Coalition games on interaction graphs

or a nice way to play with treewidth and brambles.

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joint work with Zhentao Li and Adrian Vetta

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McGill

GERAD
GROUP FOR RESEARCH IN DECISION ANALYSIS

Context

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- Unfortunately, people are **selfish** : if it is is more interesting for them, they will create a project of their own.
- Solution : distribute payoff in such a way people do not want to leave.

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Coalition game

- A set I of n agents.
- A valuation function $v : 2^n \rightarrow \mathbb{N}$. (the gain of any group deciding to work on its own project)

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Goal

Distribute money to the agents in such a way, for every coalition S, the money distributed to agents of S is at least $v(S)$.
$\Rightarrow$ No one wants to leave the *grand coalition* and work on its own project.

Core

Definition (core)

The core of the coalition game is the set of vectors of payoff satisfying the following constraints :

$$\begin{aligned} \textstyle\sum_{i \in I} x_i &= v(I) & \\ \text{and } \textstyle\sum_{i \in S} x_i &\geq v(S) & \forall S \subseteq I \\ x_i &\geq 0 & \forall i \in I \end{aligned}$$

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- **Weaken the definition of core.**

Least core

Definition (multiplicative least core)

$$
\begin{aligned}
\texttt{Least-Core:}\ \max \quad & \alpha \\
\text{s.t.} \quad & \sum_{i \in I} x_i = v(I) \\
\text{and} \quad & \sum_{i : i \in S} x_i \geq \alpha \cdot v(S) \quad \forall S \subseteq I \\
& x_i \geq 0
\end{aligned}
$$

Intuition :

Leaving the grand coalition has a cost. Thus agents stay in the grand coalition unless they have huge profit if they left it.

Relative cost of stability

Another approach (relative cost of stability) :
How much money must be injected by an external authority to stabilize the system ?
$\Rightarrow$ Our expenses minus our gains.

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Our gains :

$$\texttt{Packing-LP:}\quad \nu(\mathcal{G}) = \max \sum_{S:S\subseteq I} v(S)\cdot y_S$$
$$\text{s.t.} \quad \sum_{S\subseteq I: i\in S} y_S \leq 1 \quad \forall i \in I$$
$$y_S \text{ is integral and } \geq 0 \quad \forall S \subseteq I$$

If v is super-additive then the packing is of I is precisely $v(I)$.
$\Rightarrow$ The (integral) packing represents the amount of money we gain.

Our expenses

$$\texttt{Covering-LP:}\quad \tau(\mathcal{G})^* = \min \sum_{i \in I} x_i$$
$$\text{s.t.} \sum_{i:i \in S} x_i \geq v(S) \quad \forall S \subseteq I$$
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Simplification :

$\Rightarrow$ Since $\tau^* - \nu$ is not "stable" (by disjoint copy for instance), we consider $\frac{\tau^*}{\nu}$ instead.

Relative Cost of Stability

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By Strong Duality Theorem, we have :

$$\nu(\mathcal{G}) \leq \nu^*(\mathcal{G}) = \tau^*(\mathcal{G}) \leq \tau(\mathcal{G})$$

Thus

$$\frac{\tau^*}{\nu} = \frac{\nu^*}{\nu} \leq \frac{\tau}{\nu}$$

Hypergraph representation

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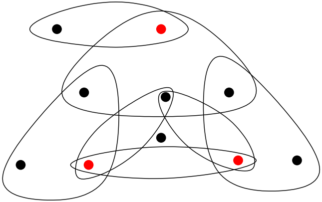

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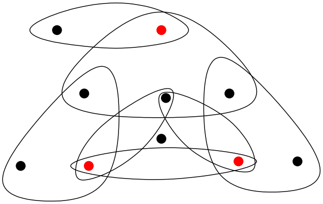

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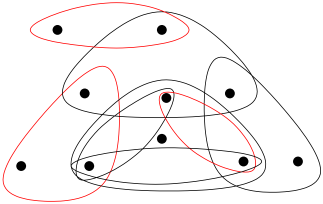

Worst case

Lemma

The Packing-Covering ratio $\frac{\tau(\mathcal{G})}{\nu(\mathcal{G})}$ (and both integrality gaps) can be arbitrarily large.

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Example :

- A set of agents $I = \{1, \ldots, n\}$
- A subset S of I is a coalition if and only if $|S| > |I|/2$.

