# Complexity Landscape for Local Certification

**Nicolas Bousquet**

(Joint work with Laurent Feuilloley, Sébastien Zeitoun)

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LACIM
Laboratoire d'algèbre, de combinatoire et d'informatique mathématique

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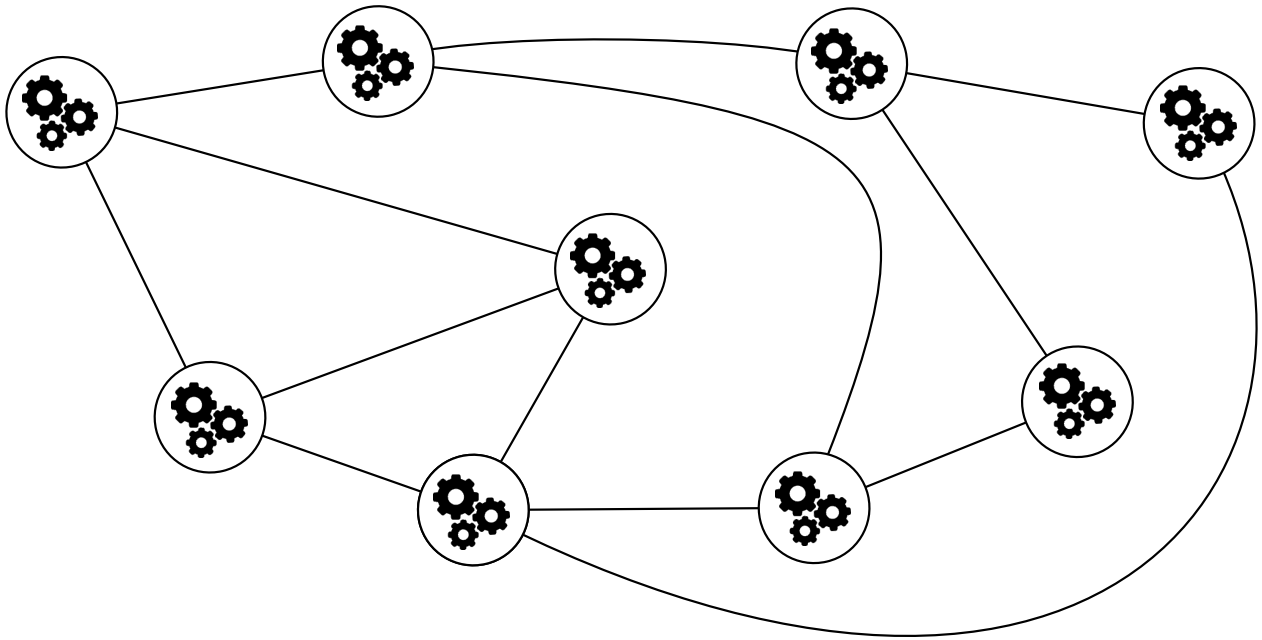
**Context :** distributed computing

**Model :** graph, $\begin{cases} \text{vertices} = \text{comp. units} \\ \text{edges} = \text{com. channels} \end{cases}$



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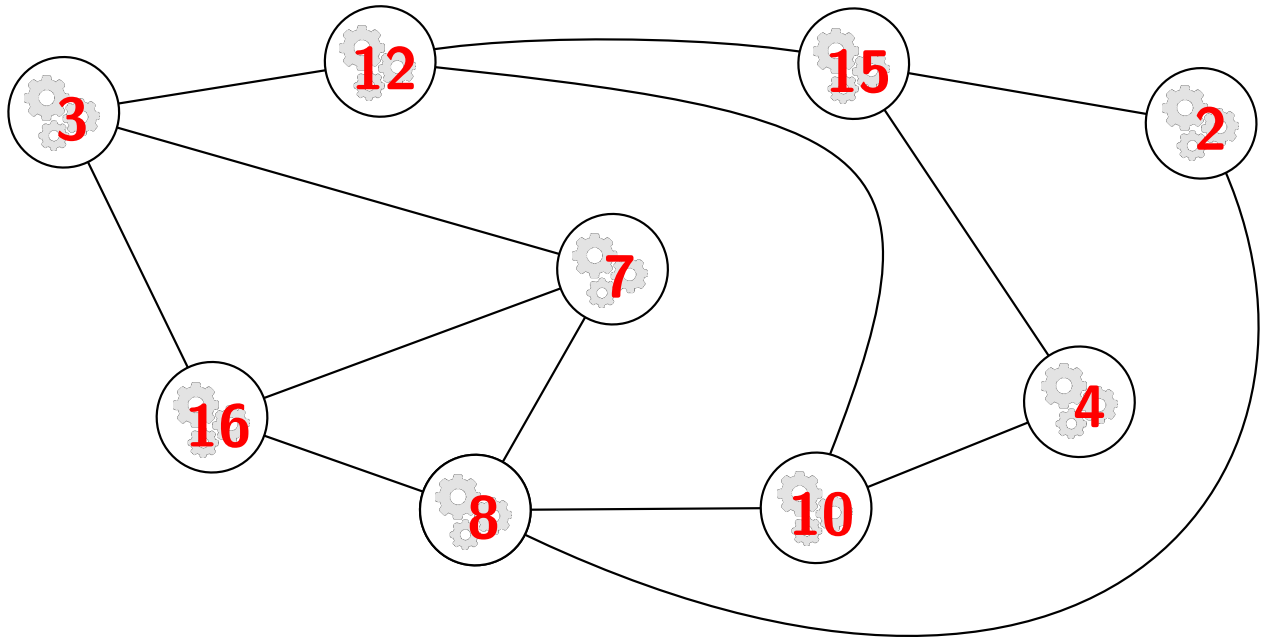

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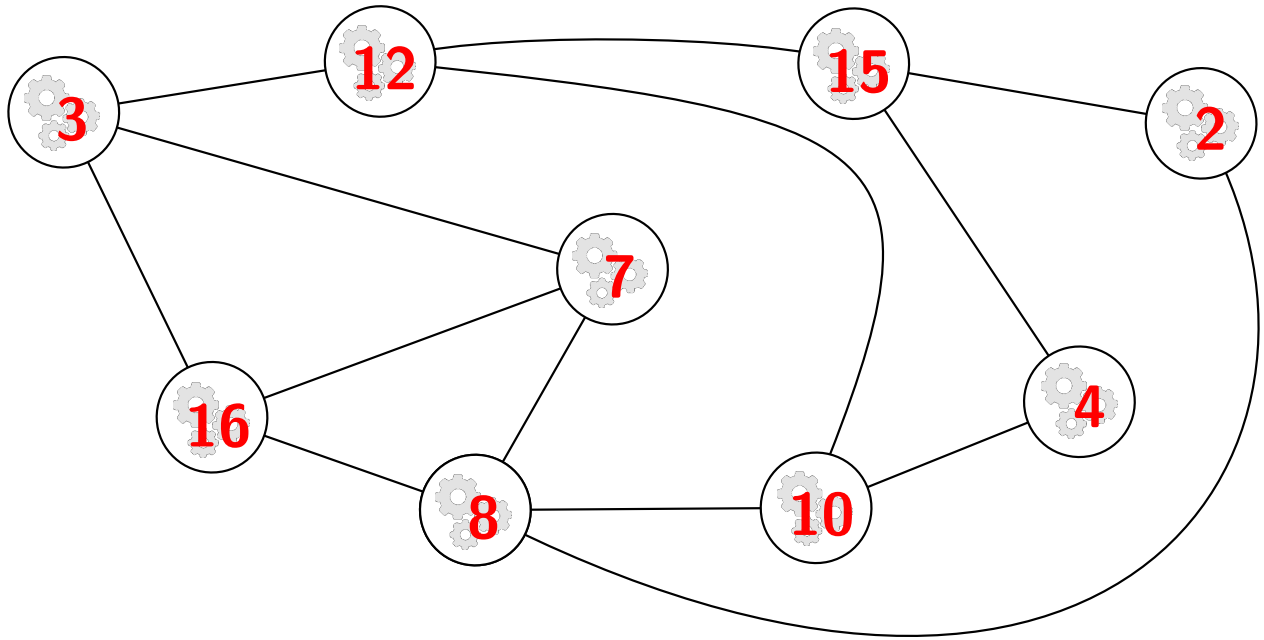

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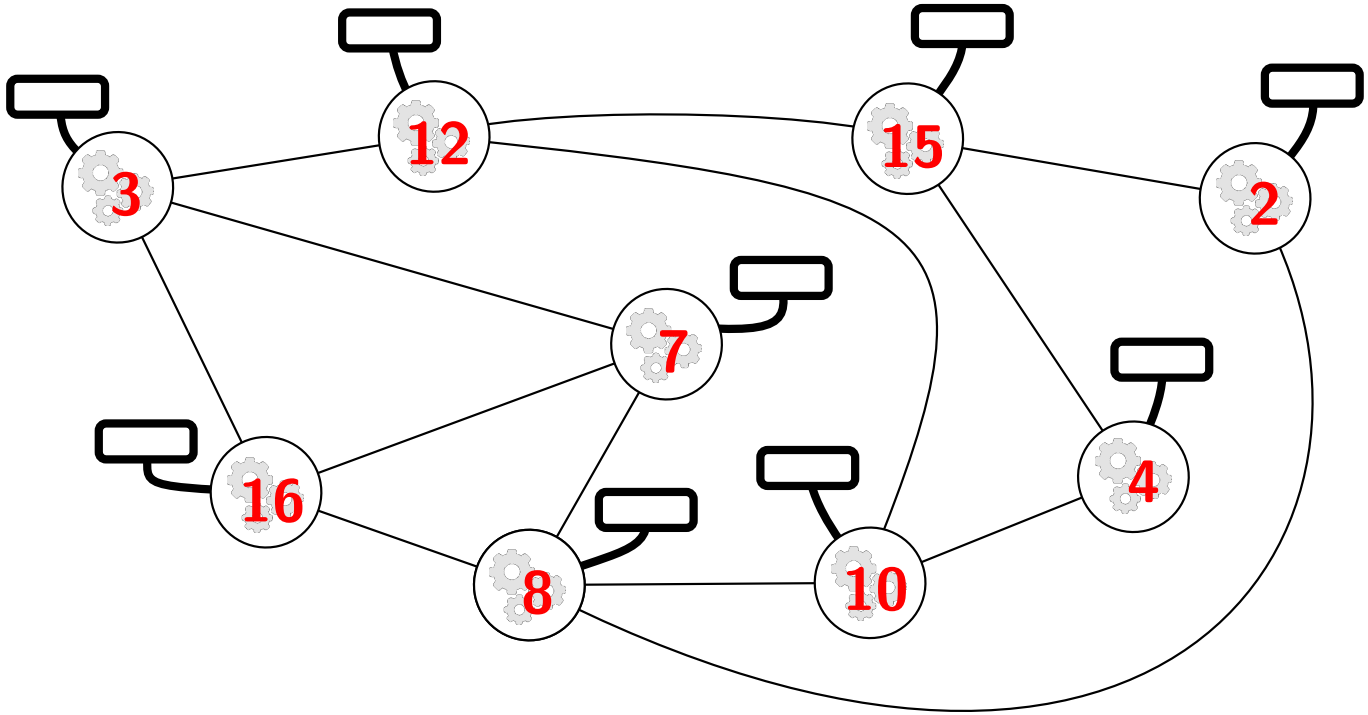

