Graphs and Discrete Structures

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Fall 2018

1 Chromatic number, independence number and clique number

The *chromatic number* $\chi(G)$ of G is the minimum number of colors needed to color properly G. A coloring is *proper* if any pair of incident vertices receive distinct colors. All along the course, we will only consider proper colorings and then we will omit the term "proper". A k-coloring is a coloring using at most k colors.

An *independent set* is a set of pairwise not incident vertices in a graph. We denote by $\alpha(G)$ the maximum size of an independent set in G.

A *coloring* is a partition into independent sets. We can formulate the coloring problem as an ILP Variables x_S for each stable set Constraints: $\forall v, \sum_{S|v \subseteq S} x_S \geq 1$ Fractional coloring = fractional covering by stable sets.

Note that it ensures that that $\chi(G) \geq \frac{|V(G)|}{\alpha(G)}$. It in particular implies that an induced odd cycle is not 2-colorable. Since there exist 3-colorings of odd cycles, they are 3 colorable, i.e. $\chi(C_{2k+1}) = 3$.

We have a natural lower bound on $\chi : \omega$, the clique number. The clique number $\omega(G)$ of G is the maximum size of a clique in G

Q1? Do we always have $\chi(G) = \omega(G)$?
The answer is no, for instance on an odd cycle or on a wheel.

Q2? If not can we bound χ by a function of ω?
Again the answer is no ! A graph is triangle-free if it satisfies $\omega = 2$.

Theorem 1. *(Mycielski) There exist triangle-free graphs of arbitrarily large chromatic number.*

Proof. The proof is by induction on k, the chromatic number. Let G_k be a triangle free graph of chromatic number k. Let $v_1, \ldots, v_n$ its vertices.

We create n new vertices $w_1, \ldots, w_n$ such that $v_i w_j$ is an edge if and only if $v_j v_i$ is an edge. We finally create a new vertex z connected to $w_1, \ldots, w_n$. Let us denote by G_{k+1} the resulting graph.

The following claim will be useful to conclude.

Claim 1. *If G k-colorable and not (k-1) colorable, for each color i, there exists a vertex v_i such that v_i has a neighbor of all the colors but i.*

Proof. Otherwise, it is an easy exercise to check that one color can be eliminated. □

As a corollary of the claim, we know that all the colors $1, \ldots, k$ have to appear on $w_1, \ldots, w_n$. And thus z, which is connected to all of them, have to receive another color, a contradiction. □

There are many other constructions of such graphs, e.g. Blanche Descartes, Zykov, Erdos, Borsuk...etc...

Q3? If not is it possible to bound χ by a function of ω in some graph classes? (ie if we add restriction) This question interests researchers for more that 50 years.

Complexity

Theorem 2. *The* 3*-coloring problem is NP-complete.*

One can also ask what is the efficiency of the greedy algorithm to color a graph G. Let G be a graph and σ and ordering of $v_1, \ldots, v_n$ of $V(G)$. The greedy algorithm colors the vertices $v_1, \ldots, v_n$ one after the other in the order σ and give the v_i the smallest possible available color. We denote by $\chi_g(G, \sigma)$ the number of colors used by the greedy algorithm with order $sigma$.

Remark 3. *There exists an ordering σ such that $\chi_g(G, \sigma) = \chi(G)$.*

We left the proof as an exercise.

We denote by $\chi_g(G)$, the grundy coloring number of G, the maximum over all the orders σ of $\chi_g(G, \sigma)$.

Theorem 4. *$\chi_g(G)$ can be arbitrarily larger than $\chi(G)$. For instance on trees, χ_g is not necessarily bounded while $\chi(T) \leq 2$ for every tree T.*

2 Graph classes

Coloring a graph is hard. A natural approach consists in trying to understand more carefully our instance and try to use the underlying structure of the instance to derive a better algorithm. For instance, on trees, the greedy algorithm migth perform arbitrarily bad but there exist simple polynomial time algorithms that color T with two colors.

