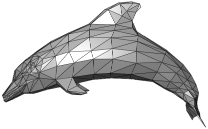
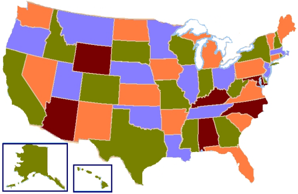
Graphs - Coloring, partition and structure

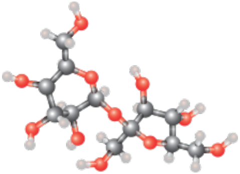
Nicolas Bousquet

Ecole Centrale Lyon

Representation of discrete structures





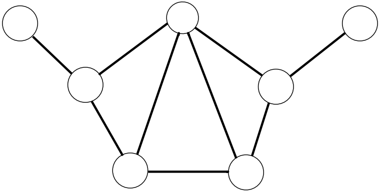


Definition

A graph is a pair (V, E) where:

- V is a set of "points" called vertices.
- E is a set of "binary relations" (segments) between pairs of vertices called edges.

Two vertices u, v are adjacent if (u, v) is an edge.



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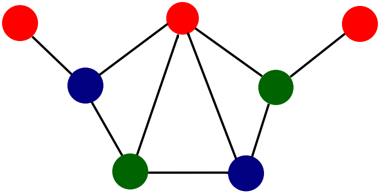

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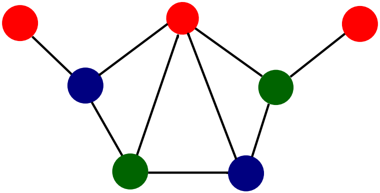

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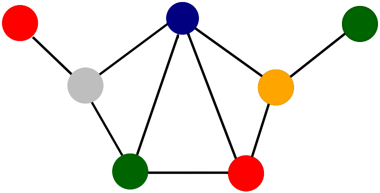

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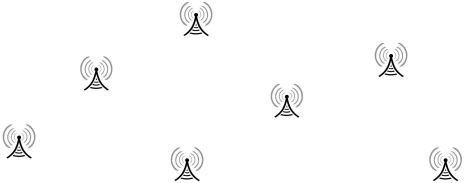

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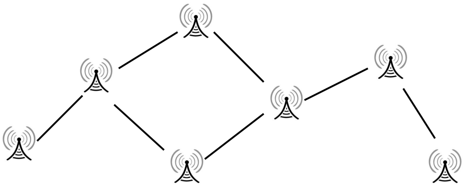
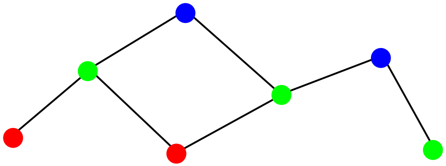


Why coloring graphs?

orange™



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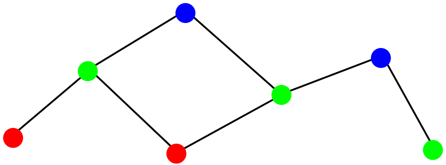
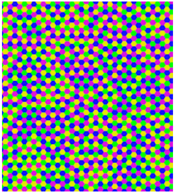
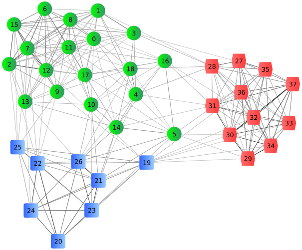


One can notice that the resulting graph is planar, i.e. represented in the plane without intersection of edges.

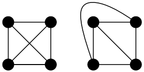
Why coloring graphs? (II)

- Avoid interference in wireless networks.
- Modelization of physical models (e.g. antiferromagnetics Potts models).
- Modelization of agents behavior in economy.
- Partitionning the graph into sets sharing some common behavior (clustering).





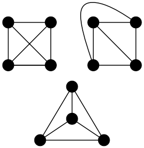
Planar graphs



Definition (planar graph)

A graph is planar if it can be represented without any crossing edge in the plane.

Planar graphs



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Lemma: For any planar graph, there exists a straight line representation.

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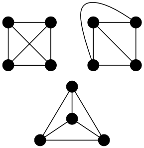

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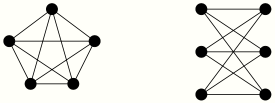
Theorem (Wagner '37)

A graph is planar if it does not "contain" one of these two graphs:



Structure of graphs and colorings

Lemma

Some graphs need an arbitrary number of colors to be properly colored.