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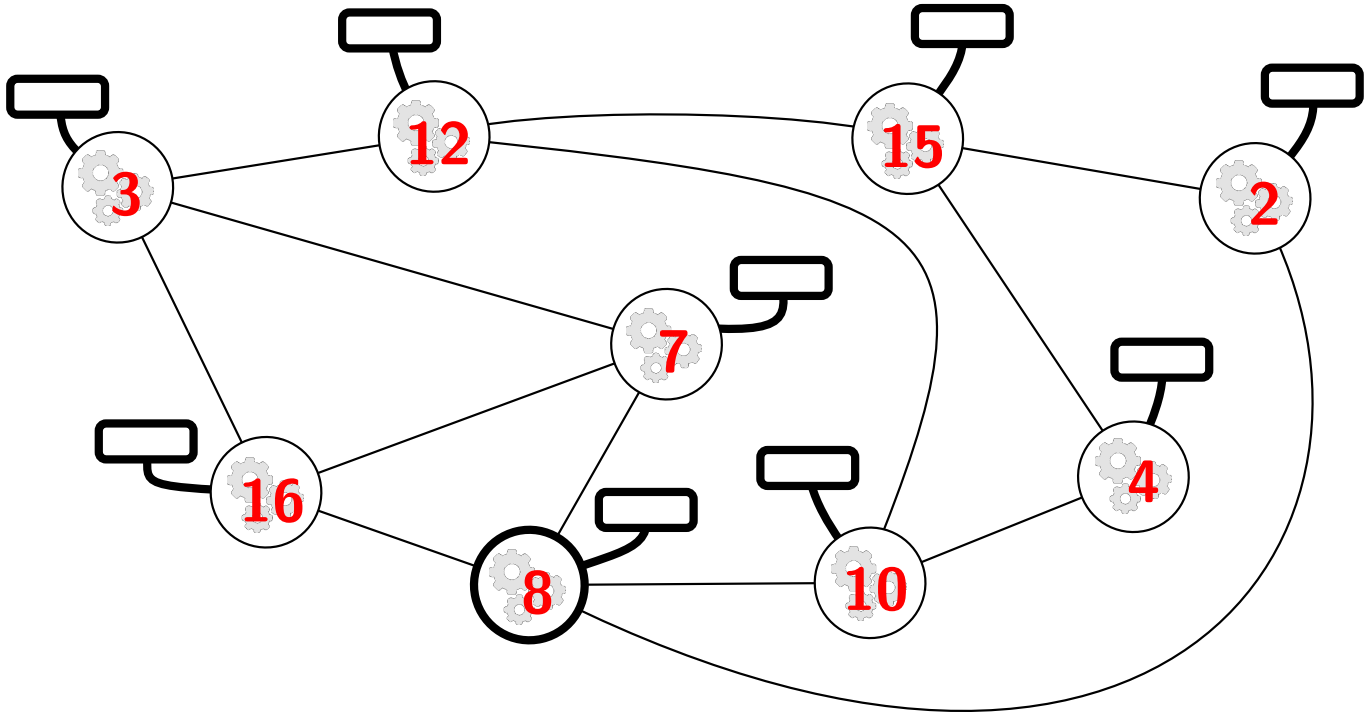

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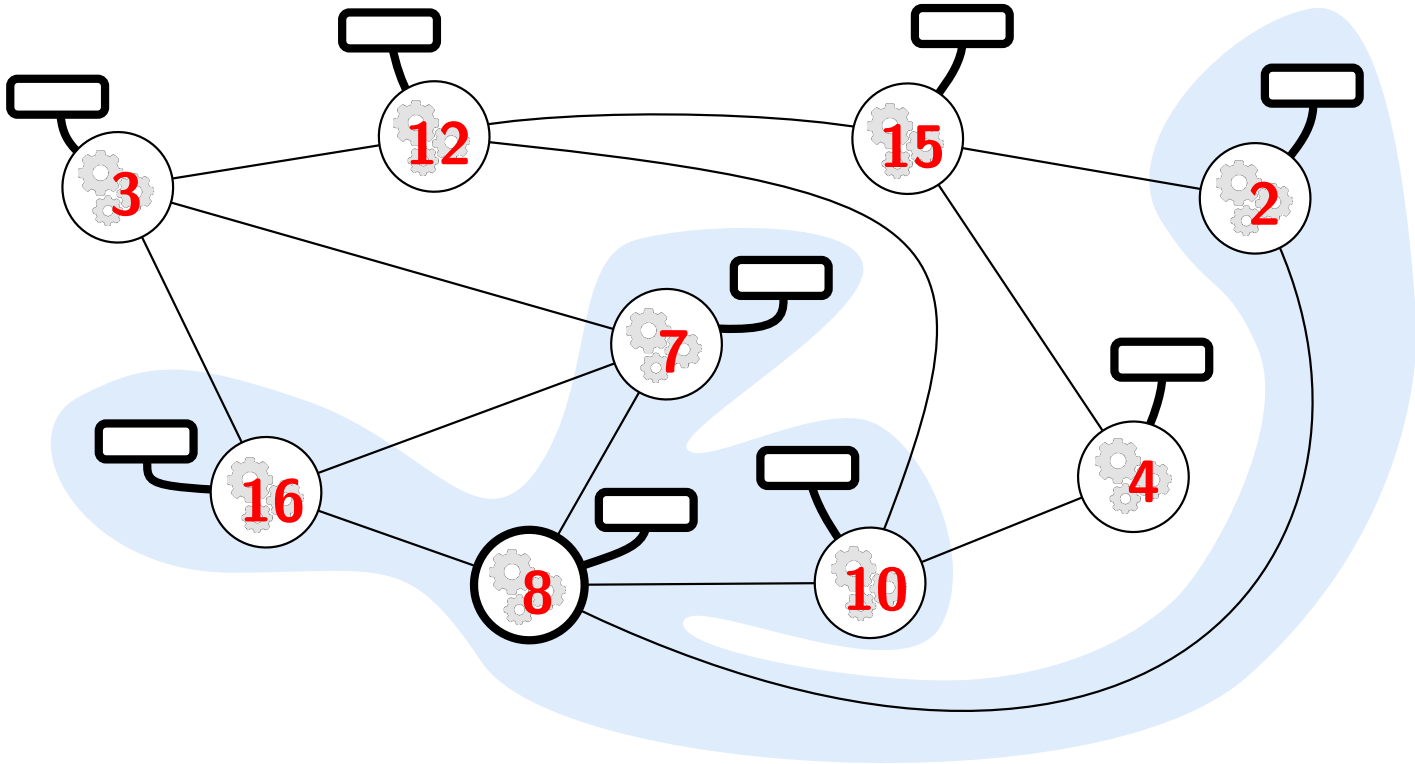

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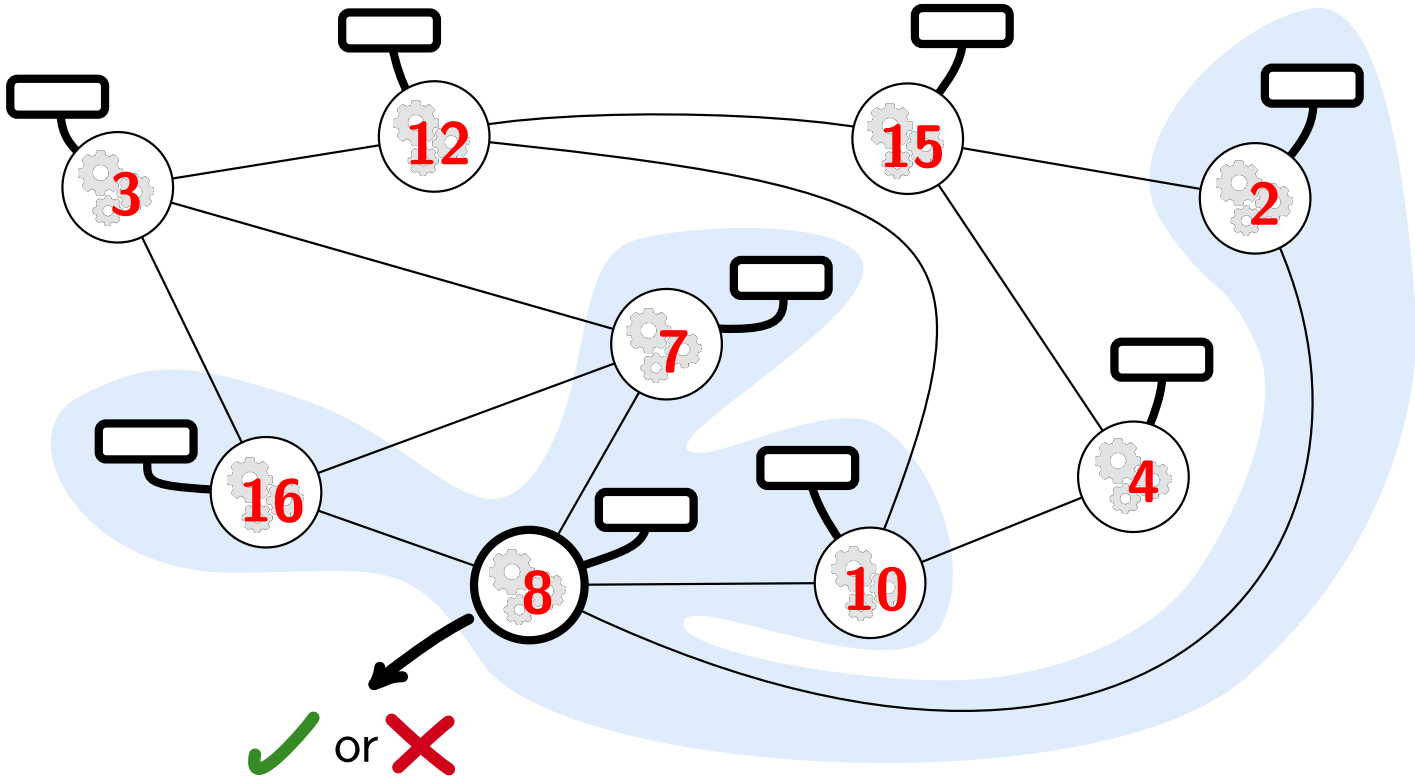

