Exact distance coloring in trees

(and other questions)

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with L. Esperet, A. Harutyunyan and R. de Joannis de Verclos

Kick-off Projet DISTANCIA

Unit distance coloring

Question (Hadwiger 1945, Nelson 1950, Gardner (1960))

What is the fewest number of colors needed to color the points of $\mathbb{R}^2$, such that every two points at unit distance have different colors ?

We denote the optimal value by $\chi(\mathbb{R}, 1)$.

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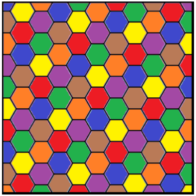

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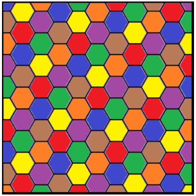
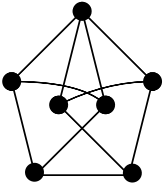


At least 4 : Moser spindle



Partial results

- [Thomassen 1999] If the coloring of corresponds to a coloring of the faces of some (infinite) planar graph, then at least 7 colors are needed.

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Remark : For some well-chosen adjacency relation in $\mathbb{R}$, $\mathbb{R}$ is either 2 colorable (with the axiom choice) or not colorable with $\mathbb{N}$ colors (without it).

Coloring hyperbolic spaces

Given a metric space (X, d), an exact coloring at distance p is a coloring c of X such that, for every $x, y \in X$, $c(x) \neq c(y)$ if $d(x, y) = p$.

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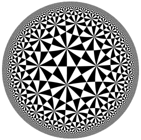

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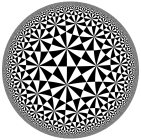
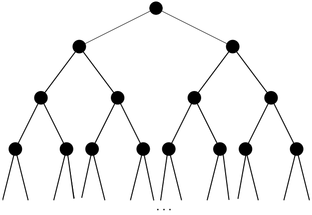


"Natural" representation of hyperbolic spaces : q-ary trees.



Coloring q-ary trees

T_q = infinite q-ary tree.

Theorem (Parlier and Petit, 17+)

If p is odd, then $\chi(T_q, p) = 2$.
If p is even, then

$$q + 1 \leq \chi(T_q, p) \leq (q - 1)(p + 1).$$

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Remark : Lower bound still true if we cut T_q at depth p.

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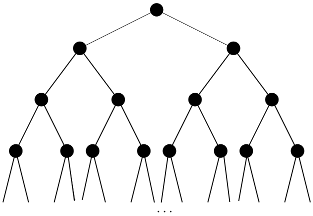
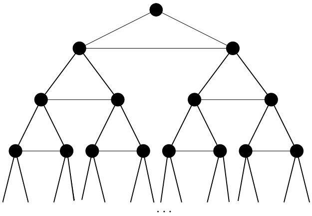
NO (B., Esperet, Harutyunyan, de Joannis de Verclos).

Proof



The graph T_2



The graph H_2

- H = Graph H_2 restricted to nodes of depth at most $p - 1$.

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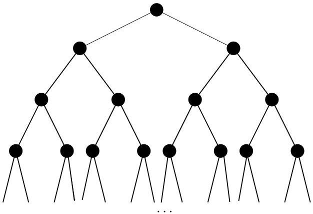

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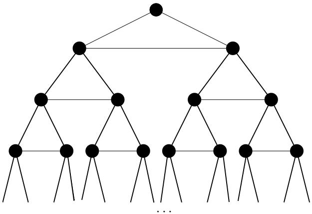

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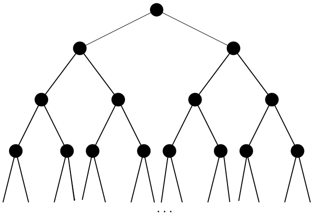

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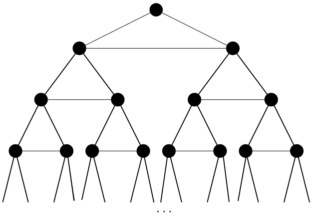

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Corollary : $\chi(H, p)$ is not bounded by a universal constant.