There exist many possible types of types of classes of graphs (ie of possible structure we might ask). For instance graphs defined as intersection of geometric object, graph that does not contain some subgraphs...etc...

Interval graphs. Interval graphs naturally appear in many applications, e.g. in scheduling. Interval graphs are intersection graphs of intervals in the plane. In other words, we create a vertex per interval and two vertices are linked by an edge if the corresponding interval intersect.

Theorem 5. *For every interval graph, we have $\chi(G) = \omega(G)$. Moreover the proof is algorithmic and provides in polynomial a $\chi(G)$-coloring of G.*

Proof. Let v be the vertex with the leftmost rightend. Let G' be the graph without the vertex v. By induction, G' can be colored with $\omega(G')$ colors. If $\omega(G') < \omega(G)$, then we have the right to use one more color in G and we use it to color v. So let us assume that $\omega(G') = \omega(G)$. One can remark that by definition of v, all the neighbors of v contain the point of the line which is the rightend point of v. So $N(v) \cup v$ is a clique. And then $N(v)$ has size at most $\omega(G) - 1$. So there exists a color that does not exist in the neighborhood of v and we can color v with this color. □

Chordal graphs More generally, we can define *chordal graphs*. A graph is chordal if it does not contain induced cycles of length at least 4. In other words, any cycle of length at least four must contain a *chord* (an edge between two non consecutive vertices of the cycle). But there is an alternative definition of chordal graphs: chordal graphs are intersection graphs of subtrees of a tree. Note that it generalizes interval graphs since interval graphs are interection graphs of subpaths of a path.

Theorem 6. *For every chordal graph, we have $\chi = \omega$.*

Another equivalent, but sometimes more easier to understand, definition of chordal graphs is the following. For every chordal graph G, there exists a tree T such that, we can associate to each node of the tree a subset of vertices of $V(G)$ called *bag* such that:

- For every $v \in V$, the set of bags in which v appears is a (non empty) subtree of T.
- $uv \in E(G)$ if and only if there exists a bag containing both u and v.

Such a tree is called a *clique tree* of G.

Exercise: Why is it equivalent with the other definition of chordal?

Proof of Theorem 6. By induction. Let T be a clique tree of G. Now let f be a leaf of T. If there is no vertex included only in f, then all the vertices in the bag of f also appear in the bag of its parent. And then we can delete the leaf of f. So there is x in the bag of f whose subtree is reduced to the leaf.

All the neighbors of x are in the bag of f, which is a clique. So x has at most $(\omega - 1)$ neighbors. So by induction $G' = G \setminus x$ can be colored and the coloring can be extended to x since at most $\omega - 1$ colors appear in the neighborhood of x. □

Note that in both cases, we prove that there is a vertex of small degree and apply induction. We can more generally do the following. A graph is *d-degenerate* if there exists an ordering of $V(G)$ $v_1, \dots, v_n$ such that $N(v_i)$ has at most d neighbors in $v_{i+1}, \dots, v_n$. Since $E \leq 3V - 6$, the degeneracy of planar graphs is at most 5.

Lemma 7. *Any d-degenerate graph is $(d + 1)$-colorable.*

Proof. We color the graph in the reverse order. Assume by induction that we have a coloring of $V \setminus v_1$. Then we can extend this coloring to v_1 since v_1 has at most d neighbors and there are $d + 1$ colors. □

Perfect graphs We have just seen two classes of graphs that satisfy $\chi = \omega$. What can we say about these graphs in general? These graphs received a considerable attention and are called perfect graphs. A graph G is *perfect* if any induced subgraph of G satisfies $\chi(G) = \omega(G)$. A graph H is an induced subgraph of G if there exists $|H|$ vertices of G whose restriction to give exactly H (not H plus some edges !). Berge conjectured in the 50ies that perfect graphs are graphs that do not contain induced odd cycles (called holes) or induced odd anticycles (complement of odd cycles - called antiholes).