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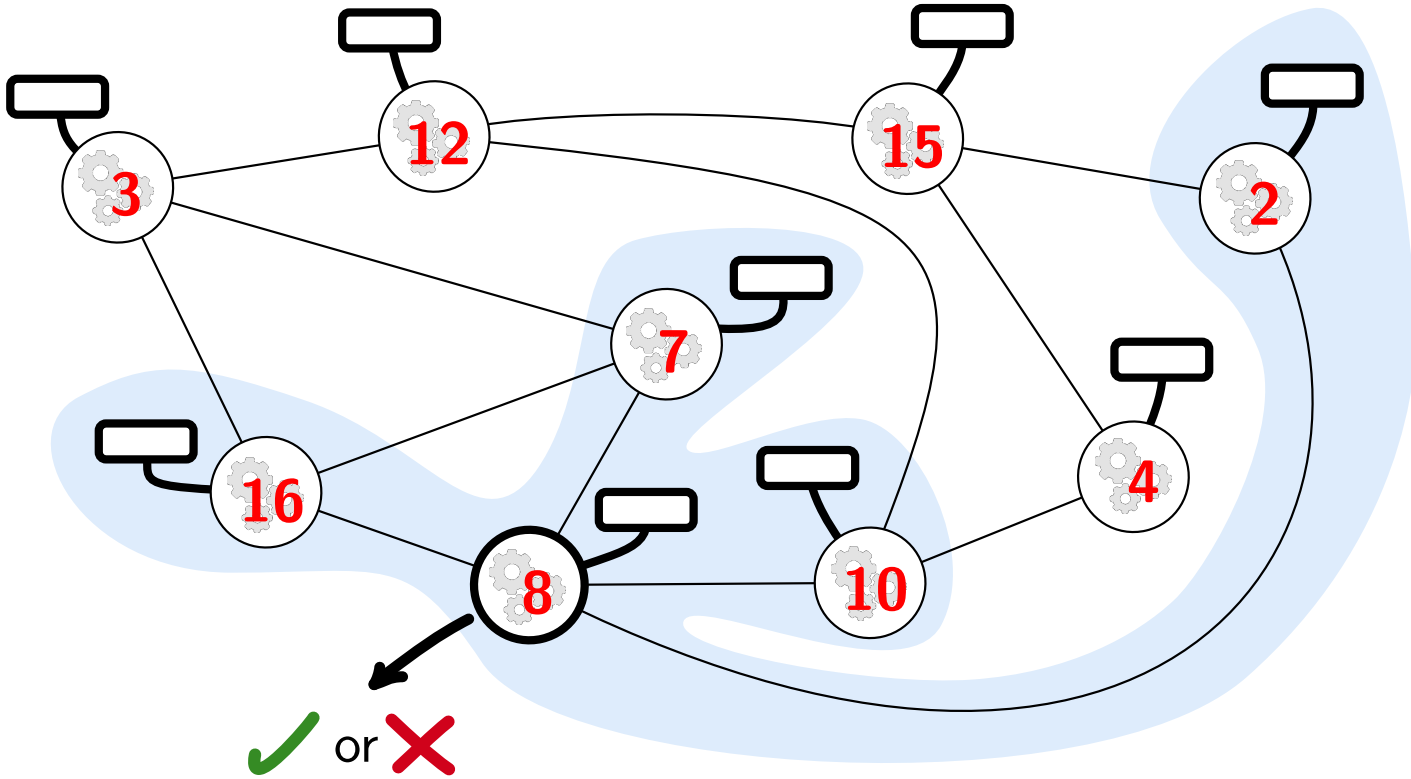


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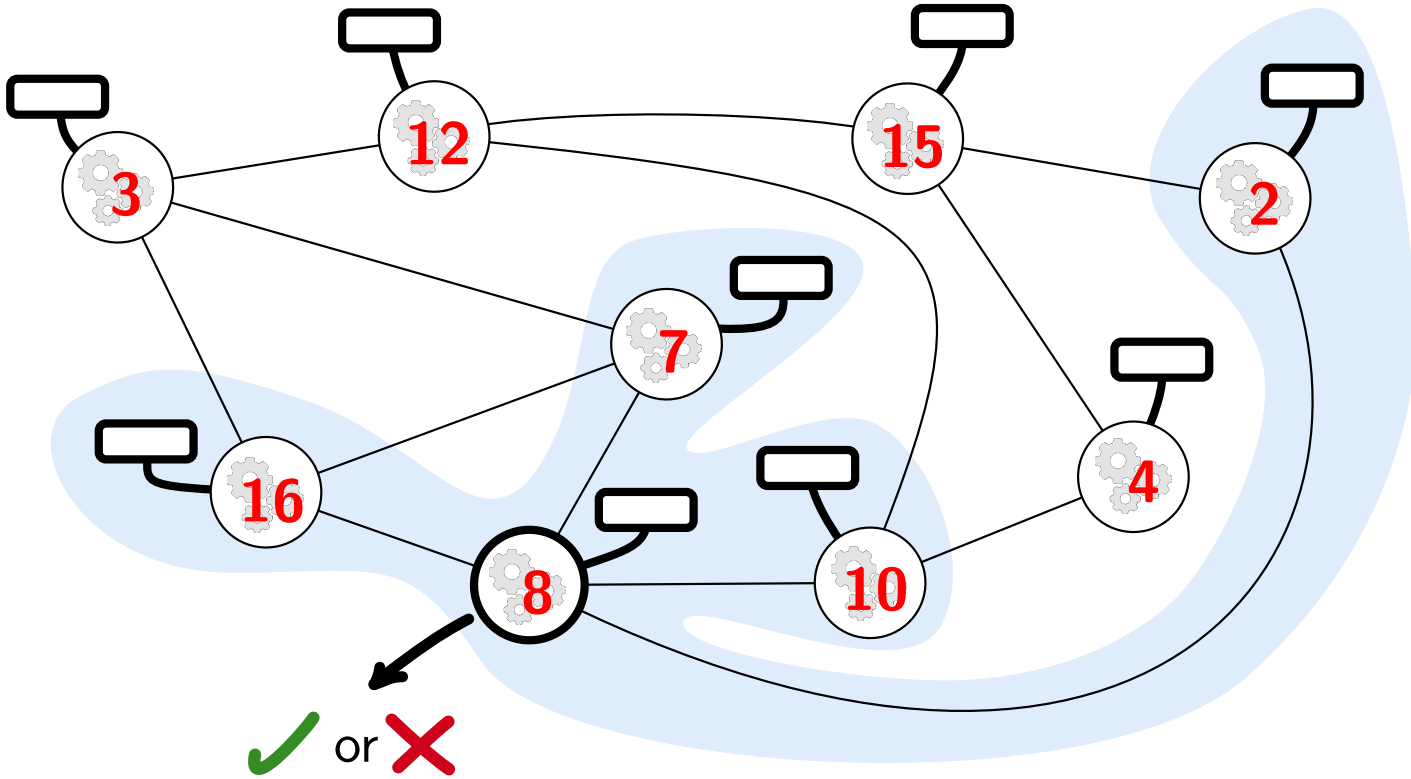


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$G$ satisfies $\mathcal{P}$ $\iff$ there exists an assignment of the certificates such that $G$ is accepted

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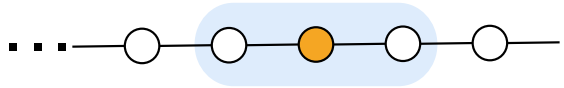

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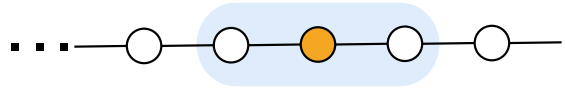


Certificate : distance to a fixed endpoint.

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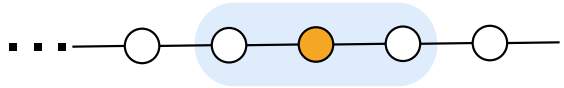


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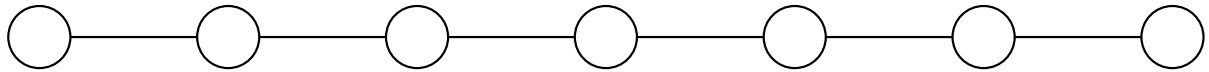

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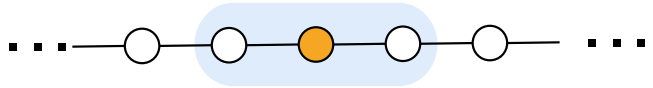


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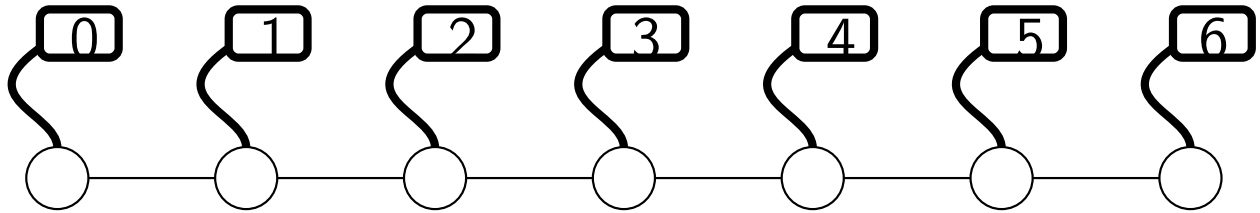

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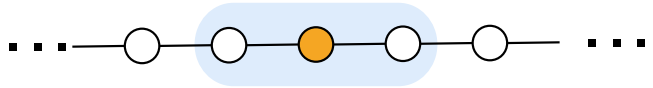