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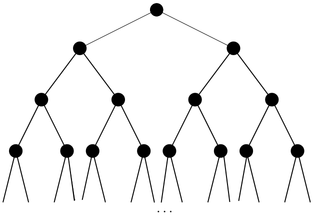

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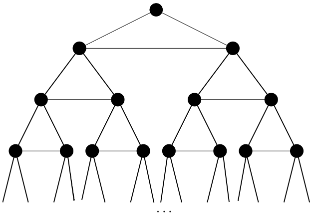

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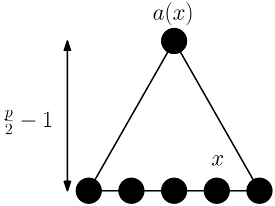
Remark : The same question was raised for chordal graphs. It also provides a counter-example to this question.

$\chi(T_q, p) \leq q + p + 1$



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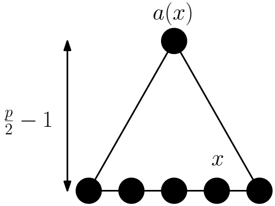

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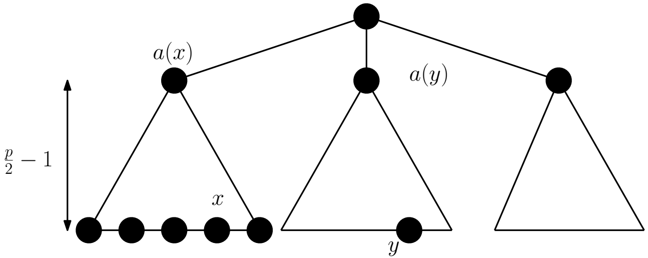

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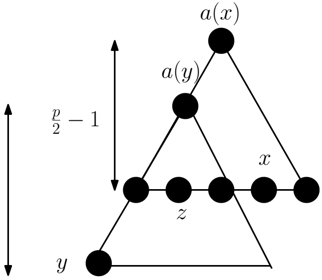

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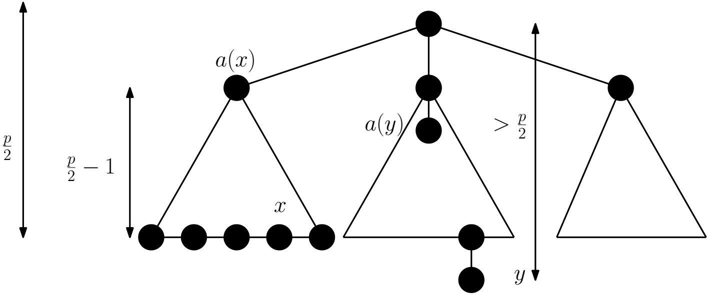

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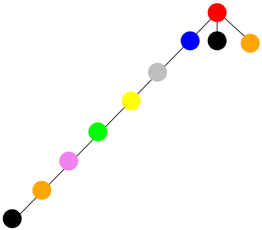
- In all the other cases $\Rightarrow xy \notin E$.

Tree coloring



Lemma :

Any coloring of the tree where :

- Siblings are colored differently,
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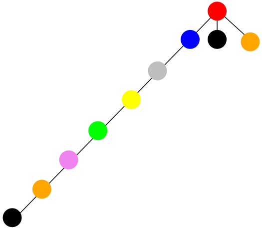

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Lemma : Such a coloring can be constructed with a BFS.

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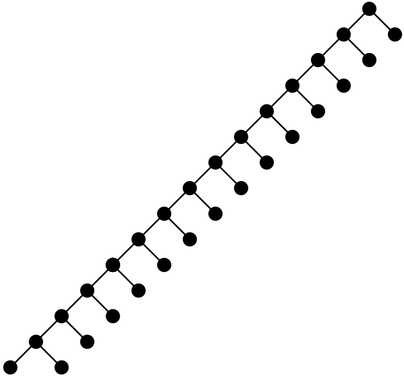

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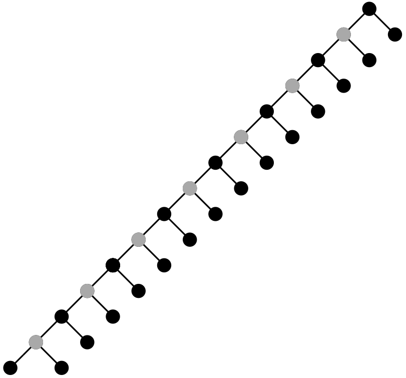

