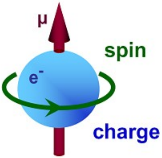
Fast transformations between colorings

Nicolas Bousquet

Cycles & Colorings 2022

Spin systems



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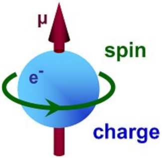

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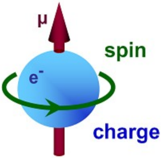

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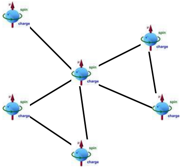

- An integer k being the number of states.
- An interaction $\{0, 1\}$ (symmetric) matrix modelizing the interaction between spins.
  - $0 =$ no interaction $=$ no link.
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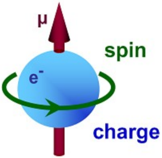

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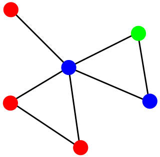

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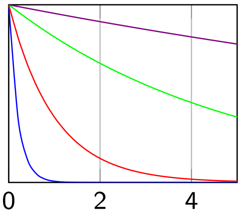
A spin configuration is a function $f : S \to \{1, \ldots, k\}^n$.
$\Leftrightarrow$ A (non necessarily proper) graph coloring.



$T = 5, 1, 0.2, 0.05$

Antiferromagnetic Potts model

$H(\sigma)$: number of monochromatic edges.

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Edges with both endpoints of the same color.

Gibbs measure at fixed temperature T :

$$\nu_T(\sigma) = e^{-\frac{H(\sigma)}{T}}$$

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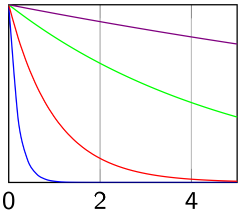

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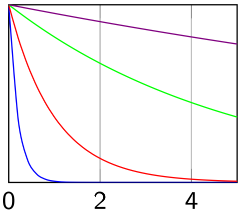

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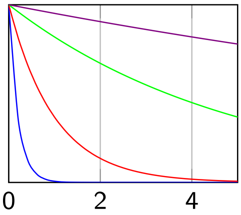

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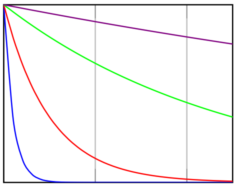

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Definition (Glauber dynamics)

Limit of a k-state Potts model when $T \to 0$.
$\Rightarrow$ Only **proper** colorings have positive measure.

How to count / sample colorings?

How to sample colorings?

Let G be a graph. Let α be a coloring of G.

- Select a vertex v and a color c at random.
- Change the color of v for c if the coloring remains proper.
- Repeat until the coloring is "random".

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Configuration Graph $\mathcal{C}_k(G)$:

- Vertices = Proper k-colorings of G.
- Edges between two colorings if they differ on exactly one vertex.

When does it work well ?

- **[Ergodicity]** Is the configuration graph connected ?

  Actually not completely exact... We need another condition which is trivially correct in our case.

- **[Mixing time]** How many steps do we need to repeat until we get a "random" coloring ?

  A classical theorem ensures that if we have a rapid mixing, then we can approximate the counting

- **[Compact representation]** How do we know if a node was already visited or not ?

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- Diameter of the configuration graph $= D$
  $\Rightarrow$ Mixing time $\geq 2 \cdot D$.
- Better lower bounds ? Look at the connectivity of the configuration graph (e.g. bottleneck ratio).

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To sum up :

If $k \geq \Delta + 2$, the random sampling / enumeration algorithms are working and otherwise, they are not working.

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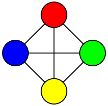
A coloring is frozen if all the vertices are frozen.



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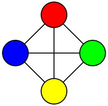

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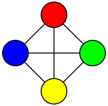

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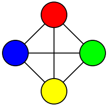

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Theorem (Feghali, Johnson and Paulusma '17)

The $(\Delta+1)$-configuration graph consists in :

- Isolated vertices (=frozen vertices).
- At most one connected component of size ≥ 2 (of diameter $\mathcal{O}(n^2)$).

