Computing maximum cliques in B_2-EPG graphs

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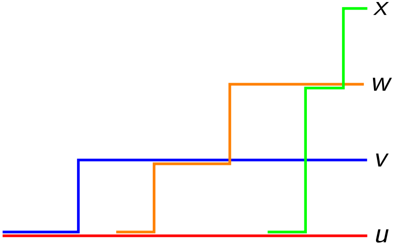
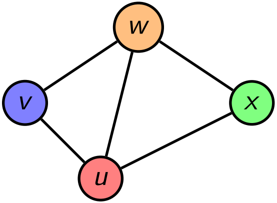
EPG graphs

Definition

EPG graphs: Edge intersection graphs of Paths in a Grid.





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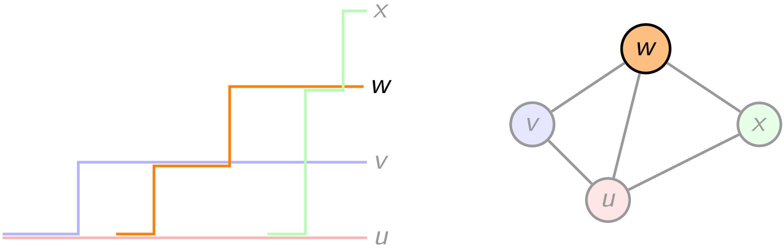

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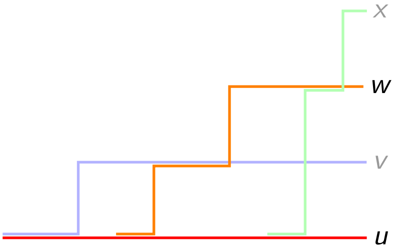

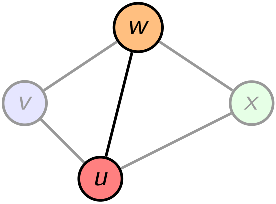

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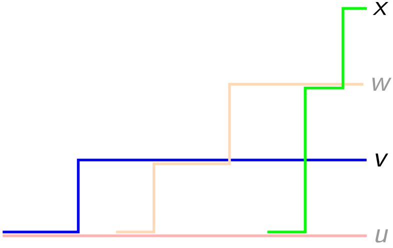

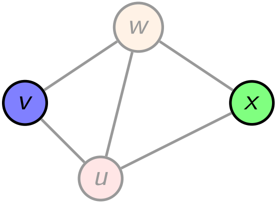

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Any graph has an EPG-representation.

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$\mathbf{B_k}$-EPG graphs: Graphs with an EPG representation using paths with at most k bends.

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Definition

$\mathbf{B_k}$-EPG graphs: Graphs with an EPG representation using paths with at most k bends.

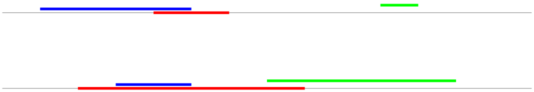
- B_0-EPG are interval graphs.
- B_1-EPG: edge intersection of $\llcorner$, $\ulcorner$, $\lrcorner$, $\urcorner$.

Definition

- **k-interval** graphs: intersection graphs of k-intervals
- **k-track** graphs: there are k-tracks, and each vertex is composed of several intervals, at most one per track.

2-interval graph:

2-track graph:



Relations with multi-track/-interval graphs

Theorem

- *k-interval* $\subseteq B_{4k-4}$*-EPG* $\subseteq (4k-3)$*-interval.*
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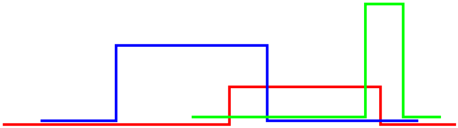

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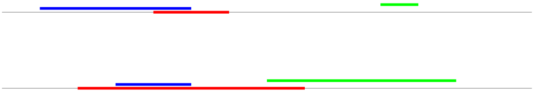

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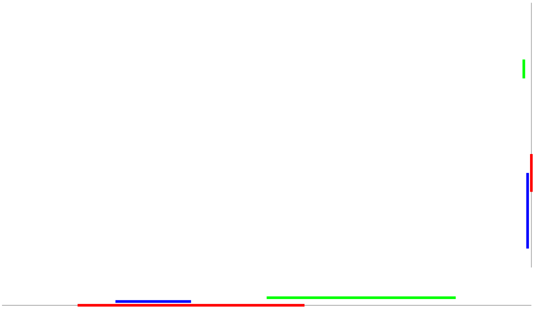

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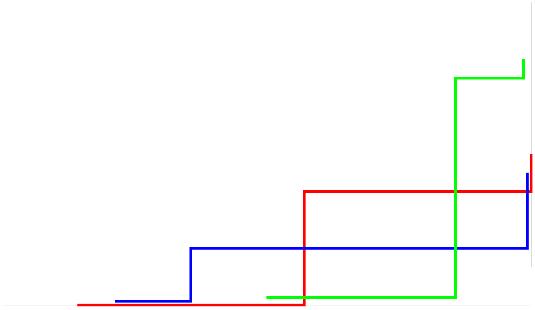

Known Results

Deciding $G \in B_k$-EPG

- Polynomial for $k = 0$.
- NP-Hard for $k = 1$, even restricted to $\llcorner$ shapes [HKU14, CCH14].
- NP-Hard for $k = 2$.