- Maximum Packing : 1.
- Minimum Covering : $\geq \frac{|I|}{2} - 1$.

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Question

Which conditions imply an Erdős-Posá property ?

Interaction graph

Myerson proposed the following model :

Definition (interaction graph)

Let G be a graph where the vertices of G are the agents of the coalition game $\mathcal{G}$.
The game $\mathcal{G}$ has interaction graph G if every coalition is connected (*i.e.*, if $v(S) > 0$ then S is connected).

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Examples :

- G is a clique : any coalition may exist.
- G is a stable set : coalitions have size one.

Treewidth and coalition game

Theorem (Meir et al.)

Let G be a graph. We have the following inequality :

$$\frac{\tau(\mathcal{G})}{\nu(\mathcal{G})} \leq_{\forall} tw(G) + 1$$

Moreover there exist graphs for which this bound is tight.

By $\leq_{\forall}$, we mean that every game $\mathcal{G}$ on interaction graph G satisfies this inequality.

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Our work : Improve this result, and look at the integrality gaps $\frac{\nu^*(\mathcal{G})}{\nu(\mathcal{G})}$ and $\frac{\tau(\mathcal{G})}{\tau^*(\mathcal{G})}$.

Reminder : treewidth

A tree T and a (bag) function $f : T \rightarrow 2^V$ is a tree decomposition of $G = (V, E)$ if :

- For every $v \in V$, the set of nodes containing v in their bags is a subtree T_v of T.
- For every edge uv, T_u and T_v intersects.

The width of a decomposition is the maximum size of a bag of the tree-decomposition minus one.

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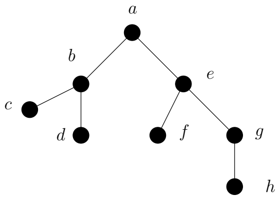

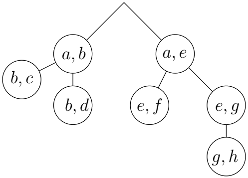

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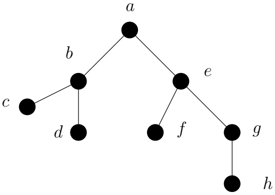
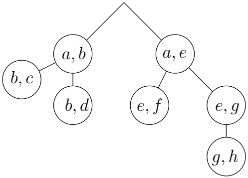
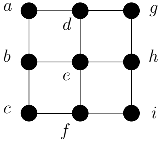
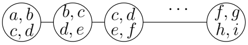
- The $d \times d$ grid has a tree decomposition of width d.









Dual notion : bramble

Definition (bramble)

A set of vertices $V_1, \ldots, V_\ell$ is a bramble of order k if

- For every i, V_i is connected.
- For every $i \neq j$, V_i and V_j intersect or share an edge.
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The $+1$ and the fact that the family does not intersect.

Examples

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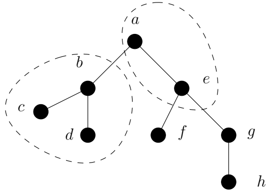

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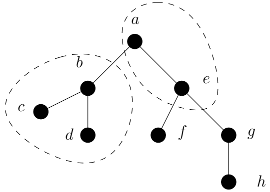
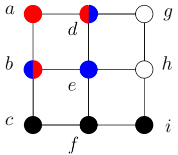
- The $d \times d$ grid has a bramble of order d (V_i is the i-row union the i-th column) :
  - Every pair row-column of the $(d-1) \times (d-1)$ grid.
  - The last row.
  - The last column minus the last vertex.





A new invariant : vinewidth

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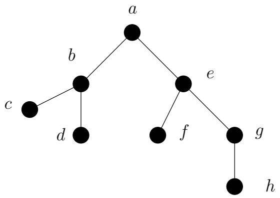

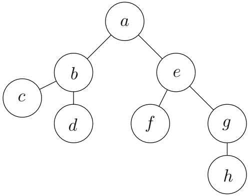

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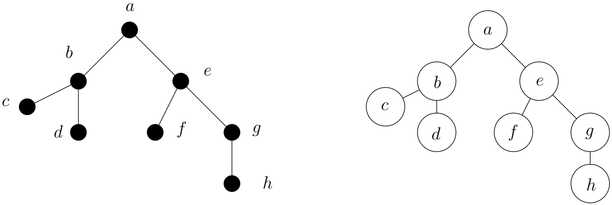

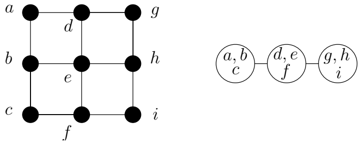

Dual notion : thicket

Definition (thicket)

A set of vertices $V_1, \ldots, V_\ell$ is a thicket of order k if

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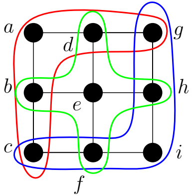

Link between treewidth and vinewidth

Lemma

Every graph G satisfies

$$\frac{tw(G)+1}{2} \leq vw(G) \leq tw(G)+1$$

Moreover both inequalities are tight.