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Theorem (B., Feuilloley, Heinrich, Rabie '22+)

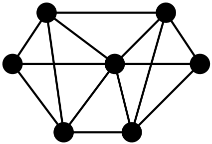
We can transform any non-frozen $(\Delta+1)$-coloring into any other in $f(\Delta) \cdot n$ steps.

Distance between colorings

Conjecture (Cereceda '08)

The $(d+2)$-recoloring diameter of any d-degenerate graph is $\mathcal{O}(n^2)$.

A graph is d-degenerate if there exists an ordering $v_1, \ldots, v_n$ such that for every i, $|N(v_i) \cap \{v_{i+1}, \ldots, v_n\}| \leq d$.



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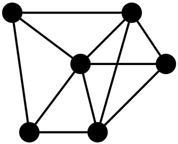

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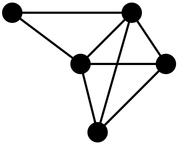

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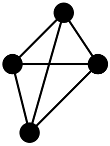

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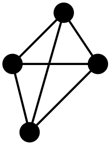

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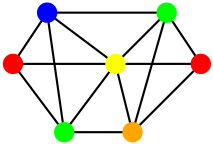

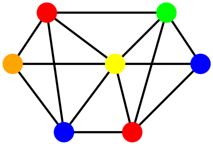

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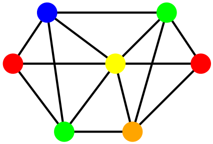

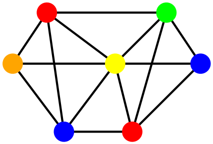

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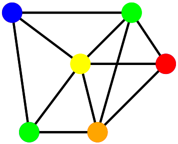

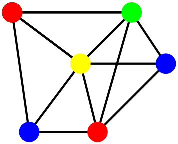

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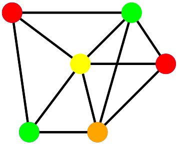

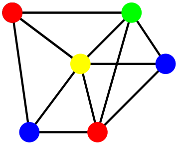

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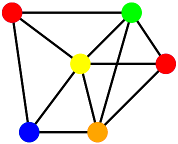

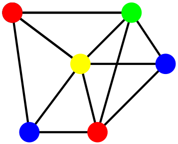

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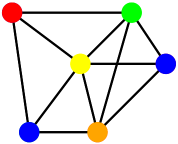

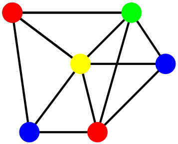

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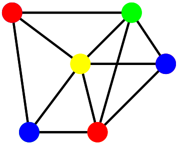

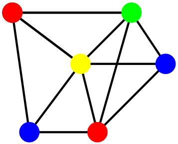

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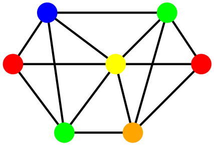

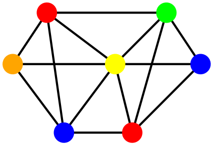

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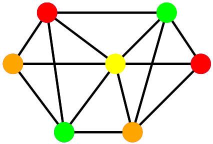

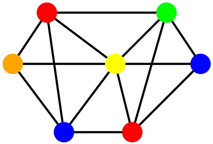

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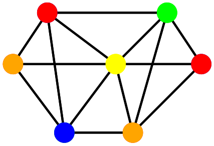

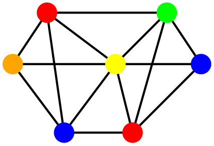

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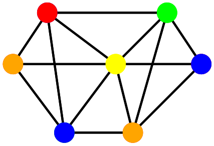

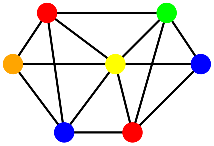

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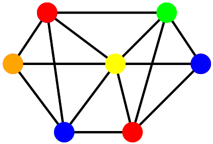

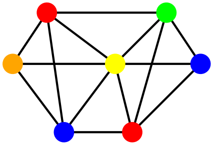

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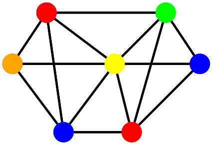

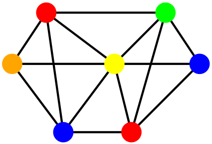

Theorem (Dyer et al. '06)

If there is an ordering of G into r consecutive independent sets $S_1, \ldots, S_r$ such that $d(S_i) \leq d$ in $G[S_{i+}]$.
Then the $(d+2)$-recoloring diameter of G is at most $2^r n$.

