Coalition games on interaction graphs

Let's play with tree decompositions

Nicolas Bousquet

joint work with Zhentao Li and Adrian Vetta

Séminaire Complex Networks

Context

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- Unfortunately, people are selfish : if it is is more interesting for them, they will create a project of their own.
- Solution : distribute payoff in such a way people do not want to leave the grand coalition (coalition of all agents).

Coalition games

Coalition game

- A set I of n agents.
- A superadditive valuation function $v : 2^n \to \mathbb{N}$. (the money generated by the coalition S if agents of S decide to work on their own project)

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Goal

Distribute payoff to the agents in such a way, for every coalition S, the money distributed to agents of S is at least $v(S)$.
$\Rightarrow$ No coalition wishes to leave the *grand coalition*.

Illustration

All the results of this talk are true for any superadditive valuation function. However, for simplicity, we will focus on simple games.

Definition (simple games)

There exists a set $\mathcal{X} = \{X_1, \ldots, X_m\}$ of non empty subsets of I called minimal coalitions such that :

- $v(X_i)$ for every i.
- The value of any set Y equals the maximum number of pairwise disjoint elements of $\mathcal{X}$ in Y.

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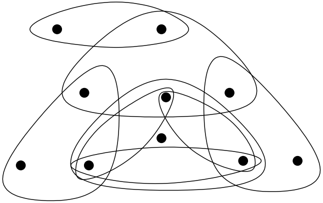

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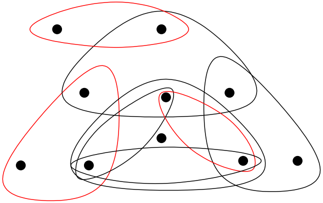

$\Rightarrow v(I) = 3.$

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$$Pack(\mathcal{G}) = \max \quad \sum_{S:S\subseteq I} y_S$$

s.t.

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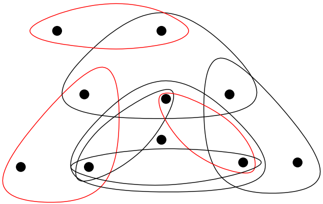

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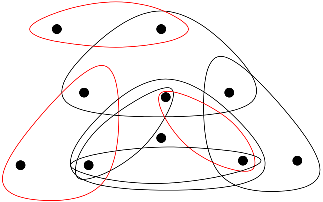

Definition (core)

The core of the coalition game is the set of payoff vectors $\mathbf{x}$ satisfying the following constraints :

$\sum_{i \in I} x_i = v(I)$		The money we can distribute
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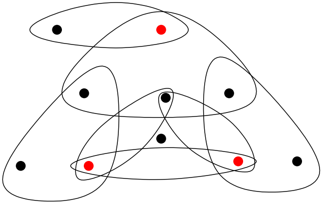

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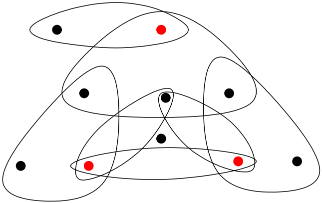

Problem : The core may be empty !

- Which conditions ensure that the core is not empty ?
- **Relax the definition of core.**

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Cov(\mathcal{G})^* &= \min \sum_{i \in I} x_i && \text{minimize the total cost} \\
\text{s.t.} \quad & \sum_{i : i \in S} x_i \geq v(S) \ \ \forall S \subseteq I && \text{stability of each minimal coalition} \\
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Stability of the notion : since $Cov(\mathcal{G})^* - Pack(\mathcal{G})$ is not "stable" (by disjoint copy for instance), we consider $\frac{Cov(\mathcal{G})^*}{Pack(\mathcal{G})}$ instead.

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Definition (relative cost of stability RCoS)

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By Strong Duality Theorem, we have :

$$Pack(\mathcal{G}) \leq Pack^*(\mathcal{G}) = Cov^*(\mathcal{G}) \leq Cov(\mathcal{G})$$

Thus

$$1 \leq \frac{Cov(\mathcal{G})^*}{Pack(\mathcal{G})} = \frac{Pack(\mathcal{G})^*}{Pack(\mathcal{G})} \leq \frac{Cov(\mathcal{G})}{Pack(\mathcal{G})}$$

Integral covering and packing

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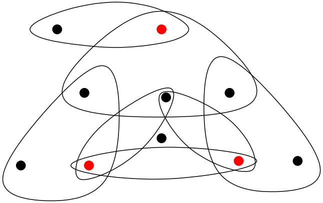

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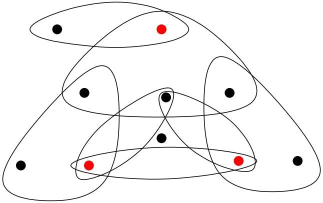

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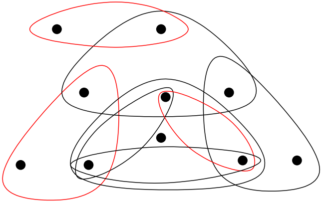

In other words the integral packing corresponds to the maximum number of disjoint minimal coalitions.