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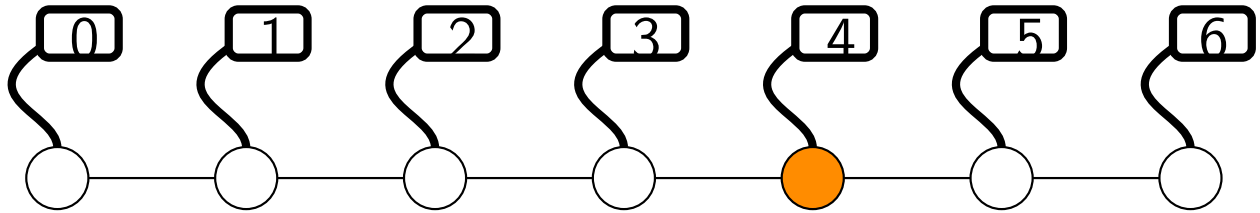

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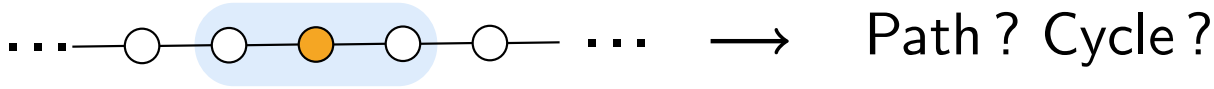


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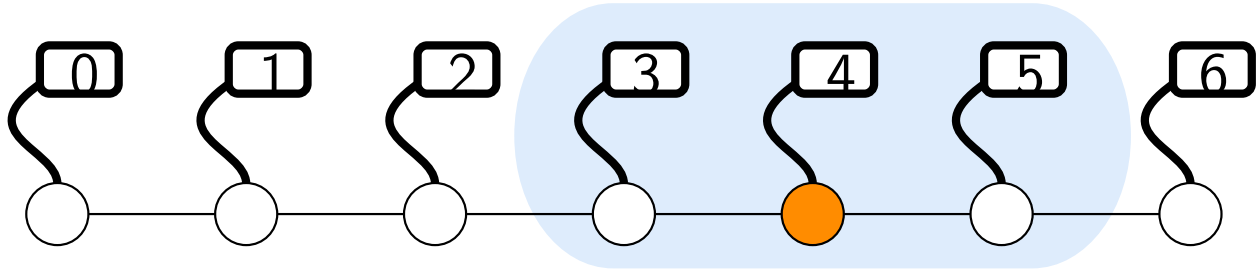

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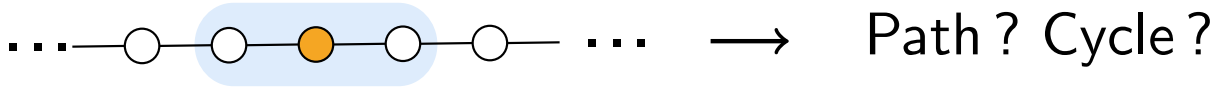


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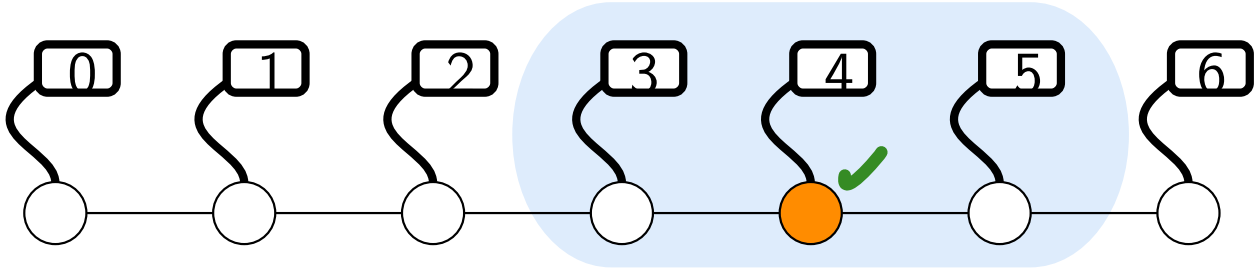

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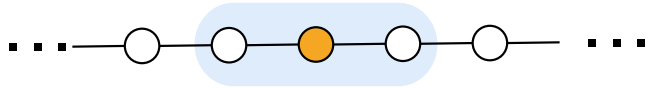

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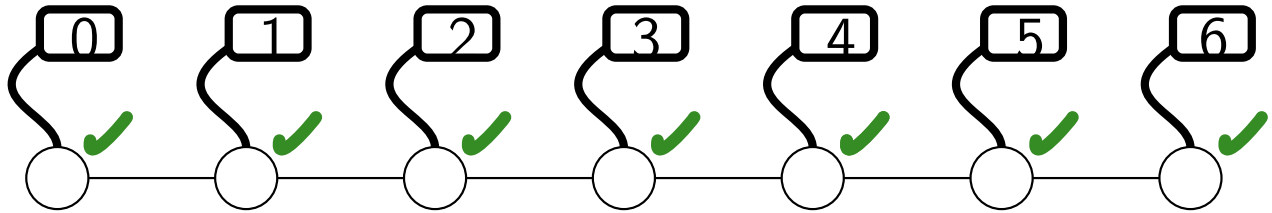

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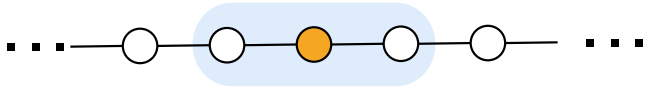

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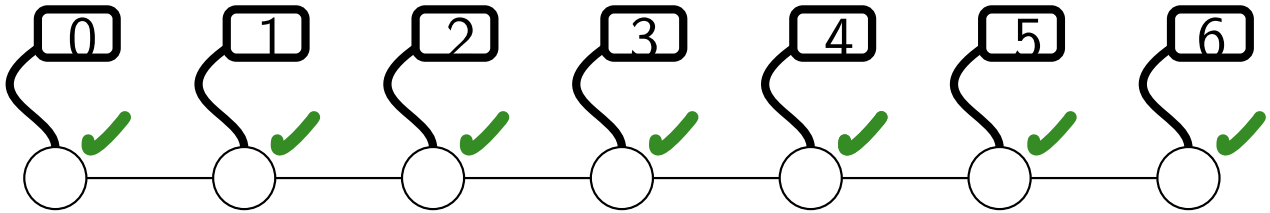


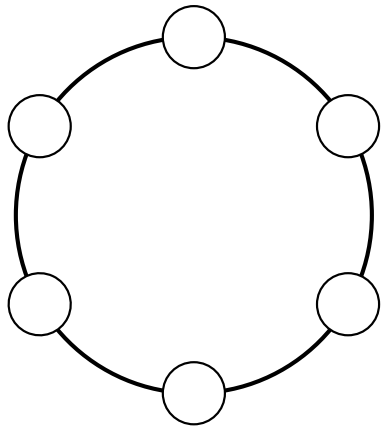

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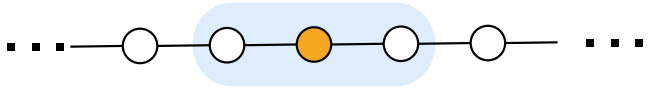

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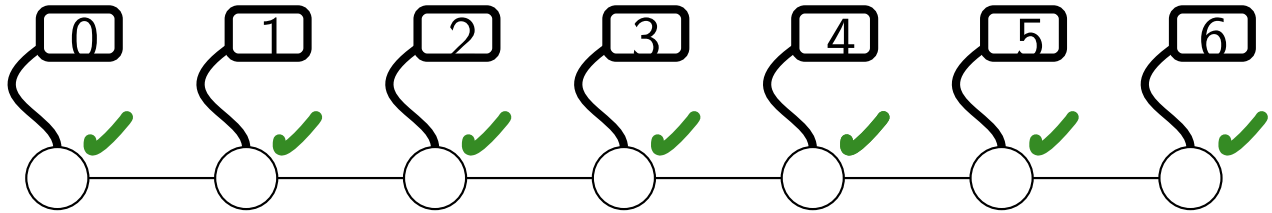


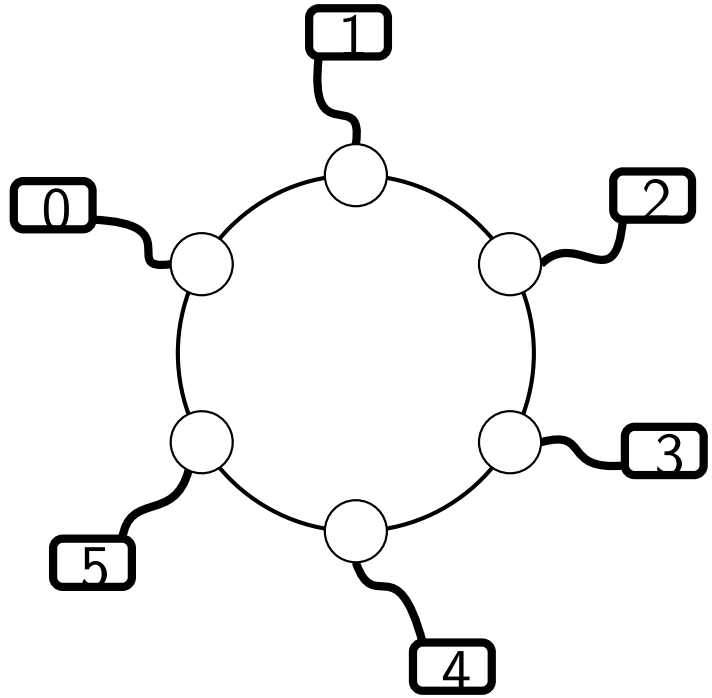

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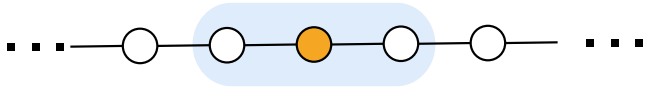

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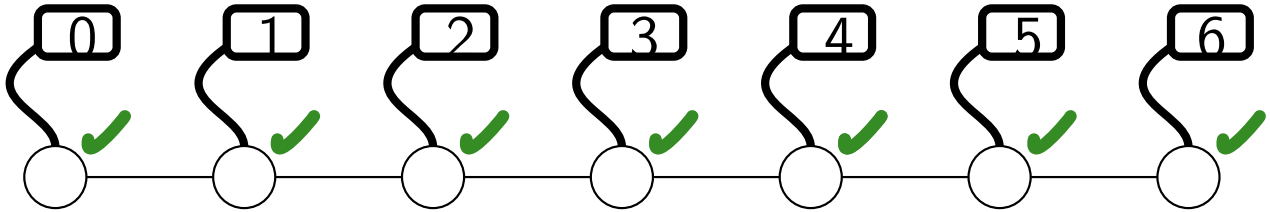


