Coalition Games on Interaction Graphs

A horticultural perspective

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EC'15

McGill

GERAD
GROUP FOR RESEARCH
IN DECISION ANALYSIS

Outline of this talk

- Coalition games, core and relative cost of stability.
- New graph parameters : vinewidth (and its dual) thicket.
- New bounds on the relative cost of stability.

Coalition games

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- A set I of n agents.
- A superadditive valuation function $v : 2^n \to \mathbb{N}$. (the money generated by the coalition S if agents of S decide to work on their own project)

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Goal

Distribute money to the agents in such a way, for every coalition S, the money distributed to agents of S is at least $v(S)$.
$\Rightarrow$ No coalition wishes to leave the *grand coalition*.

Core

Definition (core)

The core of the coalition game is the set of payoff vectors x satisfying the following constraints :

$$\sum_{i \in I} x_i = v(I) \qquad \text{The money we can distribute}$$

$$x_i \geq 0 \qquad \forall i \in I \quad \text{Non-negative salary}$$

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$\sum_{i \in S} x_i \geq v(S)$	$\forall S \subseteq I$	No coalition can benefit by deviating
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- Which conditions ensure that the core is not empty ?
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Problem : The core is usually empty !

- Which conditions ensure that the core is not empty ?
- **Relax the definition of core.**

Least core

Definition (multiplicative least core)

$$
\begin{aligned}
\texttt{Least-Core:}\ \max \quad & \alpha \\
\text{s.t.} \quad & \sum_{i \in I} x_i = v(I) \\
\text{and} \quad & \sum_{i : i \in S} x_i \geq \alpha \cdot v(S) \quad \forall S \subseteq I \\
& x_i \geq 0
\end{aligned}
$$

Intuition :

Agents of a coalition will leave only if there is a significant benefit in doing so.

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Another approach :

How much money must be injected by an external authority to stabilize the system ? Expenses / gains.

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Our gains :
$$Pack(\mathcal{G}) = \max \sum_{S:S\subseteq I} v(S)\cdot y_S$$
$$\text{s.t.} \quad \sum_{S\subseteq I: i\in S} y_S \leq 1 \quad \forall i \in I$$
$$y_S \in \mathbb{N} \quad \forall S \subseteq I$$

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Our expenses : $$Cov(\mathcal{G})^* = \min \sum_{i\in I} x_i$$
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Remark : This linear program is called the *fractional covering LP*.
$*$ refers to fractional LPs while no $*$ refers to integral ones.

Definition

Definition (Relative Cost of Stability (RCoS) [1]

The relative cost of stability of a game $\mathcal{G}$ is the ratio $\frac{Cov^*(\mathcal{G})}{Pack(\mathcal{G})}$.

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Lemma

The Relative Cost of Stability can be arbitrarily large.

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Lemma

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Constraints on the coalitions ? $\Rightarrow$ Interaction graphs

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Interaction graph

Myerson proposed the following model [2] : *the agents must be able to communicate if they want to form a viable coalition.*

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Definition (interaction graph)

Let H be a graph where :

- Vertices = agents.
- Edges = ability to communicate.

The game $\mathcal{G}$ in on interaction graph H if $v(S) > 0 \Rightarrow S$ induces a connected subgraph.

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Examples :

- H is a clique : any coalition may exist.
- H is a stable set : coalitions have size one.

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Treewidth and coalition game

Theorem (Meir et al.)[3]

Let H be a graph. The Relative Cost of stability $\frac{Cov^*(\mathcal{G})}{Pack(\mathcal{G})}$ of any coalition game $\mathcal{G}$ on interaction graph H is at most $tw(H) + 1$. Moreover there exist graphs for which this bound is tight.

3. *Bounding the cost of stability in games over interaction networks*, Meir, Zick, Elkind and Rosenschein (AAAI 2013).

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Actually, they proved the following stronger statement :

Theorem (Meir et al.)

The following inequality holds :[4]

$$\frac{Cov(\mathcal{G})}{Pack(\mathcal{G})} \leq_\forall tw(H) + 1$$

$$\begin{aligned} Cov(\mathcal{G}) = \min \quad & \sum_{i \in I} x_i \quad \text{s.t.} \\ \forall S \subseteq I \quad & \sum_{i : i \in S} x_i \geq v(S) \\ & x_i \in \mathbb{N} \end{aligned}$$

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4. $\leq_\forall$ means that every $\mathcal{G}$ on interaction graph H satisfies this inequality.

Our results

Informal Result 1

We introduce a new invariant $vw(H)$ that completely characterizes the packing-covering ratio, *i.e.* for every graph H :[4]

$$vw(H) \leq_\exists \frac{Cov(\mathcal{G})}{Pack(\mathcal{G})} \leq_\forall vw(H)$$

4. $\leq_\exists$ means that there exists a game $\mathcal{G}$ on interaction graph H which satisfies this inequality.
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Informal Result 2

There exists $\delta > 0$ such that for every graph H, we have

$$vw(H)^{\delta} \leq_{\exists} RCoS(\mathcal{G}) = \frac{Cov^{*}(\mathcal{G})}{Pack(\mathcal{G})} \leq_{\forall} vw(H)$$

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Informal Result 3

There exists a constant c such that

$$c \cdot vw(H) \leq_{\exists} \frac{Cov(\mathcal{G})}{Cov^{*}(\mathcal{G})} \leq_{\forall} vw(H)$$

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A new graph invariant : vinewidth

A tree T and a function $f : T \rightarrow 2^V$ (the bag function) is a tree decomposition of $H = (V, E)$ if :

- For every $v \in V$, the set of nodes containing v in their bags is a *subtree* T_v of T.
- For every edge (u, v), T_u and T_v intersect .