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Graph class	Number of bends	Reference
Trees	1	[GLS09]
Outer-planar	2	[HKU12]
Planar	$3 \leq \ldots \leq 4$	[HKU12]
Linegraphs	2	[BS10]

[Cameron,Chaplick, Hoàng, 2014], [Heldt, Knauer, Ueckerdt, 2014], [Golumbic, Lipshteyn, Stern, 2009], [Heldt, Knauer, Ueckerdt, 2012], [Biedl, Stern, 2010]

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Maximum clique problem

- Polynomial on interval graphs and 2-track graphs [K09].
- NP-complete (APX-hard) on 2-interval and 3-track graphs [FGO12].

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- Polynomial on B_0, B_1-EPG (without the representation) [EGM13].
- NP-Hard for $k \geq 4$, even APX-hard: contains 2-interval graphs.

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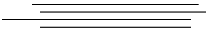
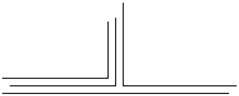
- **Polynomial for B_2-EPG (given the representation).**

[König, 2009], [Francis, Goncalves, Ochem, 2012], [Epstein, Golumbic, Morgenstern, 2013]

Maximum clique in B_1-EPG



edge clique



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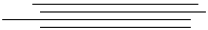

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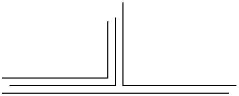
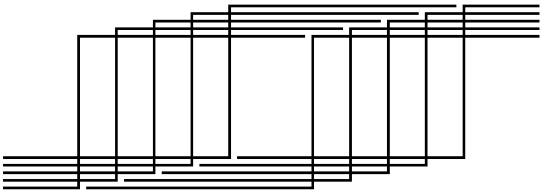


claw clique

- There is a polynomial number of maximal clique in a B_1-EPG graph.

Maximum clique in B_1-EPG



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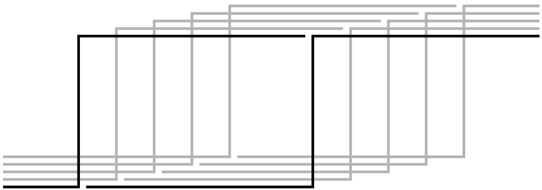

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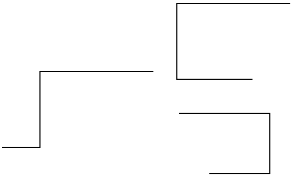
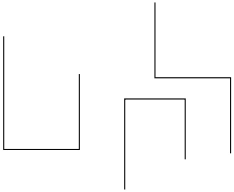
Idea of proof

Distinguish two types of vertices:



Z-vertices



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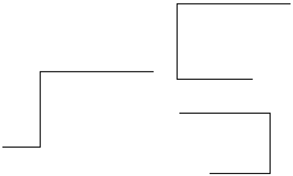

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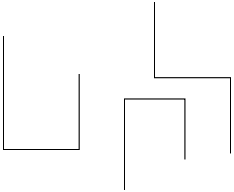

U-vertices

- Look at how the two types of vertices can interact.
- Find a maximum clique when there is only one type of vertices.

Combining Z- and U-vertices

Lemma

Let X be a clique. Either the U-vertices of X intersect at most 3 rows, or the Z-vertices of X intersect at most 3 columns.

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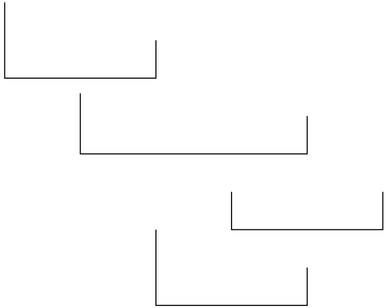

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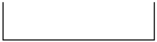

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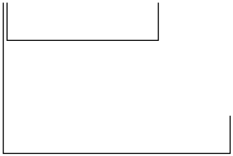

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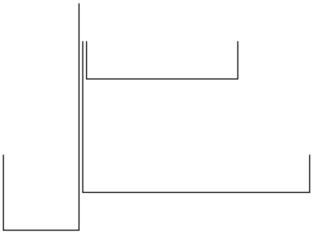