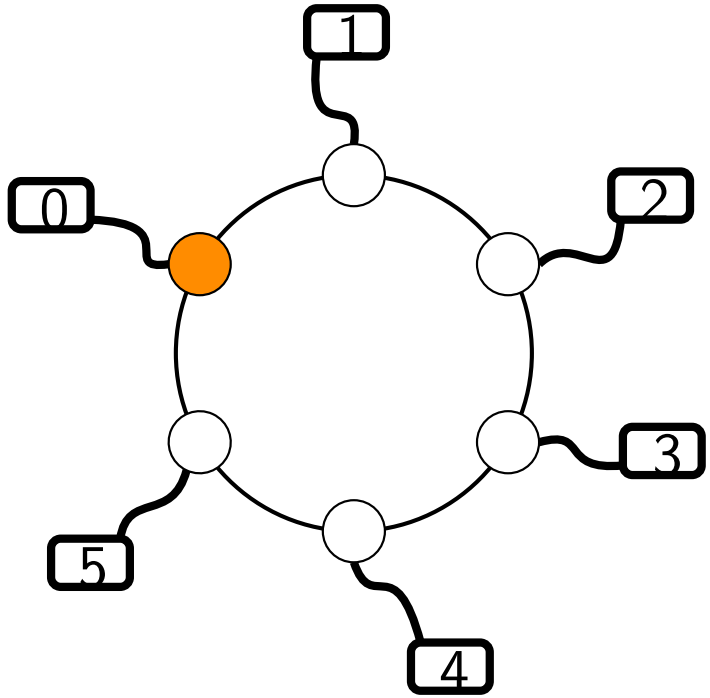

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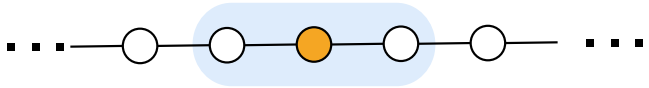

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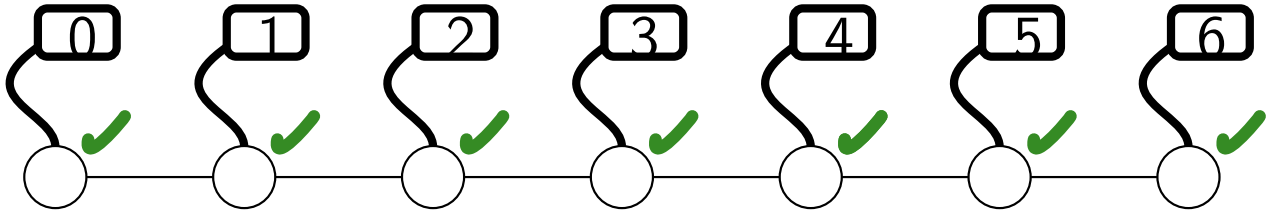


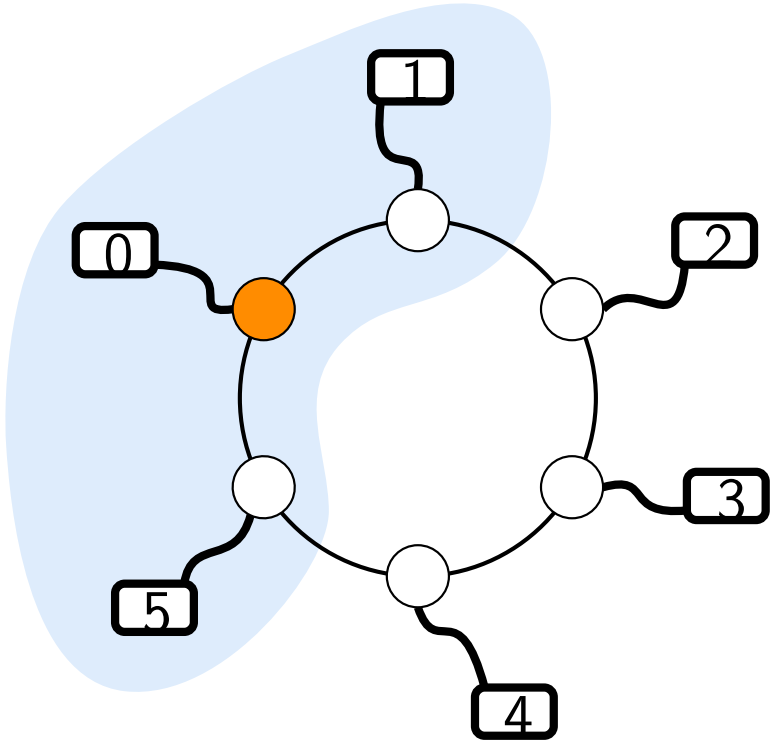

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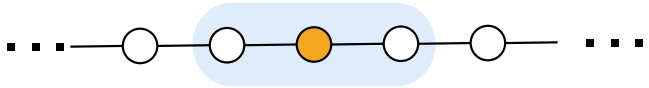

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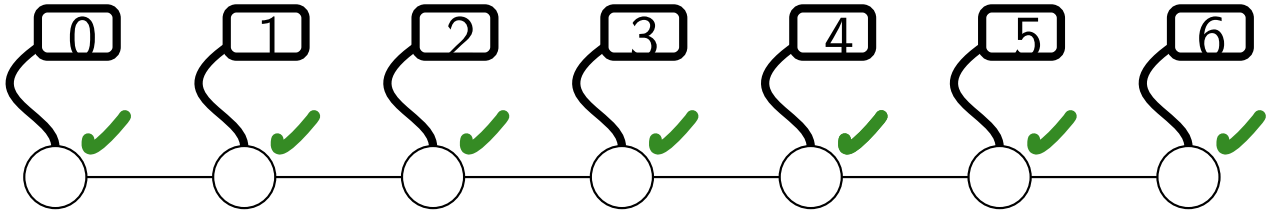


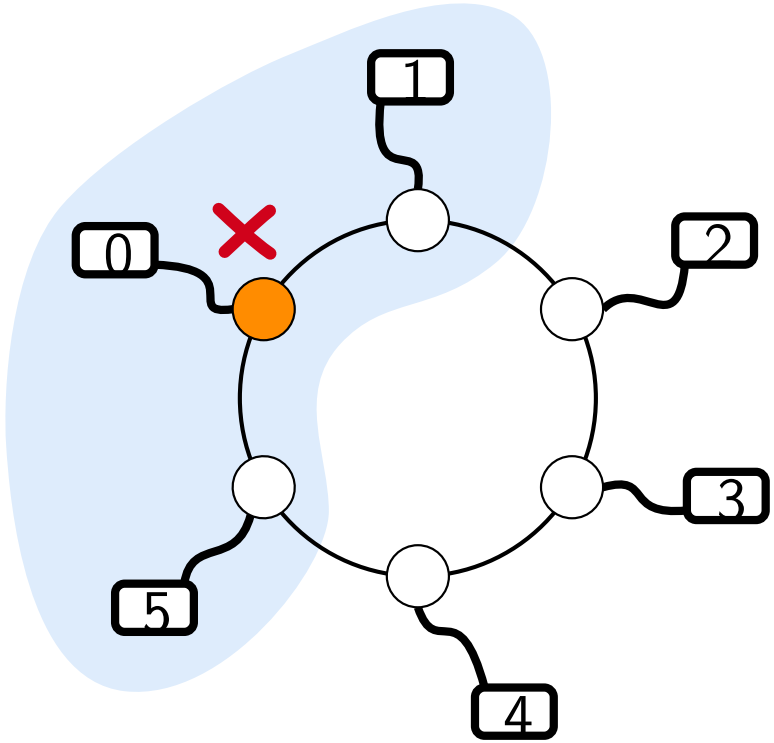

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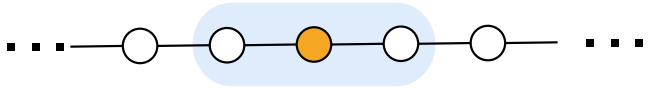


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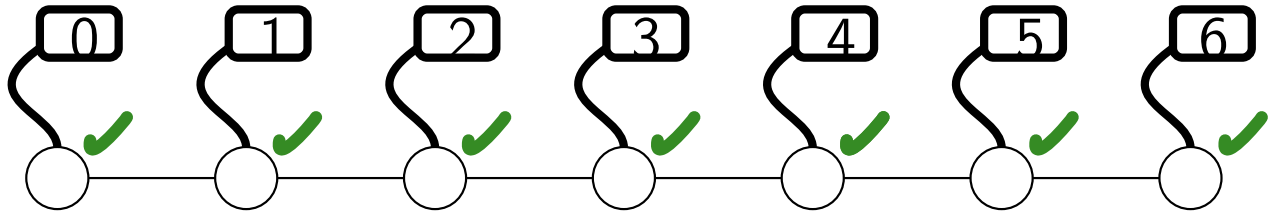


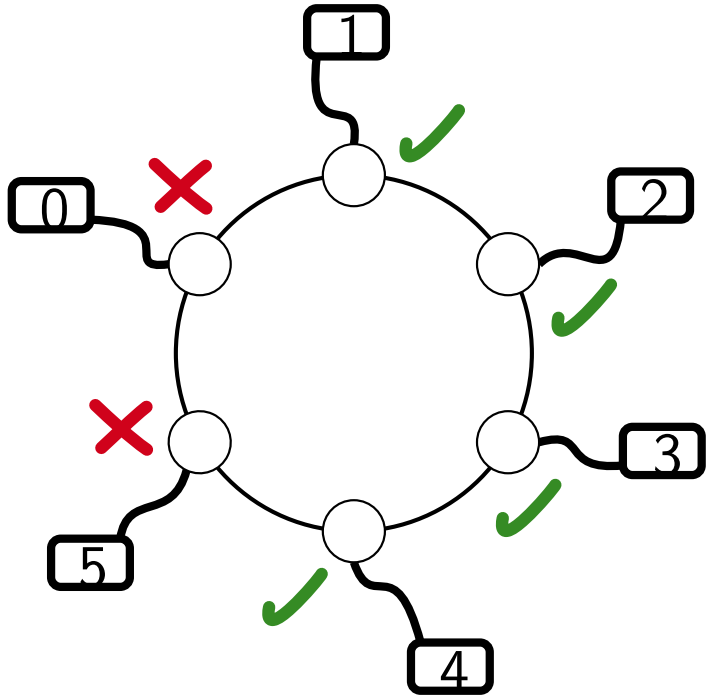

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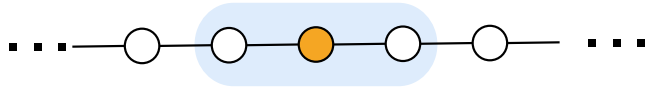