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The width of a vine decomposition is $\max_{t \in T} |f(t)|$.

Definition (vinewidth)

The vinewidth $vw(H)$ of H, is the minimum width of a vine-decomposition of H.

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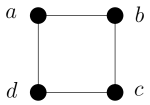

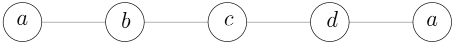

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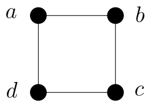

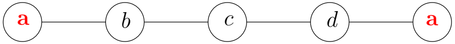

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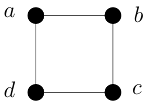

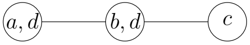

"Dual" notion : thicket

Definition (thicket)

A collection of subsets of vertices $V_1, \ldots, V_\ell$ is a thicket of order k if

- For every i, the set V_i is *connected*.
- For every $i \neq j$, V_i and V_j *intersect*.
- The minimum number of vertices intersecting all the sets $V_1, \ldots, V_\ell$ is k.

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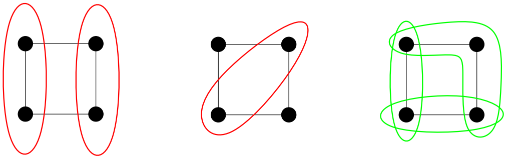

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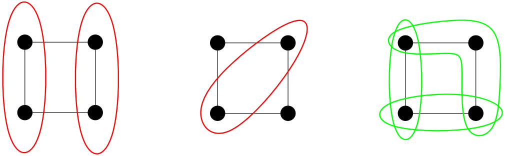

Structural Theorem (B., Li, Vetta)

The vinewidth is equal to the maximum order of a thicket.

Bounds on the Packing-covering ratio

$$Cov(\mathcal{G}) = \min \sum_{i \in I} x_i \qquad\qquad Pack(\mathcal{G}) = v(I)$$

$$\text{s.t.} \sum_{i : i \in S} x_i \geq v(S) \quad \forall S \subseteq I$$

$$\text{and} \quad x_i \in \mathbb{N} \quad \forall i \in i$$

Theorem (B., Li, Vetta)

For every graph H, we have :

$$vw(H) \leq_{\exists} \frac{Cov(\mathcal{G})}{Pack(\mathcal{G})} \leq_{\forall} vw(H)$$

$\leq_{\exists}$ means that there exists a game $\mathcal{G}$ on interaction graph H which satisfies this inequality.

$\leq_{\forall}$ means that every game $\mathcal{G}$ on interaction graph H satisfies this inequality.

Proof of $vw(H) \leq_{\exists} \frac{Cov(\mathcal{G})}{Pack(\mathcal{G})}$

Let $V_1, \ldots, V_\ell$ be a thicket of order $vw(H)$.
We consider the following $0 - 1$ game $\mathcal{G}$ where :

- For every i, every connected superset of V_i has value 1.
- The other sets receive value 0.

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Now, we have :

- For every i, the graph induced by V_i is connected : $\mathcal{G}$ is a game on interaction graph H.
- All the sets of a thicket pairwise intersect : the valuation function is superadditive.
- In a $0-1$ game, $Cov(\mathcal{G})$ is the minimum number of agents X such that every coalition contains an agent of X.
  $\Rightarrow$ By definition of the order of the thicket : $Cov(\mathcal{G}) = vw(H)$.

Thus $\frac{Cov(\mathcal{G})}{Pack(\mathcal{G})} = vw(H)$

Bounds on the Relative Cost of Stability

Theorem

There exists $\delta > 0$ such that for every graph H, we have

$$vw(H)^{\delta} \leq_{\exists} RCoS(\mathcal{G}) = \frac{Cov^{*}(\mathcal{G})}{Pack(\mathcal{G})} \leq_{\forall} vw(H)$$

δ cannot be improved beyond $\frac{1}{2}$.

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Proof of $RCoS(\mathcal{G}) \leq_{\forall} vw(H)$:

Follows from the previous theorem since

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Proof of $vw(H)^{\delta} \leq_{\exists} RCoS(\mathcal{G})$:

- We can prove that the gap is linear for grids.
- Use the polynomial grid minor theorem. [5]

5. *Polynomial bounds for the grid-minor theorem*, Chekury, Chuzhoy, STOC'14.

Conclusion

Further work :

- Other applications of the vinewidth / thicket duality ?
- What is the best coefficient δ ? (δ cannot be improved beyound $\frac{1}{2}$)
- Equivalence between the coefficient of the RCoS and of the grid minor theorem ?

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Thanks for your attention !