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Folklore

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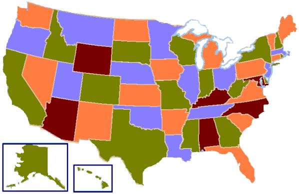
- We have $F \leq 2E/3$.
- Using Euler formula, we obtain: $E \leq 3V - 6$.
- As $\sum_{x \in S} degree(x) = 2E < 6V$
  $\Rightarrow$ There is a vertex of degree at most 5.

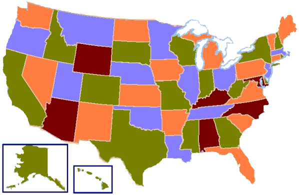
Coloring planar graphs



Conjecture

Every planar graph can be properly colored with 4 colors.

Coloring planar graphs



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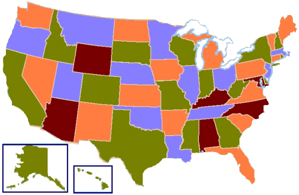

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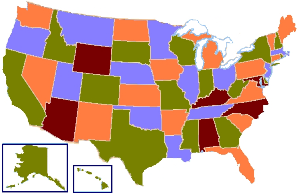
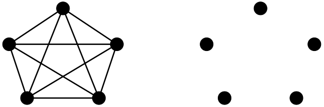


Theorem (Appel, Haken '76)

Every planar graph can be properly colored with 4 colors.

- All the known proofs of this result are computer assisted.
- Why do we need a few number of colors? One reason could be: no clique (e.g. set of vertices pairwise incident) of size at least 5.

General lower and upper bounds on the number of colors?



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A clique of size k is a subset of k pairwise incident vertices.

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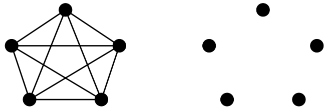

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Number of colors $\geq$ size of a maximum clique.

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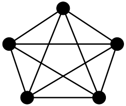

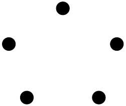

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Let G be a graph with no triangle, can we color G with a constant (10^{1000}?) number of colors?

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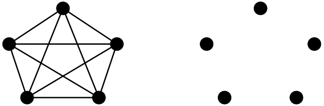

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Let G be a graph with no triangle, can we color G with a constant (10^{1000}?) number of colors? NO !

Coloring larger classes of graphs

Which conditions ensure:

min. number of colors $\leq$ function(size of the max. clique).

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- Perfect graphs (Chudnovsky et al. '03).
- Maximal triangle-free graphs with no subdivision a given graph H (B., Thomassé '12).
- Graphs with no cycle with a fixed number of chords (Aboulker, B. '15).
- ...

Related topics

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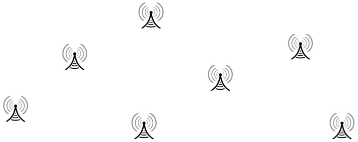
- Polyhedral combinatorics: extended formulation (Lovász '90).
- Communication complexity: Yannakakis conjecture ('89). Partial results in (B., Lagoutte, Thomassé '14).
- Existence of special patterns in graphs: Erdős-Hajnal conjecture.
  Partial results in (B., Lagoutte, Thomassé, '15).

A more dynamic model?



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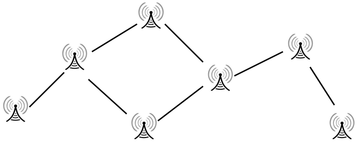

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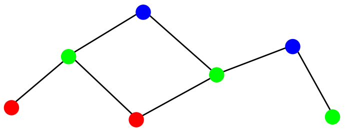

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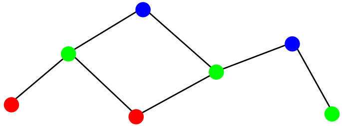

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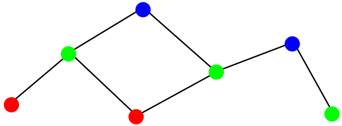
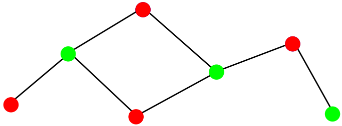
- Realocate the frequencies of the antennas.
- Condition: no interference at any time.
- Only one antenna reallocation at a time.