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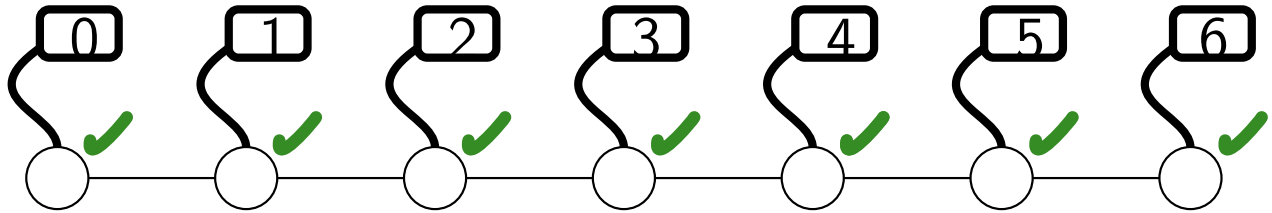


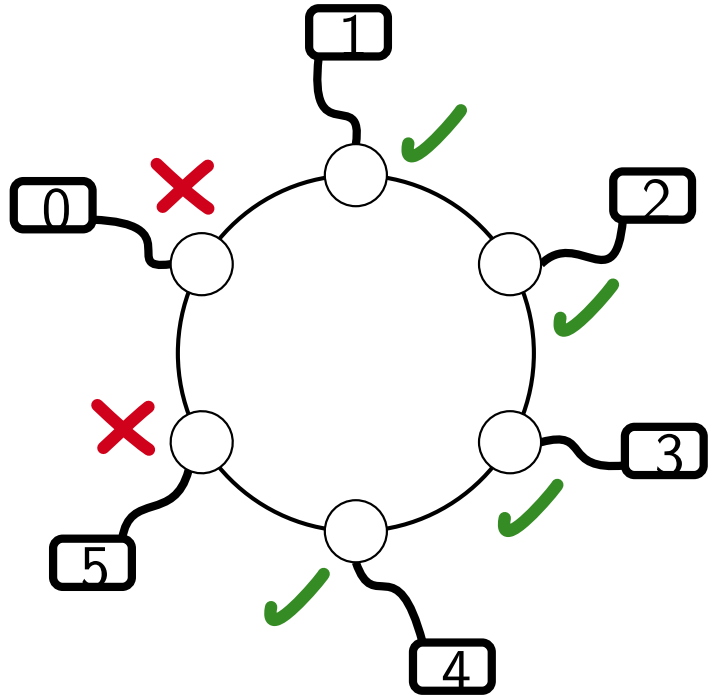


Size of the certificates : $\lceil \log n \rceil$

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**Question :** What happens below $\log n$ ?

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**No gap :**

- For caterpillars

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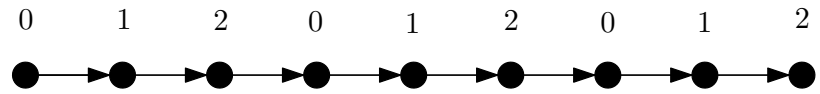

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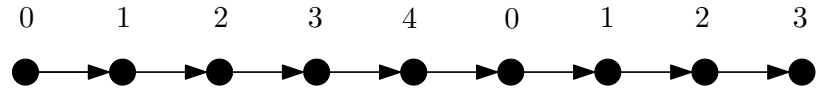

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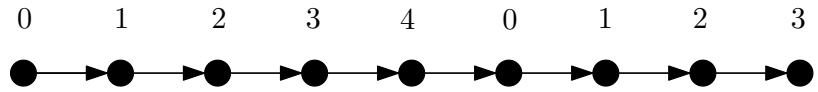


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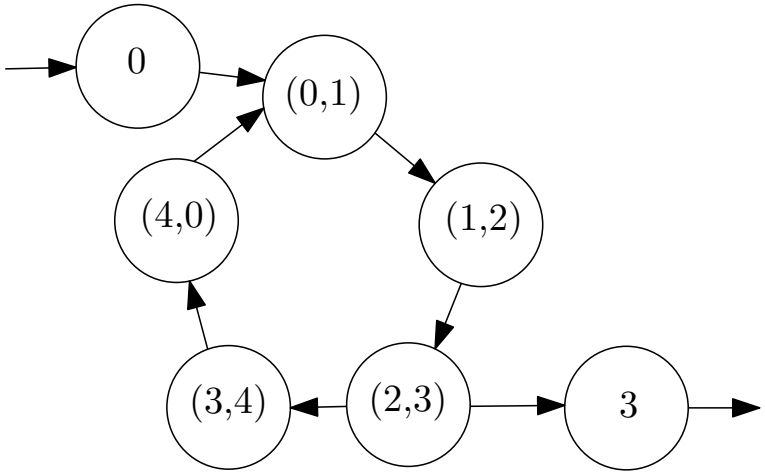

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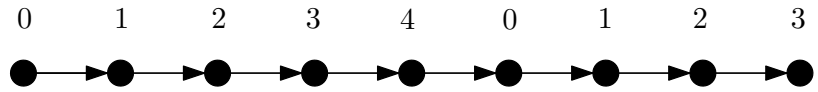


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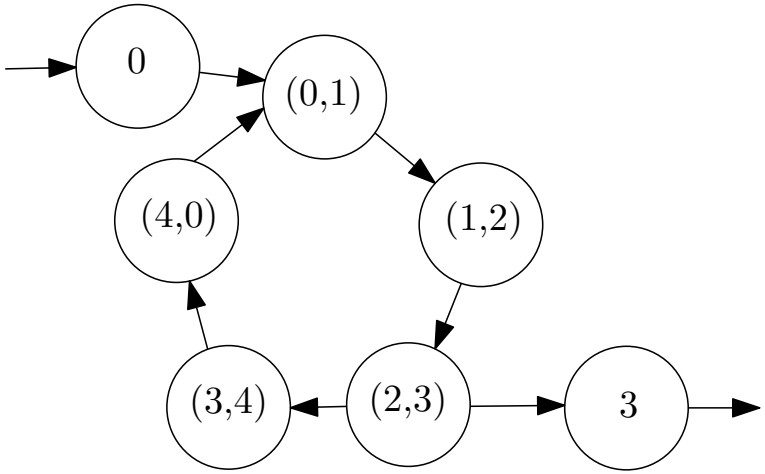


**Informal statement**

Certification with $r$ bits $\Leftrightarrow$ Automaton with $2^{2r}$ states.

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**Remark :** Constant can be improved using Chrobak's theorem

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$S = \{n \ / \ n$ is the product of the $i$ first prime (for some $i)\}$ can be certified with $O(\log \log n)$ bits.

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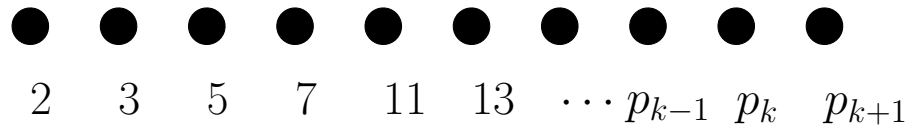

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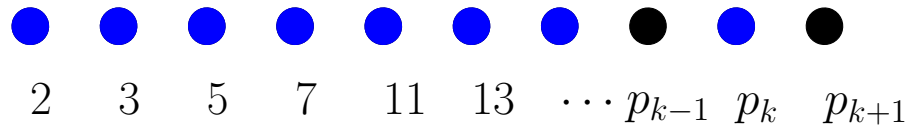

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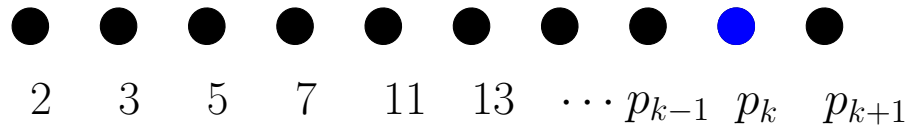


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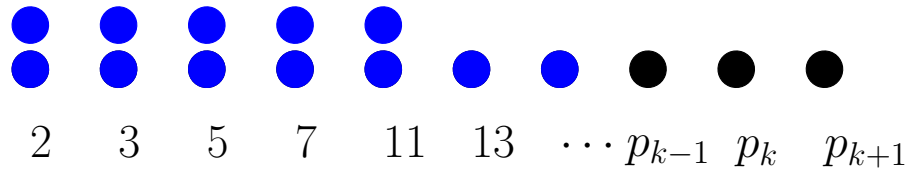


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Problem 2 : Might not exist.

# Main lemma

**Lemma :**

For $n \geq 3$, $n \notin S$ iff :

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3. $\exists p_k \leq c \log n$ and $m \leq \log n$ such that $p_k | n$, $p_{k+1} \nmid n$ and $n \neq a_k := \Pi_{i=1}^{k} p_i \mod m$

(Chinese remainder theorem)

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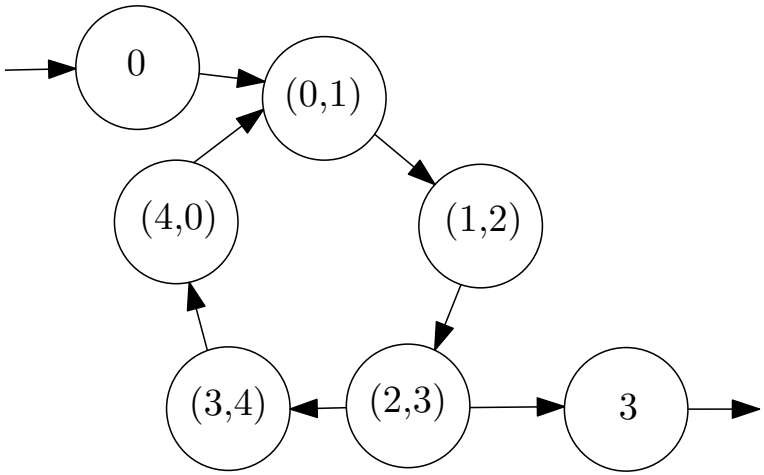
## What is different ?