Worst case

Lemma

The Packing-Covering ratio $\frac{Cov(\mathcal{G})}{Pack(\mathcal{G})}$ (and both integrality gaps) can be arbitrarily large.

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Example :

- A set of agents $I = \{1, \ldots, n\}$
- A subset S of I is a minimal coalition if and only if $|S| = |I|/2$.
- Maximum Packing : 1.
  (Any pair of coalitions intersect).
- Minimum Covering : $\geq \frac{|I|}{2} - 1$.
  Any smaller set of agents contain at least $|I|/2$ agents in their complement, and then a minimal coalition.

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Question

Which conditions imply that $\frac{Cov(\mathcal{G})}{Pack(\mathcal{G})}$ is bounded ?

Interaction graph

Myerson proposed the following model [1] : *the agents must be able to communicate if they want to form a viable coalition*.

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Definition (interaction graph)

Let H be a graph where :

- Vertices = agents.
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The game $\mathcal{G}$ in on interaction graph H if minimal coalitions are connected subgraphs of H.

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Examples :

- H is a clique : any set of minimal coalitions is possible.
- H is a stable set : minimal coalitions have size one.

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Treewidth and coalition game

Theorem (Meir et al.)[2]

Let H be a graph. The Relative Cost of stability $\frac{Cov^*(\mathcal{G})}{Pack(\mathcal{G})}$ of any coalition game $\mathcal{G}$ on interaction graph H is at most $tw(H) + 1$. Moreover there exist graphs for which this bound is tight.

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Actually, they proved the following stronger statement :

Theorem (Meir et al.)

The following inequality holds :[4]

$$\frac{Cov(\mathcal{G})}{Pack(\mathcal{G})} \leq_\forall tw(H)+1$$

$$\begin{aligned} Cov(\mathcal{G}) = \min \quad & \sum_{i \in I} x_i \quad \text{s.t.} \\ \forall S \subseteq I \quad & \sum_{i : i \in S} x_i \geq v(S) \\ & x_i \in \mathbb{N} \end{aligned}$$

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4. $\leq_\forall$ means that every $\mathcal{G}$ on interaction graph H satisfies this inequality.

Our results

Informal Result 1

We introduce a new invariant $vw(H)$ that completely characterizes the packing-covering ratio, *i.e.* for every graph H :[3]

$$vw(H) \leq_{\exists} \frac{Cov(\mathcal{G})}{Pack(\mathcal{G})} \leq_{\forall} vw(H)$$

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There exists $\delta > 0$ such that for every graph H, we have

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Informal Result 3

There exists a constant c such that

$$c \cdot vw(H) \leq_{\exists} \frac{Cov(\mathcal{G})}{Cov^{*}(\mathcal{G})} \leq_{\forall} vw(H)$$

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Reminder : treewidth

A tree T and a (bag) function $f : T \to 2^V$ is a tree decomposition of $G = (V, E)$ if :

- For every $v \in V$, the set of nodes containing v in their bags is a subtree T_v of T.
- For every edge uv, T_u and T_v intersects.

The width of a decomposition is the maximum size of a bag of the tree-decomposition minus one.

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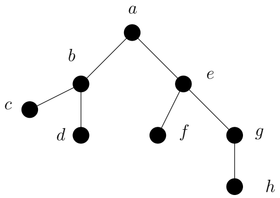

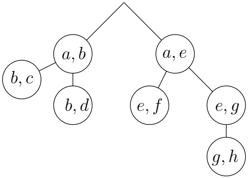

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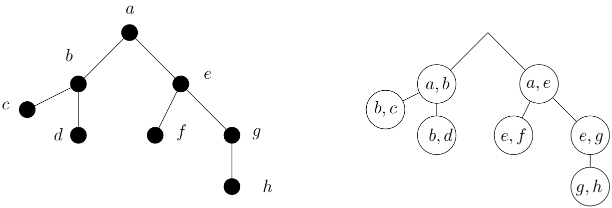
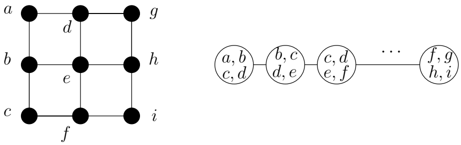
- The $d \times d$ grid has a tree decomposition of width d.





