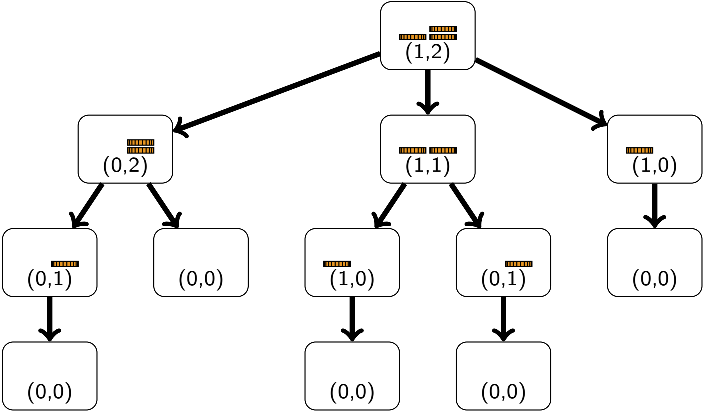

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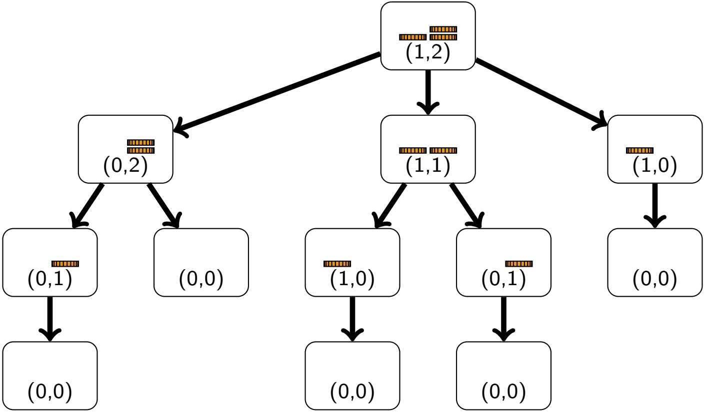

- $\mathfrak{G}$: set of all possible games.

Outcome

Goal: decide who wins under perfect play.

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Outcome of a game:

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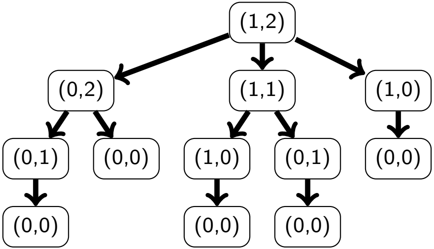

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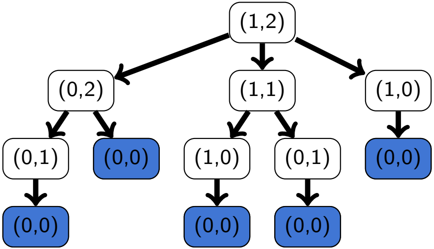

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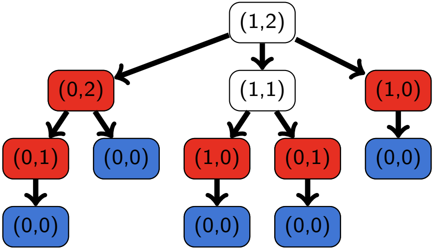

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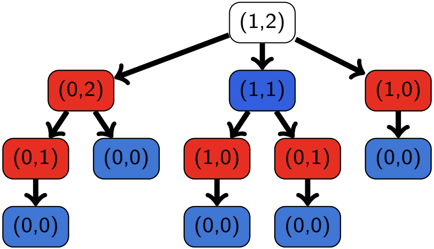

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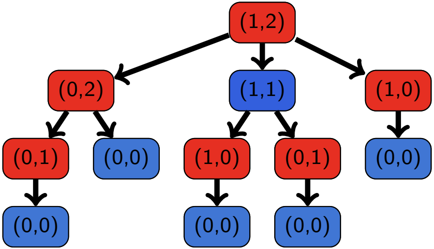

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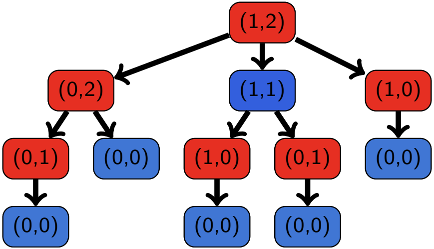

- **Problem:** Computing the whole game tree is usually too expensive

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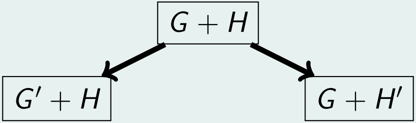

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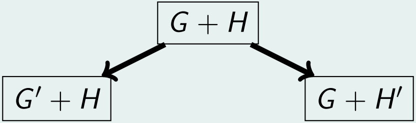

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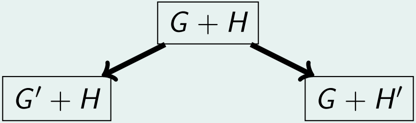

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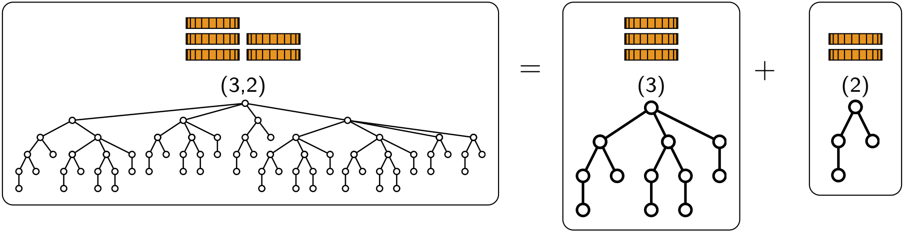

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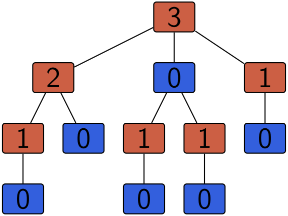

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Every game G is equivalent to a single NIM *heap of size $\mathcal{GV}(G)$.*

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- Not true for misère.