Paths

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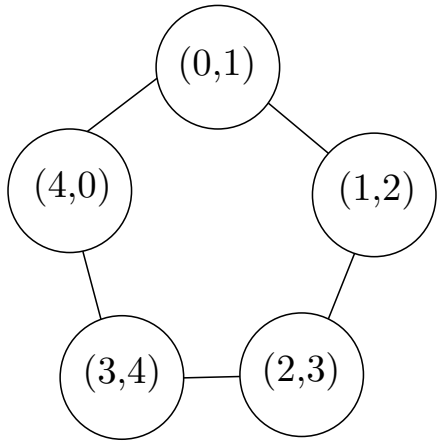

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$\mathcal{G}_k$ defined by

- Vertices : pairs of certificates of $\leq k$ bits $\Leftrightarrow \leq 2^{2k}$ vertices
- Edges : $(a, b) - (b, c)$ if there exists an accepting run where two adjacent vertices receive these certificates.

**Remark :**
$n$ accepted with $\leq k$ bits $(n \in S_k) \Leftrightarrow$ Cycle length $n$ in $\mathcal{G}_k$.

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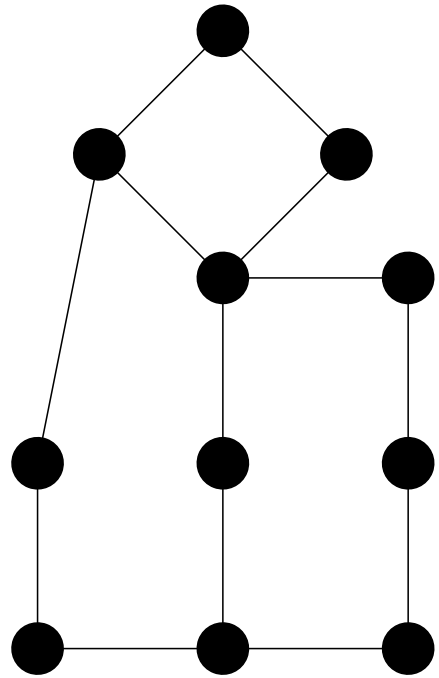

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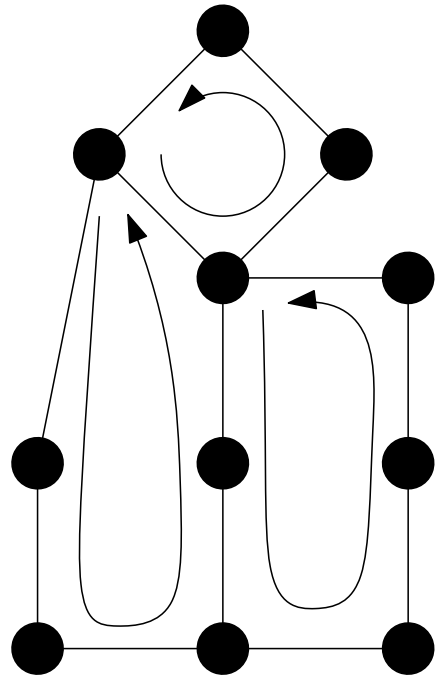

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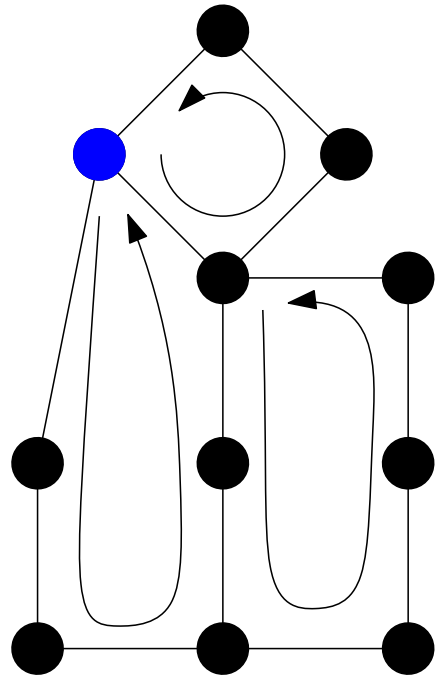

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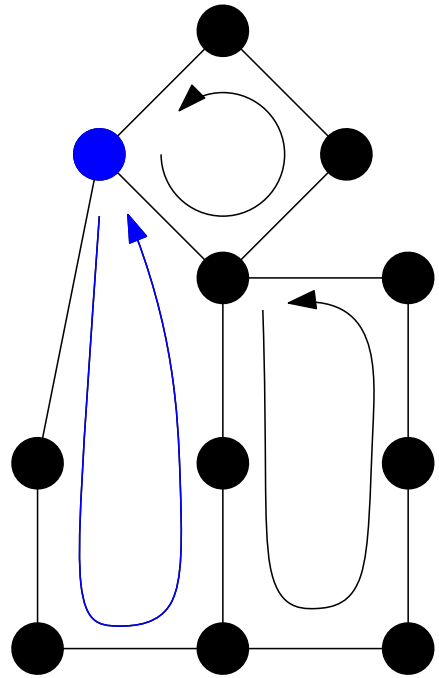

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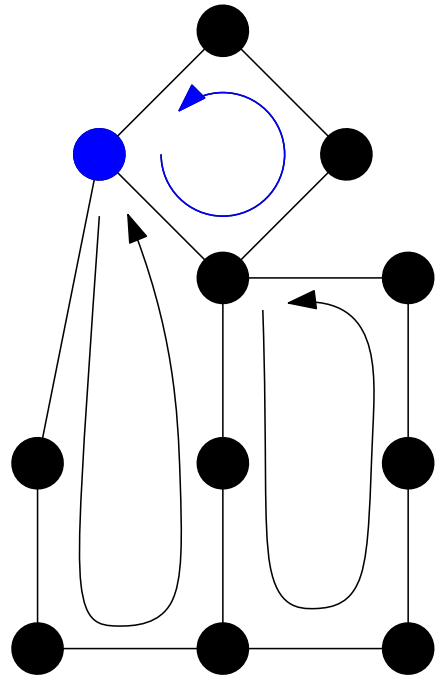

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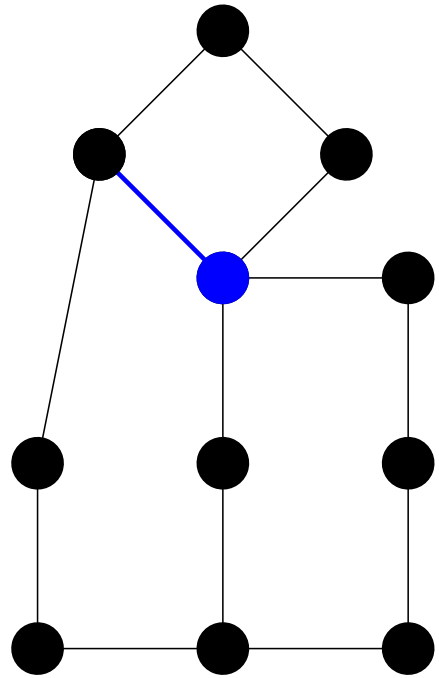

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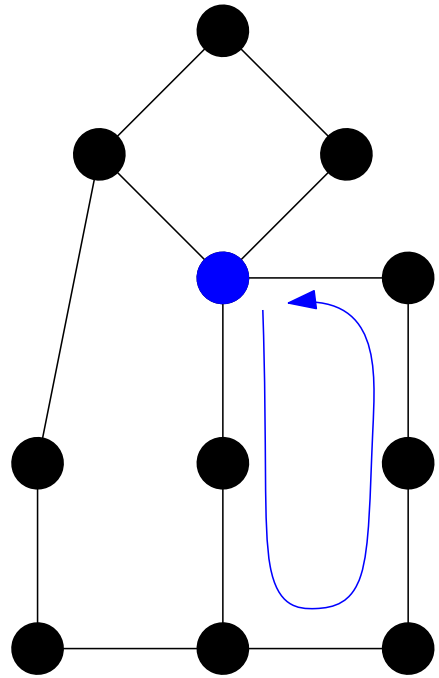

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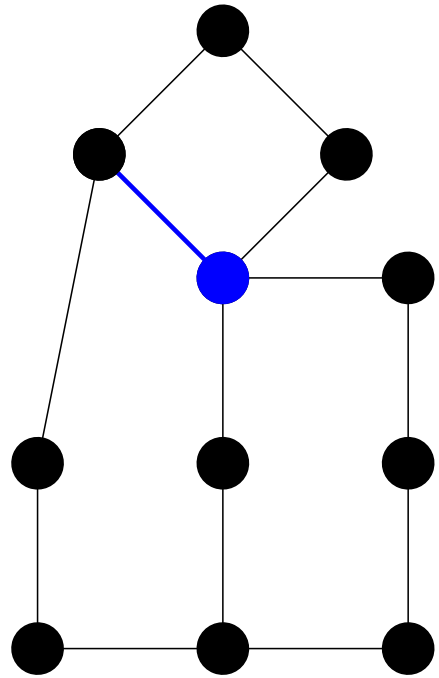

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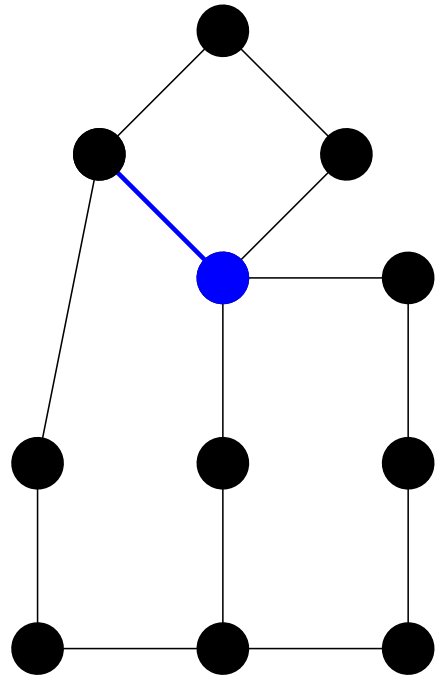

**Tool 1.** (Generalized Bezout theorem) $d = gcd(n_1, \ldots, n_k)$. If $n \geq \max(n_i)^2$ and $d|n$ then, $n = \sum a_i n_i$ where $a_i \geq 0$

Classic version : $\forall n, m \in \mathbb{N}$, $\exists a, b \in \mathbb{Z}$ such that $na + mb = gcd(n, m)$.