Remark 8. *The condition are necessary.*

A proof was provided in 2004 by Chudnovsky, Robertson, Seymour, Thomas. The proof -long and complicated- is too complicated to be explained in this lecture...

Theorem 9 (Perfect Graph Theorem). *Perfect graphs are graphs without odd holes and odd antiholes.*

χ-boundedness Gyárfás defined in 1987 the notion of χ-boundedness. A class $\mathcal{G}$ of graphs is *χ-bounded* (with binding function f) if, for every graph $G \in \mathcal{G}$, we have $\chi(G) \leq f(\omega(G))$. Note that perfect graphs are χ-bounded with $f = id$ as well as disk graphs.

We can ask a couple of questions. First, are all the intersection graphs chi-bounded? The answer is negative. Intersection of segments in the plane are NOT chi-bounded.

Let us denote by P_t the path on t vertices. Gyárfás proved the following:

Theorem 10 (Gyárfás). *P_t-free graphs are χ-bounded.*

Proof. Let us prove it in the case of triangle-free graphs (ie $\omega = 2$). Let us prove by induction that if $\chi = t$, there is a path of length at least $t/2$.

Let x be a vertex of G. Let $G' = G \setminus N[x]$. Since $N(x)$ is edge-less (otherwise there would be a triangle), the graph G' satisfies $\chi(G') \geq \chi(G) - 2$. So there exists a connected component C of chromatic number $\chi(G) - 2$. Now let y in $N(x)$ with a neighbor in C. By induction on $H = G[C] \cup y$, there exists a path of length $(\chi - 2)/2$, which completes the proof since this path can be extended to x. □

In general it can be extended to any possible value of clique, but the function is exponential in ω. Proving the existence of a polynomial function is a widely open problem.

Disk graphs A *(unit) Disk Graph* is an intersection graphs of (unit) Disks in the plane. These graphs can for instance model wireless networks or systems of antennas in telecommunication.

Theorem 11. *Every unit disk graph (resp disk graph) satisties: $\chi \leq 3\omega(G)$ (resp. $6\omega(G)$).*

Proof. Take the leftmost vertex. Partition the neighborhood of x into at most 3 cliques. Since each of these cliques have degree at most ω, the leftmost vertex has at most 3ω neighbors. So a coloring of $G \setminus x$ can be extended to x. □

Conjecture 12. *Every unit disk graph (resp disk graph) satisties: $\chi \leq 3/2\omega(G)$ (resp. $3\omega(G)$).*

Planar graphs Planar graphs are graphs that can be drawn in the plane without crossing edges.

Lemma 13 (Euler formula). *For every planar graph, we have:*

$$V - E + F = 2.$$

Corollary 14. $E \leq 3V - 6$

In particular, it implies that planar graphs are 5-degenerate, so planar graphs can be colored with 6 colors. Note that it is exactly the proof technique we used to prove that interval graphs can be colored with ω colors. We actually proved that they are $(\omega - 1)$-degenerate and then can be colored with ω colors.

Is this bound can be improved? Yes !

Theorem 15. *Every planar graph is 5-colorable.*

Sketch of the proof. By induction. If G has less than 5 vertices the conclusion holds. So G has more than five vertices. By the corollary, there exists a vertex v of degree at most 5. Let $G' = G \setminus v$ and c be a coloring of G'. If a color does not appear in the neighborhood of v in G, the conclusion holds. So the five neighbors $v_1, \ldots, v_5$ of v are colored with colors $1, \ldots, 5$.

A the *Kempe chain* (c, v) (where v is a vertex and c a color) is the component of v restricted to the vertices colored c and $c(v)$. A *Kempe change* is the exchange of the colors of the vertices in the Kempe chain. Note that the resulting coloring is proper.