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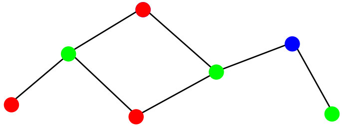

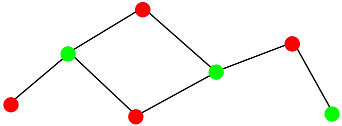

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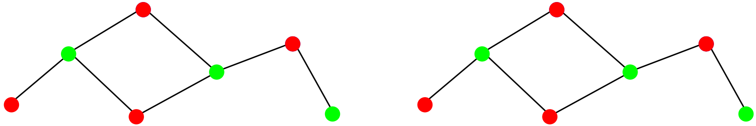

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A few results:

- For planar graphs, any 8-coloring can be transformed into any other in a polynomial number of steps. (B., Perarnau '15). Best lower bound: 7.
- Any $(tw(G)+1)$-coloring can be transformed into any other within a quadratic number of steps (Bonamy, B. '13). Tight.

Motivations (in the physics comminity)

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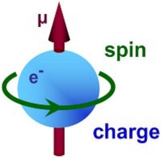
- Obtain a 'random' coloring of a graph.
- Obtain lower bounds on the mixing time of the Markov chain where
  - Elements are colorings.
  - There is a positive probability transition between two colorings if one can transform into the other by recoloring a single vertex.

Anti-ferromagnetic Potts Model



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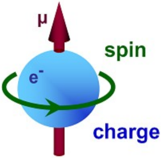

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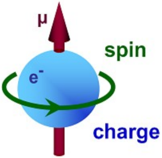

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$\Leftrightarrow$ All the k-colorings of G.

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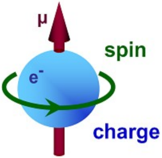

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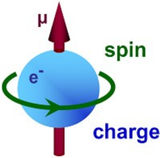

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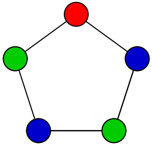
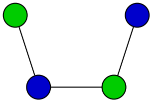
The physicists want to:

- Find the mixing time of Markov chains on Glauber dynamics.
  We need to recolor only one vertex at a time.
- Generate all the possible states of a Glauber dynamics.
  We have no constraint on the method.



Let a, b be two colors.

Kempe chains



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- A connected component of the graph induced by the vertices colored by a or b is a Kempe chain.

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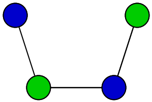

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- Permuting the colors of a Kempe chain is a Kempe change.

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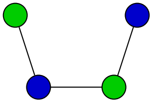

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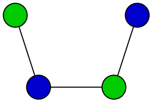

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- A connected component of the graph induced by the vertices colored by a or b is a Kempe chain.
- Permuting the colors of a Kempe chain is a Kempe change.

Remark: If a component is reduced to a single vertex, then the Kempe change consists in recoloring one vertex.
$\Rightarrow$ Kempe changes generalize single vertex recolorings.

Two colorings are Kempe-equivalent if one can transform one into the other via a sequence of Kempe changes.

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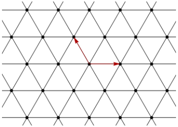
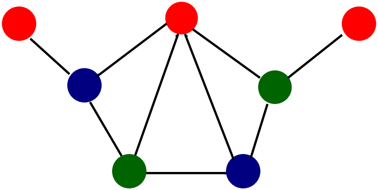
Theorem (Bonamy, B., Feghali, Johnson '15)

All the k-colorings of a connected k-regular graph with $k \geq 4$ are Kempe equivalent.



Consequence in physics: Close the study of the Wang-Swendsen-Koteký algorithm for Glauber dynamics on triangular lattices.

Graph partitionning



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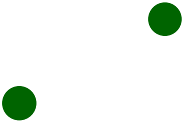

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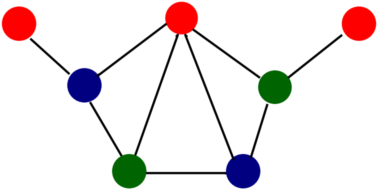
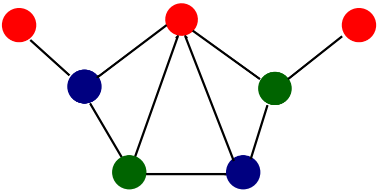


Definition (independent set)

An independent set is a subset of vertices that does not induce any edge.