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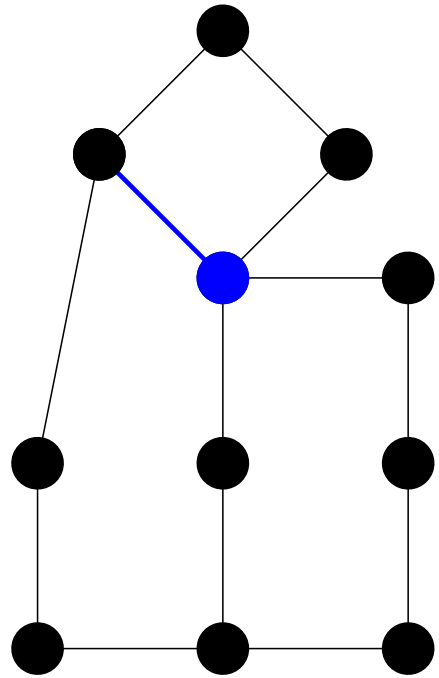

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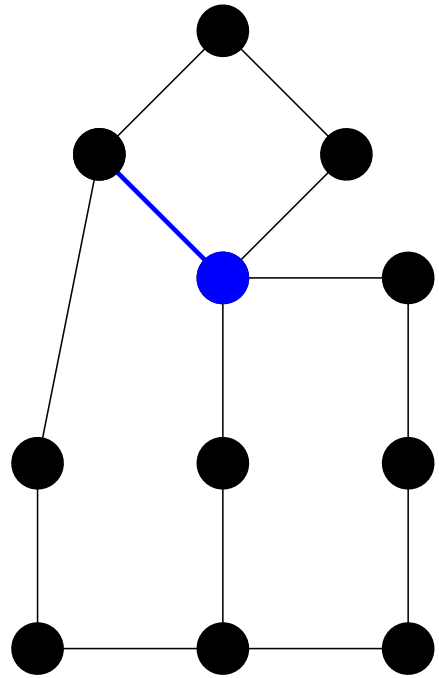


**Tool 1.** (Generalized Bezout theorem) $d = gcd(n_1, \ldots, n_k)$. If $n \geq \max(n_i)^2$ and $d|n$ then, $n = \sum a_i n_i$ where $a_i \geq 0$

Classic version : $\forall n, m \in \mathbb{N}$, $\exists a, b \in \mathbb{Z}$ such that $na + mb = gcd(n, m)$.

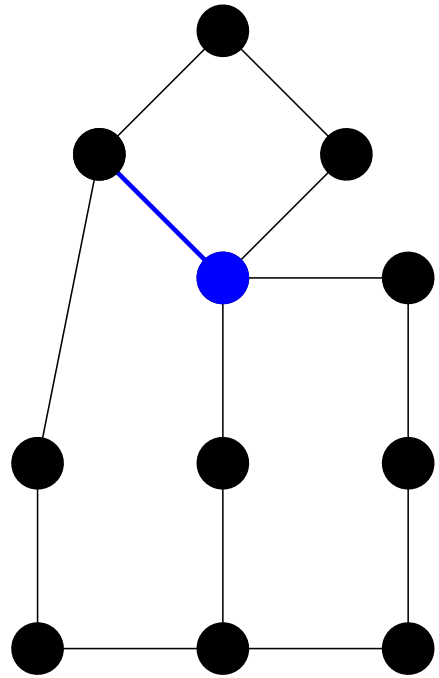
**Tool 2.** Prime numbers are "dense"

# Sketch of the proof

- $d = gcd$(cycles in (a conn. comp. of) $\mathcal{G}_k$).

$\Rightarrow \exists$ closed walk of length $\leq O(2^{O(k)})$ passing through all the cycles

- $m \geq 2^{\Omega(k)}$ and $d|m \Rightarrow m \in S_k$
- If $n >> 2^{\Omega(k)} \Rightarrow \exists$ many primes between $2^{\Omega(k)}$ and $n$.
- If min. $n \in S_k \setminus (\cup_{i \leq k} S_i)$ is too large w.r.t $2^{\Omega(k)} \Rightarrow$ Contradiction



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No gap for caterpillars (in terms of $n$).

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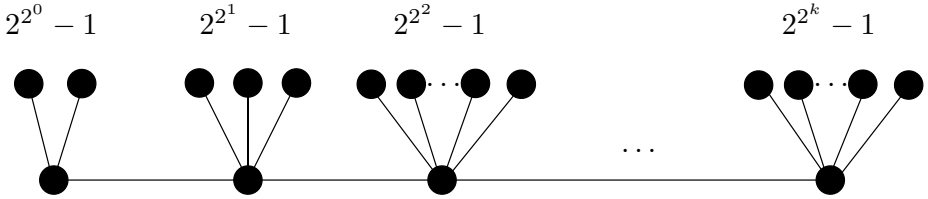

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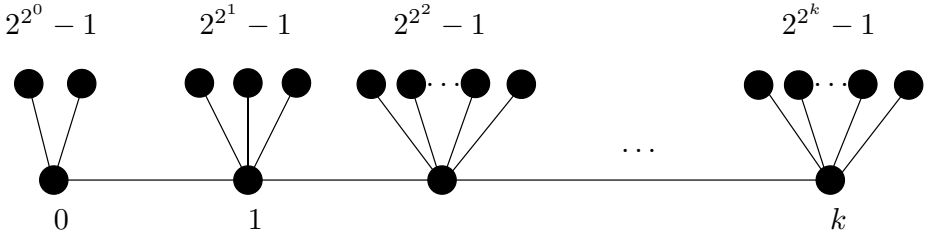

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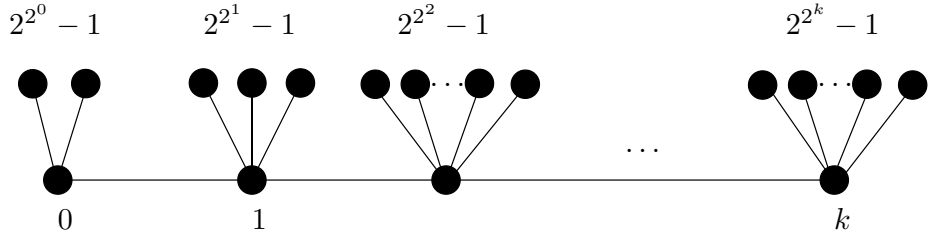


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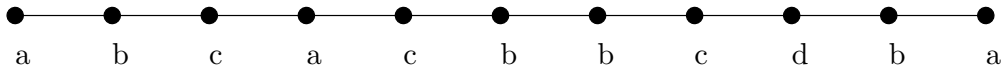

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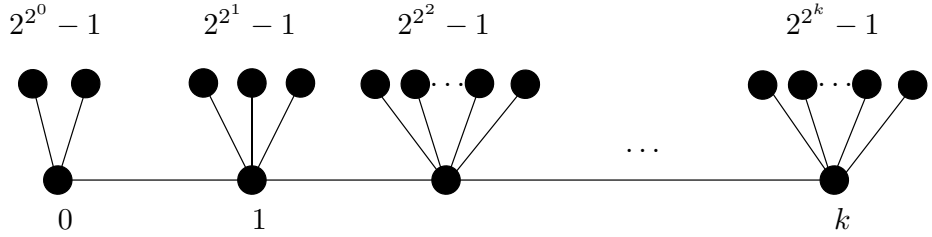


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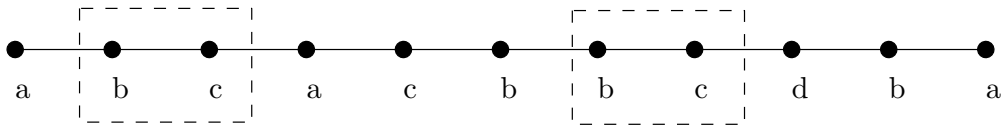

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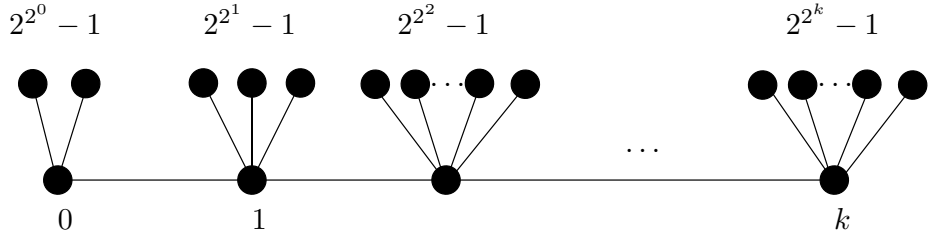


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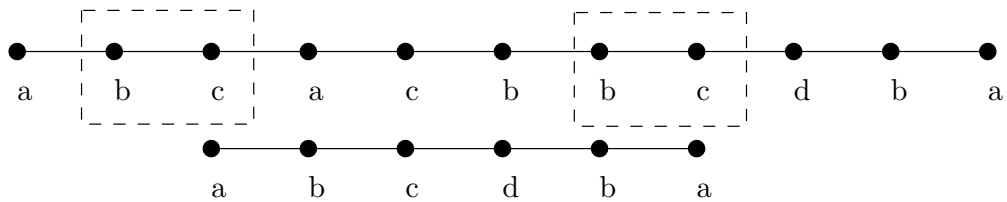

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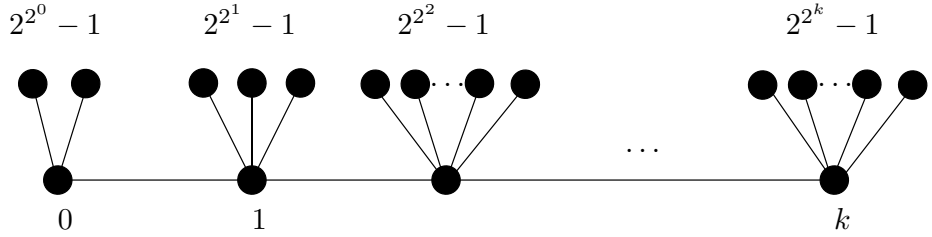


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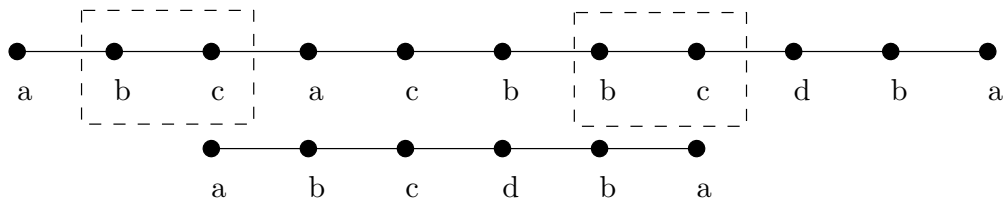


**Remark :** We can close the gap between lower and upper bound.

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**Thanks for your attention !**