Coalition games on interaction graphs

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joint work with Zhentao Li and Adrian Vetta

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The problem

Let $G = (V, E)$ be a graph.
Let $\mathcal{C}$ be a collection of connected subgraphs of G.

Maximum Packing : Maximum number of sets of $\mathcal{C}$ that are pairwise disjoint.
Notation : ν.

Minimum Covering : Minimum number of vertices of V such that any set of $\mathcal{C}$ contains one of these vertices.
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Question :
Can we bound the packing-covering ratio and/or the integrality gaps (with some graph parameters) ?

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- Unfortunately, people are selfish : if it is is more interesting for them, they will create a project of their own.
- Solution : distribute payoff in such a way people do not want to leave the grand coalition.

Coalition games

Coalition game

- A set I of n agents.
- A valuation function $v : 2^n \rightarrow \{0, 1\}$. (it actually has a more general definition but it allows us to think about it as a hypergraph)

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Goals :

- Of the external authority. Maximize the value generated by the set of agents.
- Of the agents. Maximize their payoff
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  $\Rightarrow$ Be sure that if a coalition S leaves the whole group I, their agents cannot make more money.
  $\Rightarrow$ The external authority wants stability so it will pay at least $v(S)$ to S for each S.

Reformulation of goals

External authority goal 1 : Maximizing welfare.

$$\begin{aligned}
\nu(\mathcal{G}) \quad &= \max \sum_{S:S\subseteq I} v(S)\cdot y_S \\
\text{s.t.} \quad &\sum_{S\subseteq I: i\in S} y_S \le 1 \qquad \forall i\in I \\
&y_S \in \mathbb{N} \qquad\qquad \forall S\subseteq I
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External authority goal 2 : Minimizing cost :

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Remark : It is the Fractional Hitting Set problem !

Relative Cost of Stability

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In general all these value can be arbitrarily large !

Interaction graph

Myerson proposed the following model :

Definition (interaction graph)

Let G be a graph where the vertices of G are the agents of the coalition game $\mathcal{G}$.
The game $\mathcal{G}$ has interaction graph G if every coalition is connected (*i.e.*, if $v(S) > 0$ then S is connected).

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Examples :

- G is a clique : any coalition may exist.
- G is a stable set : coalitions have size one.

Treewidth and coalition game

Theorem (Meir et al.)

Let G be a graph. We have the following inequality :

$$\frac{\tau(\mathcal{G})}{\nu(\mathcal{G})} \leq_{\forall} tw(G) + 1$$

Moreover there exist graphs for which this bound is tight.

By $\leq_{\forall}$, we mean that every game $\mathcal{G}$ on interaction graph G satisfies this inequality.

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Our work : Improve this result, and bounds on the relative cost of stability.

Our contribution

1. Provide lower bounds of the type "*for every graph of treewidth BLABLA, the packing-covering ratio is at least BLUBLU for some coalition game on this graph*").

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1. Provide lower bounds of the type "*for every graph of treewidth BLABLA, the packing-covering ratio is at least BLUBLU for some coalition game on this graph*").
2. Refine the invariant : introduce an invariant (close to treewidth) that precisely catch the exact value of the packing-covering ratio.
3. Find sharper bounds on the integrality gaps : Can we use this new invariant to obtain similar results for integrality gaps (and in particular relative cost of stability).

Main statement

$$\tau = \min \sum_{i \in I} x_i \quad \text{s.t.} \sum_{i: i \in S} x_i \geq v(S) \quad \forall S \subseteq I$$

$$\nu = \max \sum_{S: S \subseteq I} v(S) \cdot y_S \quad \text{s.t.} \sum_{S \subseteq I: i \in S} y_S \leq 1 \quad \forall i \in I$$

Theorem (B. Li Vetta '14)

For every graph G, we have :

$$\frac{tw(G)+1}{2} \leq_{\exists} \frac{\tau(\mathcal{G})}{\nu(\mathcal{G})} \leq_{\forall} tw(G)+1$$

By $\leq_{\forall}$, we mean that every game $\mathcal{G}$ on interaction graph G satisfies this inequality.
By $\leq_{\exists}$, we mean that there exists a game $\mathcal{G}$ on interaction graph G which satisfies this inequality.

Actually with our new graph invariant, lower and upper bounds match

Definition (bramble)

A set of vertices $V_1, \ldots, V_\ell$ is a bramble of order k if

- For every i, V_i is connected.
- For every $i \neq j$, V_i and V_j intersect or share an edge.
- The minimum number of vertices intersecting all the sets $V_1, \ldots, V_\ell$ is $(k+1)$.

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Theorem (Robertson, Seymour)

The treewidth of the graph G is equal to the maximum order of a bramble of G.

Better objects for us : Thicket

Definition (thicket)

A set of vertices $V_1, \ldots, V_\ell$ is a thicket of order k if

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By "retro-engineering", we can define vinewidth and prove :

Theorem (B., Li, Vetta)

The vinewidth of the graph is equal to the maximum size of a thicket.

Vinewidth

A tree T and a function $f : T \to 2^V$ is a vine decomposition of $G = (V, E)$ if :

- For every $v \in V$, the set of nodes containing v in their bag is a subtree T_v of T.
- For every edge uv, T_u and T_v **intersects or share an edge**.

The width of a decomposition is the maximum size of a bag of the vine-decomposition.

Definition (vinewidth)

The vinewidth of G, is the minimum width of a vine-decomposition of G.

Overview of the other results

Informal Result 1

We introduce a new invariant $vw(H)$ that completely characterizes the packing-covering ratio, *i.e.* for every graph H :[1]

$$vw(H) \leq_{\exists} \frac{Cov(\mathcal{G})}{Pack(\mathcal{G})} \leq_{\forall} vw(H)$$

1. $\leq_{\exists}$ means that there exists a game $\mathcal{G}$ on interaction graph H which satisfies this inequality.
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Informal Result 2

There exists $\delta > 0$ such that for every graph H, we have

$$vw(H)^{\delta} \leq_{\exists} RCoS(\mathcal{G}) = \frac{Cov^*(\mathcal{G})}{Pack(\mathcal{G})} \leq_{\forall} vw(H)$$

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Informal Result 3

There exists a constant c such that

$$c \cdot vw(H) \leq_{\exists} \frac{Cov(\mathcal{G})}{Cov^{*}(\mathcal{G})} \leq_{\forall} vw(H)$$

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- Bramble / Thickets, it's cool !
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- Does it exist a "good" invariant which characterizes the relative cost of stability ?
- On which interaction graph can we obtain a linear bound in terms of vinewidth ?

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Thanks for your attention !