Why quadratic ?

Theorem (Bonamy et al. '11)

The configuration graph of 3-colorings of P_n has diameter $\Omega(n^2)$.

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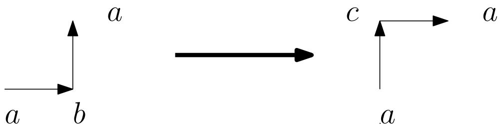
What is the modification of a color change ?



Remark.

The area under the curve is not modified by 1 or -1.

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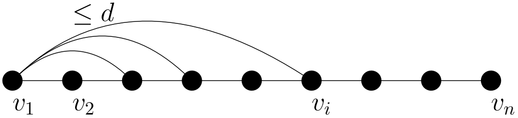

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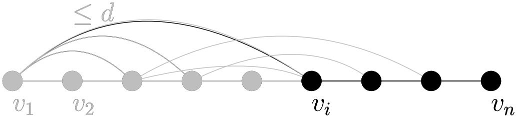

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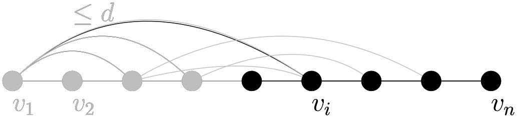

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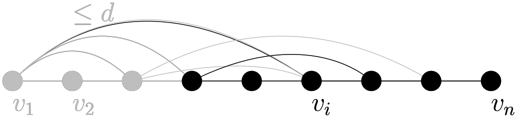

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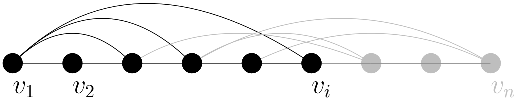

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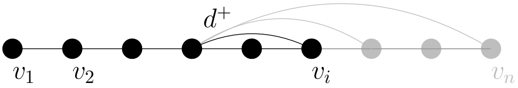

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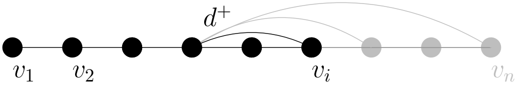

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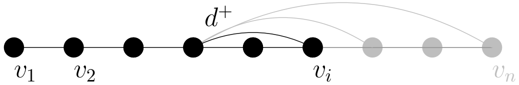

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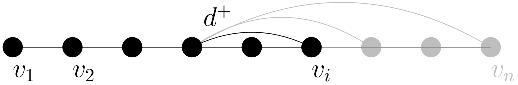

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[Feghali, Johnson, Paulusma '17] $d = 2$ and $\Delta = 3$ is true.

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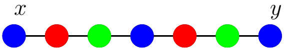

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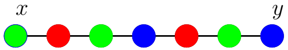

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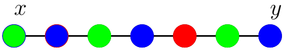

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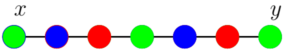

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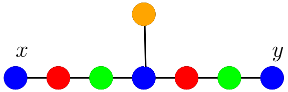

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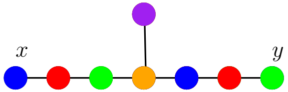

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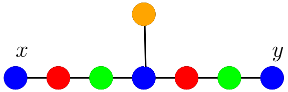

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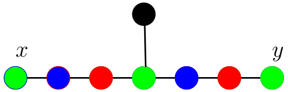

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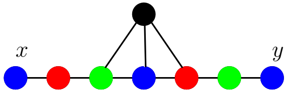
Let $x, y \in V$ s.t. $d(x, y) = 7$ and x is unfrozen.
We can unfreeze x and y by only recoloring vertices in $B(x, 6)$.