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Moreover both inequalities are tight.

Proof sketch :

- Any tree decomposition is a vine decomposition.
  $\Rightarrow vw(G) \leq tw(G)+1$.
- We make the "union" of every pair of adjacent bags to be sure that T_u and T_v intersect.
  $\Rightarrow tw(G) \leq 2 \cdot vw(G) - 1$.

Main statement

$$\tau = \min \sum_{i \in I} x_i \quad \text{s.t.} \sum_{i : i \in S} x_i \geq v(S) \quad \forall S \subseteq I$$

$$\nu = \max \sum_{S : S \subseteq I} v(S) \cdot y_S \quad \text{s.t.} \sum_{S \subseteq I : i \in S} y_S \leq 1 \quad \forall i \in I$$

Theorem (BLV'14)

For every graph G, we have :

$$vw(G) \leq_{\exists} \frac{\tau(\mathcal{G})}{\nu(\mathcal{G})} \leq_{\forall} vw(G)$$

By $\leq_{\forall}$, we mean that every game $\mathcal{G}$ on interaction graph G satisfies this inequality.
By $\leq_{\exists}$, we mean that there exists a game $\mathcal{G}$ on interaction graph G which satisfies this inequality.

Proof of $vw(G) \leq_{\exists} \frac{\tau(\mathcal{G})}{\nu(\mathcal{G})}$.

Take a thicket $V_1, \ldots, V_\ell$ of order $vw(G)$.
We consider the following $0-1$ game $\mathcal{G}$ where :

- For every i, V_i is a coalition of value 1.
- The other sets receive value 0.

Now, we have :

- For every i, the set V_i is connected : $\mathcal{G}$ is a game on interaction graph G.
- All the sets of a thicket intersect : $\nu(\mathcal{G}) = 1$.
- By definition of thicket : $\tau(\mathcal{G}) = vw(G)$.

Thus $\frac{\tau(\mathcal{G})}{\nu(\mathcal{G})} = vw(G)$

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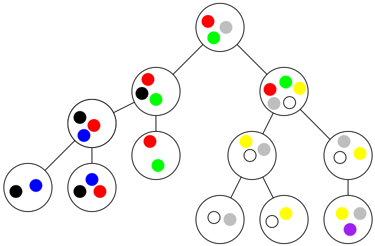
Bottom-up from the leaves of a (rooted) vine decomposition.

- If all the vertices of a coalition C are in the bag B_f of a leaf f, add vertices of B_f in the covering and C in the packing. Delete the coalitions containing one vertex of B_f.
- Otherwise, delete all the leaves of T.



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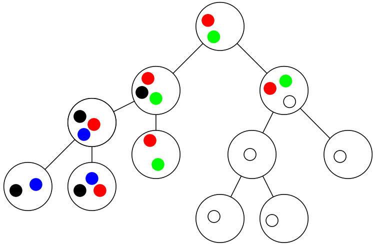

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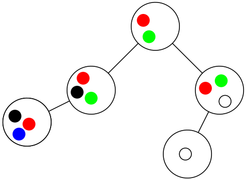

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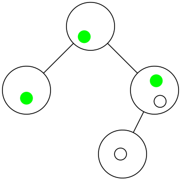

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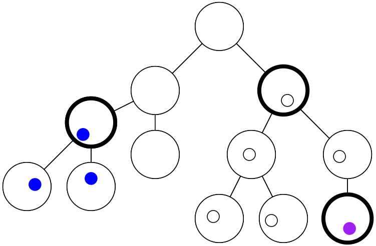

- We have selected k disjoint coalitions : $\nu(\mathcal{G}) \geq k$.
- We have deleted at most $vw(G) \cdot k$ vertices.

Other integrality gaps

Question

Since the vinewidth correctly evaluates the packing-covering ratio, does it also correctly evaluates both integrality gaps ?