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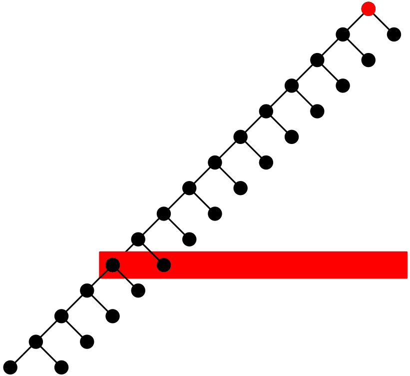

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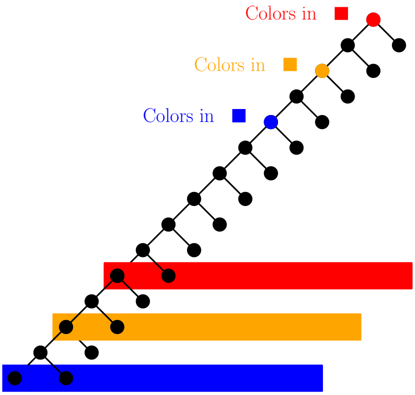

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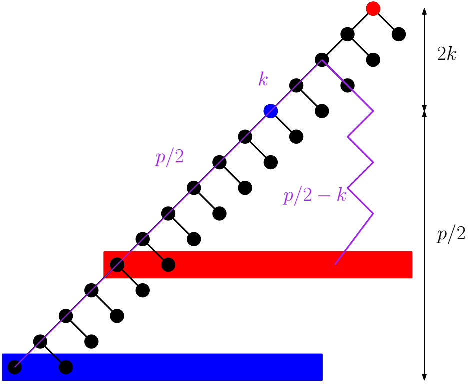

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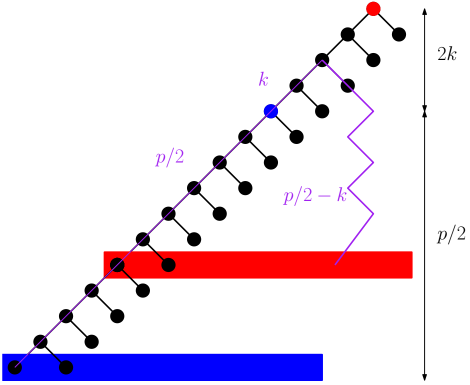

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If all the vertices have distinct sets : $\Rightarrow \geq \log p$ colors.

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For each color c we define : $L_c(u)$ = List of depths (in the original graph) between $p/2$ and p of vertices colored c in the subtree rooted in u.

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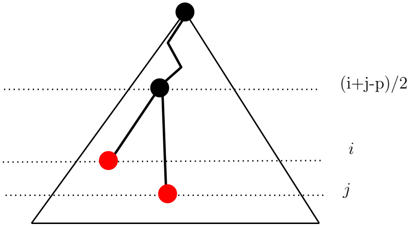

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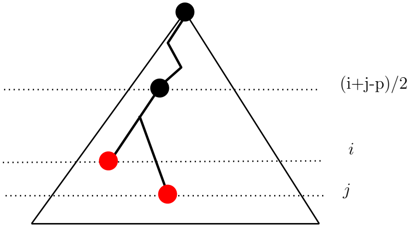

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Technical lemma : $\mathbb{E}(\mathcal{E}_{u_k})$ is decreasing when we perform a (descending) random walk.

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Corollary : The average number of vertices in $G^c_{u_{p/2}}$ is at most $\frac{\log(d/2)}{\log(q-1)}$.

End of the proof

$$\mathcal{E}_u = \sum_{i \in G_u^c} (q-1)^{deg(v)}.$$

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Corollary of the corollary :

$$\chi(T_q, p) \geq \Omega\left(\frac{p \log(q-1)}{\log p}\right).$$

Conclusion

- Solve the problem on the plane.
- Find a human checkable proof for the non 4-colorability of some unit-distance graph in $\mathbb{R}^2$.
- Number of colors for a hyperbolic plane ?
  There is no large bipartite graphs in this case.

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Thanks !