Approximate Maximum Cliques in Disk and Unit Ball Graphs

Nicolas Bousquet with M. Bonamy, E. Bonnet [1], P. Charbit, S. Thomassé

Séminaire OC

1. Thanks to Édouard Bonnet for most of the figures.

Disk graphs

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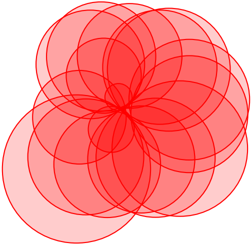
A unit disk graph is the intersection graph of disks in the plane.



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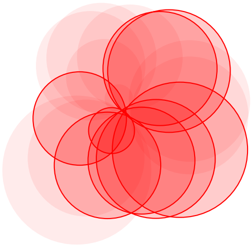

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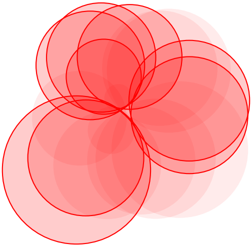

Computing Maximum Cliques

Theorem (Clark, Colbourn, and Johnson '99)

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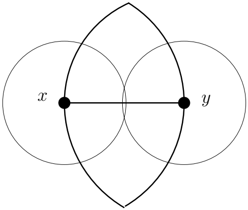

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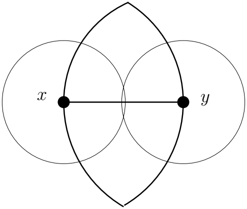

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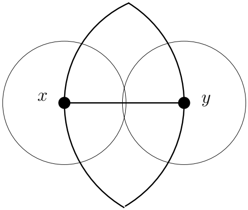
- MAX CLIQUE in a co-bipartite graph $\Leftrightarrow$ MAX INDEPENDENT SET in a bipartite graph.



What about disk graphs ?

Open problem :
Complexity status of Maximum Clique on disk graphs.

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Question :
Can we at least improve the approximation ratio ?

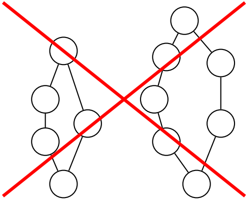
Can we obtain a 1.99-approximation algorithm if there are only two radii ? [Cabello]

Breakthrough : forbidden structure

Complement $\overline{G}$ of $G = uv \in E(\overline{G})$ iff $uv \notin E(G)$.

Theorem (Bonnet et al.)

The complement of a disk graph does not contain two anticomplete odd cycles.



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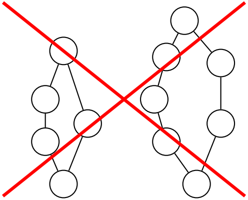

Remark : False for even cycles, false for ellips (with arbitrarily small excentricity).

Corollaries [Bonnet et al.]

There exist a subexponential algorithm and a QPTAS.

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- Compute an MIS on a bipartite graph.

Independent Set of linear size

Piercing number of geometric objects $\mathcal{C}$ = Min. number of points intersecting $\mathcal{C}$.

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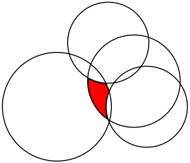

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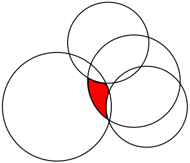

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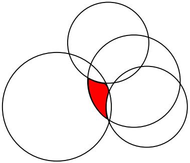

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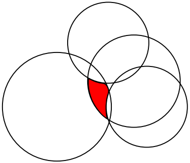
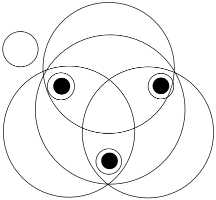


Remark : It is the classical 2-approximation algorithm.

VC-dimension



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A set $X \subseteq V$ is shattered iff all the traces on X exist.

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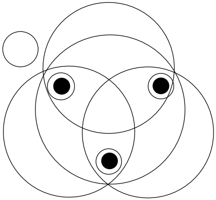

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Theorem (Haussler, Welzl '73)

Every hypergraph H of VC-dimension d has an ϵ-net of size at most $\mathcal{O}(\frac{d}{\epsilon}\log(\frac{1}{\epsilon}))$. Furthermore, any set of size at least $\frac{10d}{\epsilon}\log\frac{1}{\epsilon}$ is an ϵ-net w.h.p.

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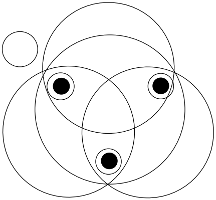

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Hypergraph for us : **closed neighborhood hypergraph**.

MIS, iocp and VC-dimension

iocp(G) = maximum number of anticomplete odd cycles.
VC(G) = VC-dimension of G.
$\alpha(G)$= size of a maximum MIS.