Graph partitionning



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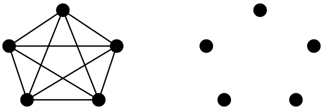
A proper coloring is a partition of the vertices into independent sets.

Graph partitionning



Clique partitions have many applications:

- In the theoretical world: a point from which we can start.
- In machine learning.
- Communities in social network.
- Big data.

Coalition game

- A set I of n agents.
- A superadditive valuation function $v : 2^n \to \mathbb{N}$. (the money generated by the coalition S if agents of S decide to work on their own project)

Coalition games

Coalition game

- A set I of n agents.
- A superadditive valuation function $v : 2^n \to \mathbb{N}$. (the money generated by the coalition S if agents of S decide to work on their own project)

Goal

Distribute money to the agents in such a way, for every coalition S, the money distributed to agents of S is at least $v(S)$.
$\Rightarrow$ No coalition wishes to leave the *grand coalition*.

Definition (core)

The core of the coalition game is the set of payoff vectors x satisfying the following constraints:

$$\sum_{i \in I} x_i = v(I) \qquad \text{The money we can distribute}$$

$$x_i \geq 0 \qquad \forall i \in I \qquad \text{Non-negative salary}$$

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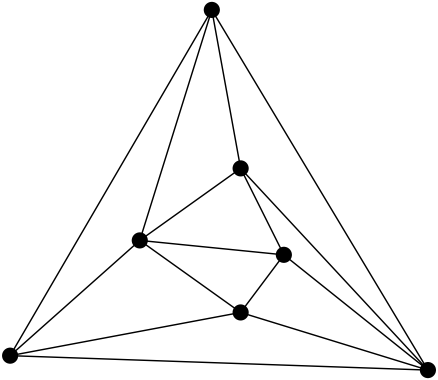
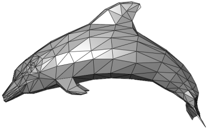
$\Rightarrow$ New bounds on some relaxations of the core (B., Li, Vetta).

Planar triangulations [1]





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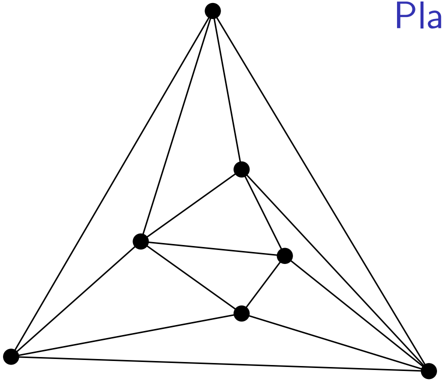

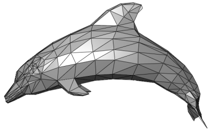

Euler : $v - e + f = 2$

Triangulation : $3f = 2e$ $\implies$

$$\begin{aligned} e &= 3v - 6 \\ (e-3) &= 3(v-3) \\ e_{int} &= 3v_{int} \end{aligned}$$

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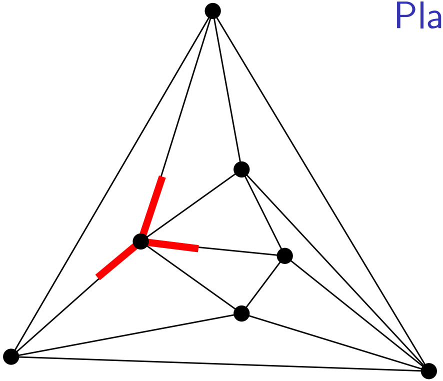

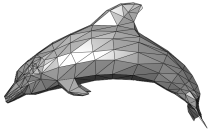

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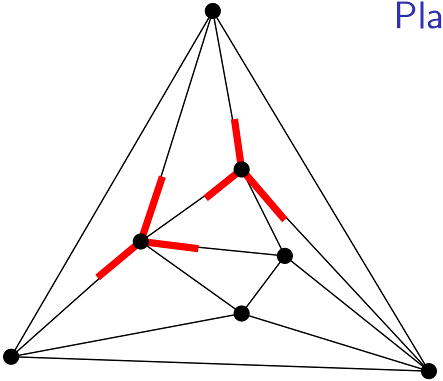