Answer : YES (even if it is not tight) !

- One integrality gap is linear in terms of the vinewidth.
- One integrality gap is polynomial in terms of the vinewidth.

Fractional packings and coverings

$$\begin{aligned} \nu^* = \max \quad & \sum_{S:S\subseteq I} v(S)\cdot y_S \\ \text{s.t.} \quad & \sum_{S\subseteq I: i\in S} y_S \le 1 \quad \forall i\in I \\ & y_S \ge 0 \quad \forall S\subseteq I \end{aligned}$$

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The vector y can be seen as a weight function on the coalitions such that :

- The total weight is maximized.
- For each vertex v, the sum of the weights of the coalitions containing v is at most 1.

The vector x can be seen as a weight function on the vertices such that :

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By Strong Duality Theorem, we have $\tau^* = \nu^*$.

Relative cost of stability

Reminder of the motivation :
We want to bound the relative cost of stability $\frac{\tau^*}{\nu}$.

Theorem

Let G be a graph. There exists $\delta > 0$ such that

$$vw(G)^\delta \leq_\exists \frac{\nu^*(\mathcal{G})}{\nu(\mathcal{G})} \leq_\forall vw(G)$$

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Proof of $vw(G)^{\delta} \leq_{\exists} \frac{\nu^*(\mathcal{G})}{\nu(\mathcal{G})}$:

- Prove that the gap is linear for grids.
- Use grid minor theorem.

Relative cost of stability on grids

Theorem (Chekuri,Chuzhoy)

There exist c' and $\delta > 0$ such that every graph G has a grid minor of size at least $c' \cdot tw(G)^{\delta}$.

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Proof :

Consider the game $\mathcal{G}$ where coalitions are $R_i \cup C_i$ for every i.

- $\nu(\mathcal{G}) = 1$ since all the coalitions intersect.
- $\nu^*(\mathcal{G}) \geq \frac{d}{2} = \frac{vw(G_d)}{2}$ since if we allocate weight $\frac{1}{2}$ to each coalition, the constraints are satisfied.

$\Rightarrow \frac{1}{2} \cdot vw(G) \leq_{\exists} \frac{\nu^*(\mathcal{G})}{\nu(\mathcal{G})}$

Questions

$$vw(G)^{\delta} \leq_{\exists} \frac{\nu^*(\mathcal{G})}{\nu(\mathcal{G})} \leq_{\forall} vw(G)$$

Questions :

- What is the best constant δ (we cannot beat $\frac{1}{2}$, on cliques) ?
- Which conditions ensure a linear bound ?
- Does it exist a "nice" invariant catching the relative cost of stability ?

The other integrality gap

Theorem

Let G be a graph. The following inequalities are satisfied :

$$\frac{1}{4} \cdot vw(G) \leq_{\exists} \frac{\tau(\mathcal{G})}{\tau^*(\mathcal{G})} \leq_{\forall} vw(G)$$

Moreover, the right inequality is tight (in the limit).

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Moreover, the right inequality is tight (in the limit).

- In the LHS, $vw(G)$ can be replaced by $(tw(G) + 1)$ (which is slightly better).
- $\frac{1}{4} \cdot tw(G)$ tight on cliques (and on grids ?).

Proof of $\frac{1}{4} \cdot vw(G) \leq_{\exists} \frac{\tau(\mathcal{G})}{\tau^*(\mathcal{G})}$:

Let $V_1, \ldots, V_\ell$ be a thicket of order k. Let $Y = \{y_1, \ldots, y_k\}$ be a hitting set of $V_1, \ldots, V_\ell$.

Let $\mathcal{G}$ be the game such that S is a coalition iff :

- $S = \cup_{i \in J \subseteq \{1,\ldots,\ell\}} V_i$.
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We have :

- $\tau(\mathcal{G}) \geq k/2$ since if $Y' \subseteq Y$ satisfies $|Y'| > k/2$ then $\cup_{y_j \in Y'} V_{i_j}$ is a coalition.
- $\tau^*(\mathcal{G}) \leq 2$ since the weight function assigning weight
    - $2/k$ to each vertex of Y,
    - 0 to the other vertices,

  satisfies the constraints.

Conclusion

- Other applications of the vinewidth / thicket ?
- Does it exist a "good" invariant which characterizes the relative cost of stability ? The other integrality gap ?
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Thanks for your attention !