Main Theorem (Bonamy, Bonnet, B., Charbit, Thomassé)

If
- $\alpha(G) = \Omega(n)$;
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Disks graphs satisfy

- $\alpha(G) \geq n/4$ (restricted to the set of candidates)
- VC-dimension ≤ 4 [Aronov, Donakonda, Ezra, Pinchasi]
- $iocp(G) = 1$ [Bonnet et al.]

What about higher dimension ?

A k-ball graph = intersection graph of balls in $\mathbb{R}^k$.
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Questions :

- Complexity status ?
  Only known to be hard if $k \geq \log n$ [Afshani, Hatami].

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- Approximation algorithms ?
  2.553-approximation algorithm for unit 3-ball graphs. [Afshani, Chan]
  No known approximation in general.

Hardness results

Theorem (Bonamy, Bonnet, B., Charbit, Thomassé)

- No $(1+\epsilon)$-approximation algorithms in time $2^{0.99n}$ for k-ball graphs for $k \geq 3$ (under ETH).
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Sketch of the proof :

- If a class $\mathcal{G}$ contains all the 2-subdivisions, then $\mathcal{G}$ has no $(1+\epsilon)$-approximation algorithm in time $2^{0.99n}$ for MIS (under ETH) [Bonnet et al.].
- Unit ≥ 4-ball graphs and ≥ 3-ball graphs contain all the co-2-subdivisions.

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- Apply the Main Theorem.

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How can we find X algorithmically ?

- Enumerate all the sets of size $\frac{10d}{\epsilon^3}\log(\frac{1}{\epsilon^3})$ in G (PTAS).
- Sample $\frac{10d}{\epsilon^3}\log(\frac{1}{\epsilon^3})$ vertices at random (rand. EPTAS).

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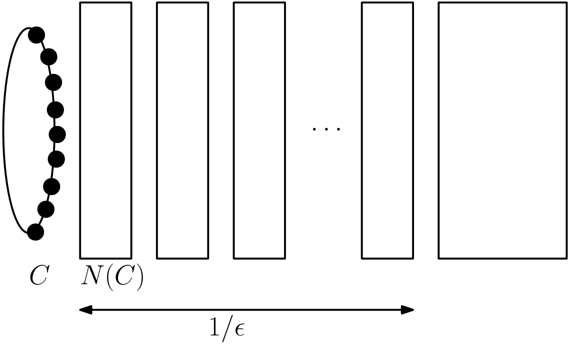
- Delete $C \cup N(C)$.
- It remains a bipartite graph with $\alpha \geq (1-\epsilon)|I|$.
  since there is no two anticomplete odd cycles.

Odd cycles of length $\geq \frac{1}{\epsilon}^2$



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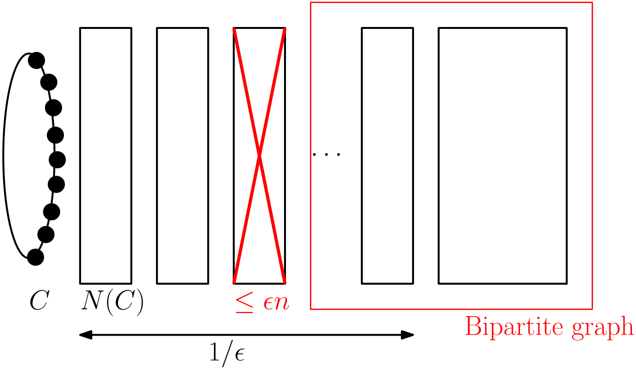

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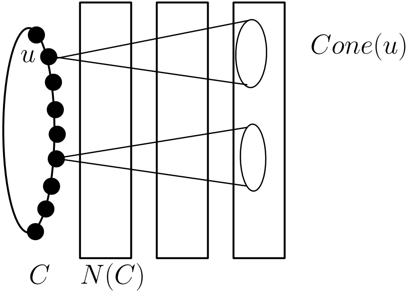

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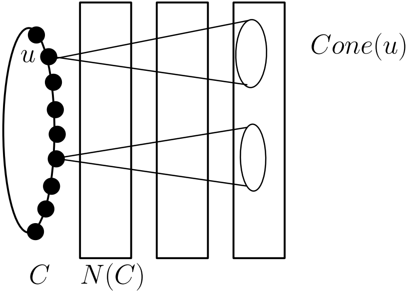

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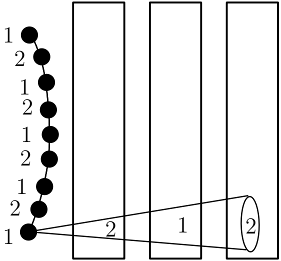

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- After the deletion of this cone, the resulting graph is bipartite.

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- Derandomization of the EPTAS.
  Derandomization of [Haussler, Welzl] is possible.
- MIS if iocp=1 ? (with no VC-dimension assumption)

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Thanks for your attention !