Now for every $i, j \leq 5$, consider the Kempe chain (v_i, j) in the graph G'. If it does not contain v_j, then we perform the Kempe change and the color i does not appear anymore in the neighborhood of v and then v can be colored with i.

We claim that it is actually impossible. Indeed, assume that in the plane $v_1, \ldots, v_5$ appear in this order. Then an alternating path $(1, 3)$ in the component $(v_1, 3)$ plus the vertex v is a cycle. And v_2 is "in" this cycle and v_4 is "out". So there cannot be a path colored $(c(v_2), c(v_4)$ between v_2 and v_4, and then the conclusion holds. □

Theorem 16 ((Appel, Haken)). *Any planar graph is 4-colorable.*

It is the first computer assisted proof in the history (as far as I know !). We still do not know any proof that can be checked by human beings !

Not covered - Graph minors It is sometimes interesting to define classes of graphs with their sets of forbidden subgraphs. For instance forests are cycle-free graphs, chordal graphs are graphs without induced cycles of length at least 4. How can we define planar graphs? First there are two natural obstructions.

Remark 17. *K_5 is not planar.*

Proof. Use the corollary to get a contradiction. □

A graph H is a *minor* of G is H can be obtained G by:

- vertex deletion, and
- edge deletion, and
- edge contraction.

For example a forest is a graph that does not contain K_3 as a minor.

Another possible definition of minors is the following. For every vertex h of H, we can associate a subset V_h of V such that:

- For every $h \in H$, V_h is connected.
- For every $h, h' \in H$, V_h and $V_{h'}$ are disjoint.
- For every edge $hh' \in E(H)$, there exists $v \in V_h$ and $w \in V_{h'}$ such that $vw \in E(G)$.

Theorem 18 ((Kuratowski)). *A graph is planar if it does not contain K_5 or $K_{3,3}$ as a minor.*

Minor closed classes are usually "simple" classes of graphs in the sense that many hard problems become simpler on graphs excluding a minor. There is a very famous conjecture about coloring of minor free graphs:

Conjecture 19. *Every K_t-free graphs can be colored with $(t-1)$ colors.*

True for $t = 2$ (independent sets are 1-colorable), $t = 3$ (forests are 2-colorable), $t = 4$ and $t = 5$ (a long proof based using a as blackbox the 4-colorabily of planar graphs).

But we know that a function exists...

Lemma 20. *Every K_t-minor free graph has at most $C_t n$ edges.*

Asymptotically (for large n and t, we know that this function is of order $\mathcal{O}(t \log tn)$.

Lecture 4 - Ramsey Theory One can ask the following question: is it true that, if the graph is large enough, the graph contains some structure in it? The Ramsey theorem gurantees that it is true.

Theorem 21. *For every k, there exists an integer $R(k)$ such that for every $n \geq n_0$ and every edge-coloring of K_n with 2 colors, there exists a subset X of k vertices such that the edges of the subgraph induced by X is a monochromatic clique.*

The integer $R(k)$ is called the Ramsey number of order k. It is sometimes denoted by $R(k,k)$ (we will understand why a bit later).

Theorem 22.

$$R(k) \leq 4^k$$

Proof. In order to prove it, let us prove that every edge coloring of K_n with 2 colors contains a monochromatic clique of size (almost) $\log n/2$. Let G be a clique. Let S be a set initialized as the empty set. Let x_1 be a vertex. Add x_1 to S. Since the edges of G are colored with 2 colors, either half of the edges incident to x_1 are colored 1 or half of the edges incident to x_1 are colored 2. Wlog assume that it is 1. Let X_i be the set of neighbors y of x such that x_1y is colored i. Since $X_1 \cup X_2 = V \setminus x$, X_1 contains half of the vertices of G (almost). Let G_2 be the restriction of G to the vertices of X_1. Repeat this operation until the graph is the empty graph.