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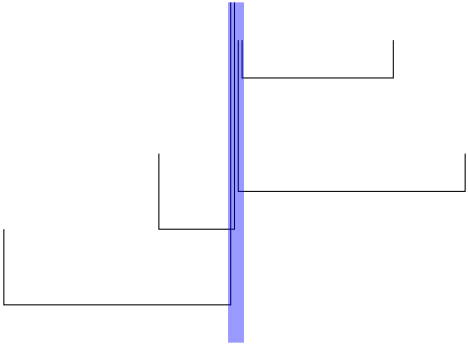

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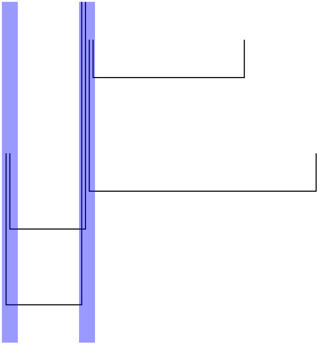

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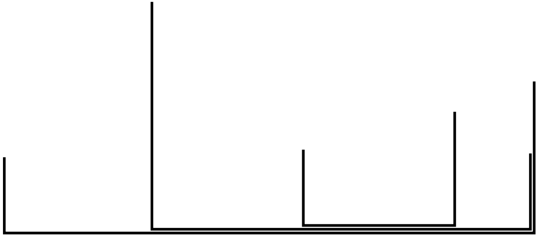

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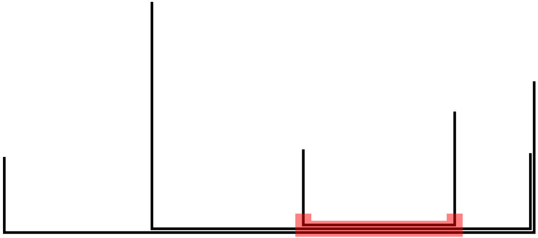

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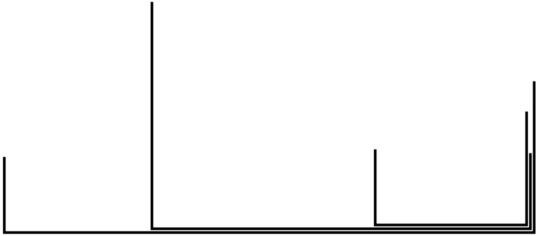

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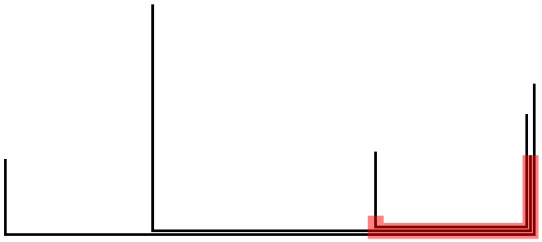

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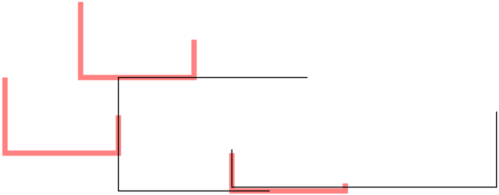

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4. Compute a maximum clique of G_Z and G_U.

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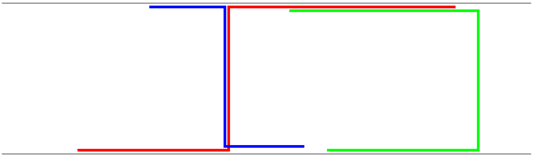
Good subgraphs



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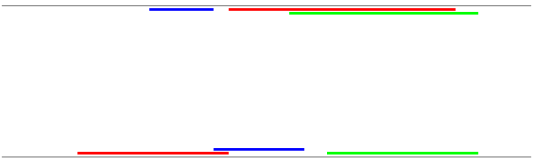

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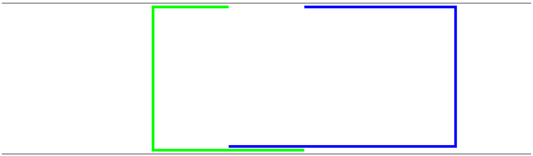

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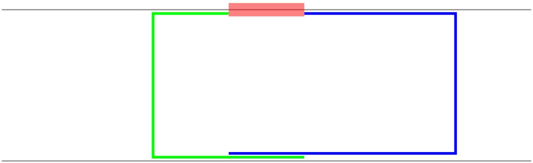

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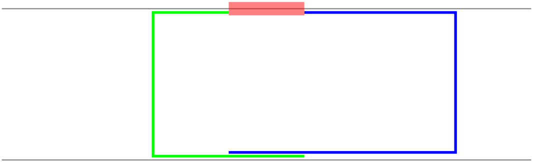

- Good subgraphs defined in terms of vertices containing/intersecting certain grid edges.
- Join of clique, 2-track and complement of bipartite.

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- What can you do without the representation?
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