Sketch of the proof :

- P : min. path from x to y
- Recolor vertices of the path to unfreeze the last vertex.
- If a vertex of $N(P)$ is unfrozen, start the path from it.
- If a vertex of $N(P)$ is unfrozen at the end, use it to defreeze x.
- We can change the path !



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Let I be an independent set at distance 28. It is possible to unfreeze all the vertices of I in $O(n)$ steps (if ≥ 1 vertex is non-frozen).

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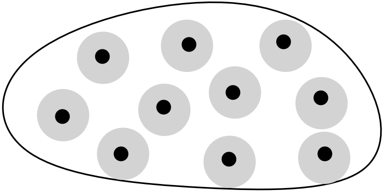

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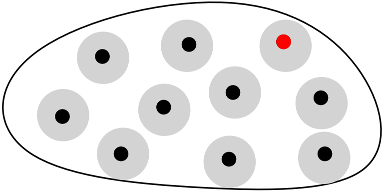

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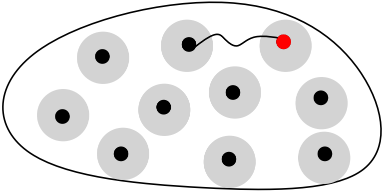

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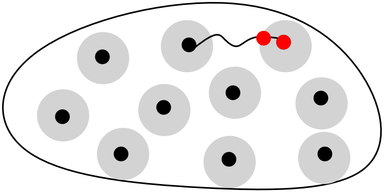
- Using Local Warming defreeze the border vertex.



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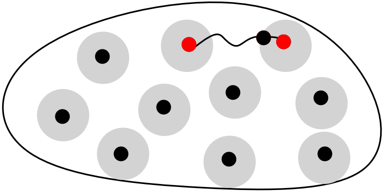

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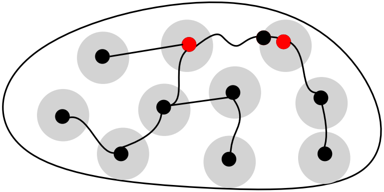
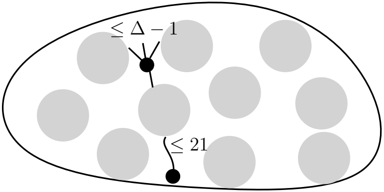
- Using a BFS defreeze the vertices one after another



Combining the ingredients

Next step

We can recolor G into a coloring where $V \setminus B(I, 7)$ is colored with the target coloring.



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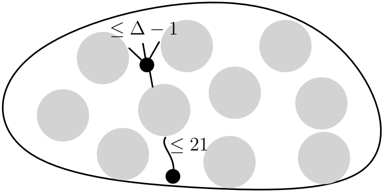

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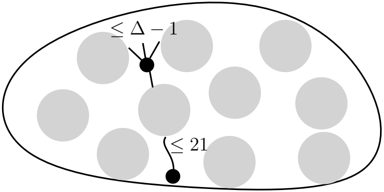

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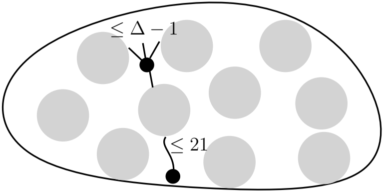

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$\Rightarrow$ Obtain a coloring of G where $V \setminus B(I,7)$ is colored as in the target coloring.

(Using again local warming, trust me)

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Recolor $B(x, 7)$ for every $x \in I$ without modifying the rest of the coloring.

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- If $B(x, 7)$ contains a vertex of degree $\Delta - 1$ or two neighbors in $V \setminus B(x, 7)$ colored the same : ✓

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Idea of the proof :

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- Twist the target coloring to be sure that this property holds.

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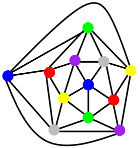

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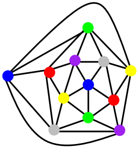

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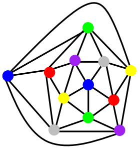

Thanks for your attention !