We claim that at the end of the algorithm the size of S is $\approx \log n$ vertices. Indeed, at each step, we extract one vertex and repeat on a graph of size at least $n(G)/2$. So the number of repetition is of order $\log n$.

Now, for each extracted vertex x_i and for every $j, j' > i$, we have that x_ix_j and $x_ix_{j'}$ are of the same color. Indeed by definition, we have at the end of step i restricted the graph to the neighbors of x_i such that the color of the edge to reach them is the same. We say that x_i is extracted with color r if we have restricted the graph to the neighbors colored r. At least half of the vertices of S are extracted with the same color r in $\{1,2\}$.

It is easy to check that these vertices induce a monochromatic clique. □

Let us now prove that we are not far from the tight bound:

Theorem 23. *For every $k \geq 3$, we have:*

$$R(k) \geq \sqrt{2}^k$$

Proof. The proof is a "probabilistic" proof. It means that we will "count" the average number of monochromatic clique on a clique of size n if the colors of the edges are chosen independently at random with probability $1/2$. If the expected value is less than one it means that there exists at least a graph that does not contain any monochromatic clique. Indeed, using Markov' Lemma we have: $\mathbb{E}($ number of monochromatc $K_k) \leq \mathbb{P}G$ contains at least $K_k/1$.

Now let K_n be a clique on n vertices and for every edge, let us color with color 1 with probability $1/2$ and with color 2 with probability $1/2$. Let X be a set of k vertices.

$$\mathbb{P}(X \text{ is monochromatic }) = 2 \cdot (\frac{1}{2})^{k(k-1)/2}$$

So, by union bound, the average number of monochromatic K_k is at most

$$\begin{aligned} mathbbE(\text{Number of monochromatic } K_k) &\leq \binom{n}{k} \cdot \mathbb{P}(X \text{ is monochromatic }) \\ &\leq \binom{n}{k} \cdot 2 \cdot (\frac{1}{2})^{k(k-1)/2} \end{aligned}$$

Since $\binom{n}{k} \leq n^k/k!$, if $n \leq 2^{k/2}$, we have:

$$mathbbE(\text{Number of monochromatic } K_k) \leq \frac{2^{k^2}}{k!} \cdot 2/2^{k^2} \cdot 2^{k/2}$$
$$\leq \frac{2^{k/2+1}}{k!} < 1$$

which concludes the proof since the last inequality holds as long as $k \geq 3$. □

Variants We can do a lot of variants of Ramsey theorems, for instance using more colors, or to desequilibrate the size of the target monochromatic cliques..

Theorem 24. *For every $k_1, \ldots, k_\ell$, there exists an integer $R(k_1, \ldots, k_\ell)$ such that for every $n \geq n_0$ and every edge-coloring of K_n with ℓ colors, there exists a subset X of vertices such that the edges of the subgraph induced by X is a monochromatic clique of color i and of size k_i for some $i \leq \ell$.*

Exercice: Adapt the proof to prove it. Which function do you obtain?

Applications

Theorem 25. *If $n > R(k)$, any graph on n vertices admits a clique or a stable set on k vertices.*

Proof. Color edges with color 1 and non edges with color 2. □

There are many other applications of Ramsey theory to various fields. For instance the following Schur theorem:

Theorem 26 (Schur theorem)**.** *Let $[|1, n|]$ be the set of integers between 1 and n. Let us color the integers with colors of $[|1, n|]$ with k colors. If $n \geq n_0$, there exists three integers x, y, z colored with the same color such that $x + y = z$.*

Proof. Consider the clique on n vertices where the edge ij with $j > i$ is colored with the color of $j - i$. If $n \geq R(3, 3, \ldots, 3)$, then G contains a monochromatic clique of size 3. Let x, y, z be the three vertices with $x < y < z$. We claim that $z - x, z - y$ and $y - x$ are colored the same in the original instance and $(y - x) + (z - y) = z - x$, which completes the proof. □