Dual notion : bramble

Definition (bramble)

A set of vertices $V_1, \ldots, V_\ell$ is a bramble of order k if

- For every i, V_i is connected.
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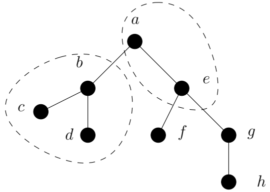

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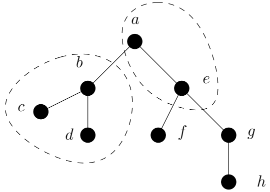
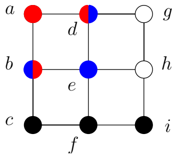
- The $d \times d$ grid has a bramble of order d (V_i is the i-row union the i-th column) :
  - Every pair row-column of the $(d-1) \times (d-1)$ grid.
  - The last row.
  - The last column minus the last vertex.





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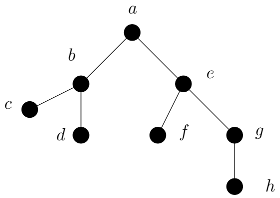
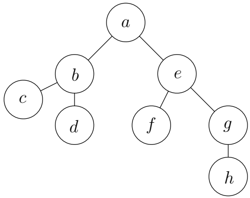
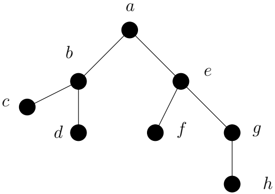
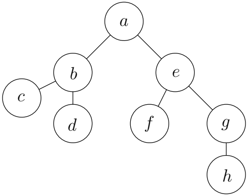


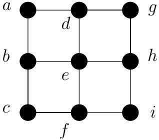


Examples

- K_n has a vine decomposition of width $\lceil n/2 \rceil$.
- Trees have vine decompositions of width 1.
- The $d \times d$ grid has a vine decomposition of width d.







Dual notion : thicket

Definition (thicket)

A set of vertices $V_1, \ldots, V_\ell$ is a thicket of order k if

- For every i, the set V_i is connected.
- For every $i \neq j$, V_i and V_j **intersect**.
- The minimum number of vertices intersecting all the sets $V_1, \ldots, V_\ell$ is k.

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  - Covering : $\lceil k/2 \rceil$.
- $d \times d$ grid.
  - Every pair row-column is in the thicket.
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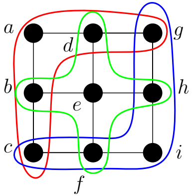

Link between treewidth and vinewidth

Lemma

Every graph G satisfies

$$\frac{tw(G)+1}{2} \leq vw(G) \leq tw(G)+1$$

Moreover both inequalities are tight.

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Proof sketch :

- Any tree decomposition is a vine decomposition.
  $\Rightarrow vw(G) \leq tw(G)+1$.
- We make the "union" of every pair of adjacent bags to be sure that T_u and T_v intersect.
  $\Rightarrow tw(G) \leq 2 \cdot vw(G) - 1$.

Main statement

$$Cov = \min \sum_{i \in I} x_i \quad \text{s.t.} \sum_{i:i\in S} x_i \geq v(S) \;\; \forall S \subseteq I$$

$$Pack = \max \sum_{S:S\subseteq I} y_S \quad \text{s.t.} \sum_{S\subseteq I:i\in S} y_S \leq 1 \;\; \forall i \in I$$

Theorem

For every graph G, we have :

$$vw(G) \leq_{\exists} \frac{Cov(\mathcal{G})}{Pack(\mathcal{G})} \leq_{\forall} vw(G)$$

Reminder :

By $\leq_{\forall}$, we mean that every game $\mathcal{G}$ on interaction graph G satisfies this inequality.

By $\leq_{\exists}$, we mean that there exists a game $\mathcal{G}$ on interaction graph G which satisfies this inequality.

Proof of $vw(G) \leq_{\exists} \frac{Cov(\mathcal{G})}{Pack(\mathcal{G})}$.

Take a thicket $V_1, \ldots, V_\ell$ of order $vw(G)$.
We consider the following $0-1$ game $\mathcal{G}$ where :

- Minimal coalitions are the coalitions $V_1, \ldots, V_\ell$.

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We have :

- For every i, the set V_i is connected : $\mathcal{G}$ is a game on interaction graph G.
- All the sets of a thicket intersect : $Pack(\mathcal{G}) = 1$.
- By definition of thicket : $Cov(\mathcal{G}) = vw(G)$.