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Corollary

The outcome of a position for a given subtraction game can be computed in polynomial time.

Rules compound

- Combine rulesets instead of games.

[Duchêne, Heinrich, Larsson, Parreau, 2018]

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$\mathcal{R}_1$ and $\mathcal{R}_2$ two rulesets. Push-compound $\mathcal{R}_1 \circledcirc \mathcal{R}_2$:

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- generalisation of pass moves,
- variations of classical games.

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Push-subtraction games:

- Rulesets of the form $\textsc{Sub}(S_1) \circledcirc \textsc{Sub}(S_2)$
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- This is not a disjunctive sum.
- Pushing the button changes the rules in both components.

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Let $\mathcal{R} = \textsc{Sub}(\{1,2\}) \circledcirc \textsc{Sub}(\{1\})$. The second player has a winning strategy on $(n_1, \ldots, n_k)$ if and only if:

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- Similar to the Grundy values.

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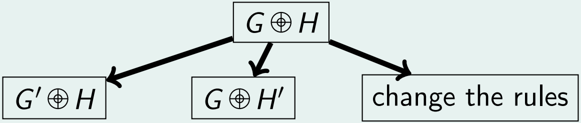

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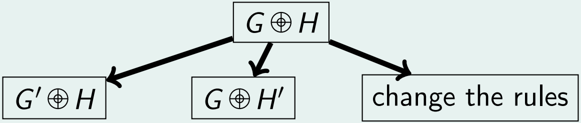

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- Easier than misère.

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The sequence of canonical representatives for $\textsc{Sub}(\{1,2\}) \circledcirc \textsc{Sub}(\{1\})$ *on a single heap has infinitely many values.*

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- The values for $\mathcal{R} \circledcirc \textsc{Sub}(\{x\})$ can be further simplified.

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The sequence of canonical representatives for $\textsc{Sub}(\{1,2\}) \circledcirc \textsc{Sub}(\{1\})$ *on a single heap has infinitely many values.*

Observation

For $\textsc{Sub}(\{1,2\}) \circledcirc \textsc{Sub}(\{1\})$, removing one token is never a good move.

- The values for $\mathcal{R} \circledcirc \textsc{Sub}(\{x\})$ can be further simplified.

Theorem

Let S be a finite set. The 'simplified values' of $\textsc{Sub}(S) \circledcirc \textsc{Sub}(\{x\})$ *are ultimately periodic.*

Perspectives

- General solution for multi-heap push-subtraction games?
- Find applications to other games.
- Restrictions on the rulesets.

Conclusion

Reconfiguration problems

(1-player games)

- Reconfiguration of graph colourings
- Reconfiguration of perfect matchings

Combinatorial Games

(2-player games)

- Partizan Subtraction Games
- Rules composition

Algorithmic Applications

- Sampling Colourings
- Online colouring

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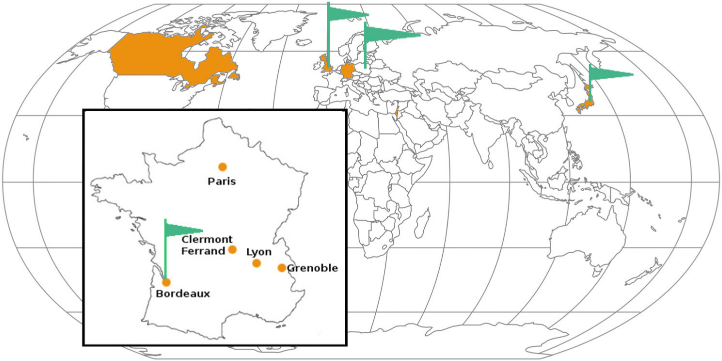
Accepted papers

Submitted papers

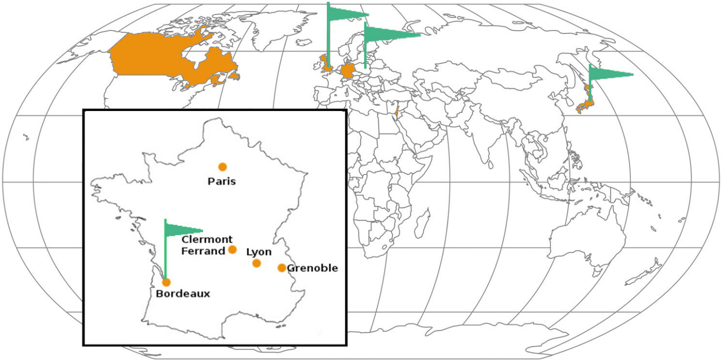
Writing in progress

Collaborations



17 co-authors

Collaborations



17 co-authors

Research stays:

- Warsaw
- Birmingham
- Bordeaux
- Tokyo

Other works

Computing maximum cliques in B_2-EPG graphs, Nicolas Bousquet, Marc Heinrich, WG, 2017.

A generalization of Arc-Kayles, Antoine Dailly, Valentin Gledel, Marc Heinrich, International Journal of Game Theory, 2018.

Enumerating minimal dominating sets in triangle-free graphs, Marthe Bonamy, Oscar Defrain, Marc Heinrich, Jean-Florent Raymond, STACS, 2019.

Thank You!