Thus $\frac{Cov(\mathcal{G})}{Pack(\mathcal{G})} = vw(G)$

Proof of $\frac{Cov(\mathcal{G})}{Pack(\mathcal{G})} \leq_{\forall} vw(G)$.

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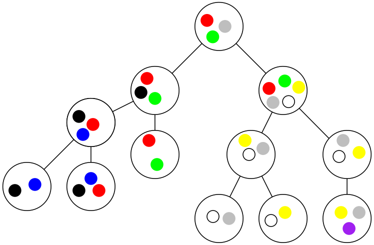
Bottom-up from the leaves of a (rooted) vine decomposition.

- If all the vertices of a coalition C are in the bag B_f of a leaf f, add vertices of B_f in the covering and C in the packing. Delete the coalitions containing one vertex of B_f.
- Otherwise, delete all the leaves of T.



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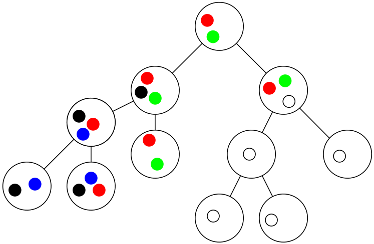

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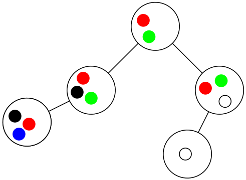

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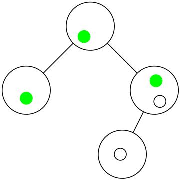

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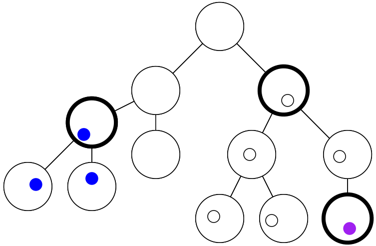

- We have selected k disjoint coalitions : $Pack(\mathcal{G}) \geq k$.
- We have deleted at most $vw(G) \cdot k$ vertices.

Other integrality gaps

Question

Since the vinewidth correctly evaluates the packing-covering ratio, does it also correctly evaluates both integrality gaps ?

Answer : **YES** (even if it is not tight) !

- One integrality gap is linear in terms of the vinewidth.
- One integrality gap is polynomial in terms of the vinewidth.

Fractional packings and coverings

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\begin{aligned}
Pack^* = \max \quad & \sum_{S:S\subseteq I} y_S \\
\text{s.t.} \quad & \sum_{S\subseteq I: i\in S} y_S \leq 1 \quad \forall i \in I \\
& y_S \geq 0 \quad \forall S \subseteq I
\end{aligned}
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The vector y can be seen as a weight function on the coalitions such that :

- The total weight is maximized.
- For each vertex v, the sum of the weights of the coalitions containing v is at most 1.

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By Strong Duality Theorem, we have $Cov^* = Pack^*$.

Relative cost of stability

Reminder of the motivation :
We want to bound the relative cost of stability $\frac{Cov^*}{Pack}$.

Theorem

Let G be a graph. There exists $\delta > 0$ such that

$$vw(G)^\delta \leq_\exists \frac{Pack^*(\mathcal{G})}{Pack(\mathcal{G})} \leq_\forall vw(G)$$

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Follows from the previous theorem since
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Proof of $vw(G)^\delta \leq_\exists \frac{Pack^*(\mathcal{G})}{Pack(\mathcal{G})}$:

- Prove that the gap is linear for grids.
- Use grid minor theorem.

Relative cost of stability on grids

Theorem (Chekuri,Chuzhoy '14)

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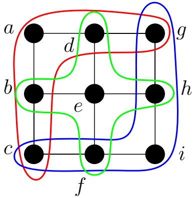

Proof :

Game $\mathcal{G}$ with minimal coalitions are $R_i \cup C_i$.

$Pack(\mathcal{G}) = 1$ (coalitions pairwise intersect).

$Pack^*(\mathcal{G}) \geq \frac{d}{2} = \frac{vw(G_d)}{2}$ (allocate weight $\frac{1}{2}$ to each coalition).

$\Rightarrow \frac{1}{2} \cdot vw(G) \leq_{\exists} \frac{Pack^*(\mathcal{G})}{Pack(\mathcal{G})}$

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Further work :

- Other applications of the vinewidth / thicket duality ?
- What is the best coefficient δ ? (δ cannot be improved beyond $\frac{1}{2}$)
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Thanks for your attention !