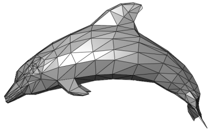

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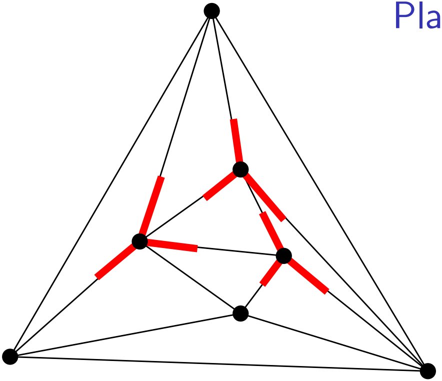

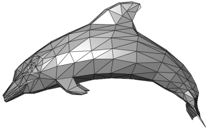

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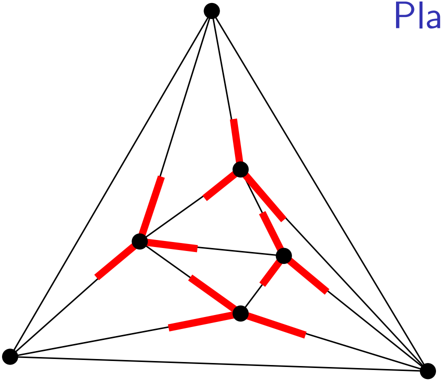

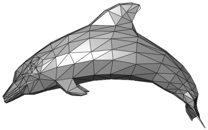

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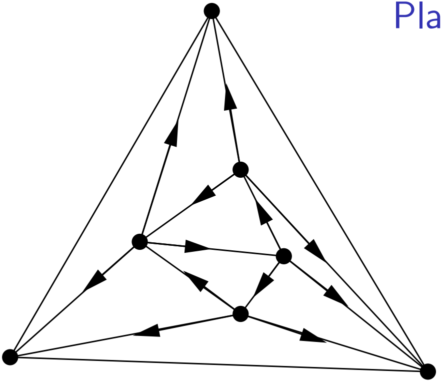

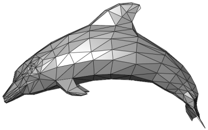

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$\rightsquigarrow$ Associate to each internal vertex three incident edges and deduce a 3-orientation.

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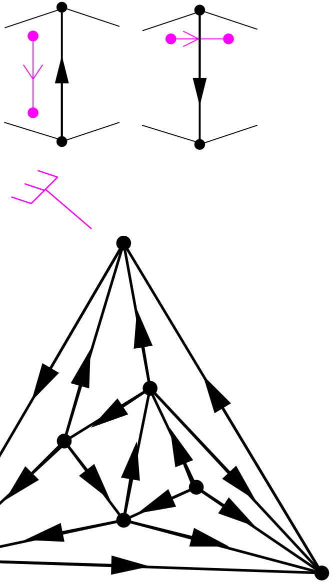
Encoding a planar graph



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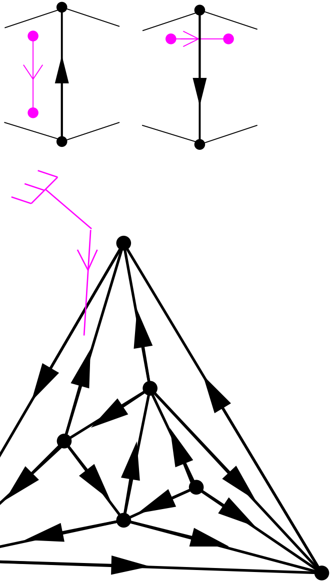

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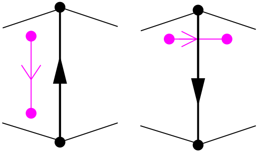

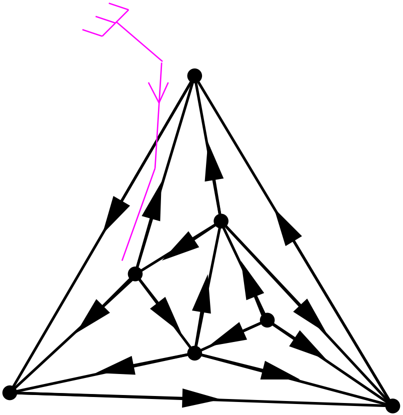

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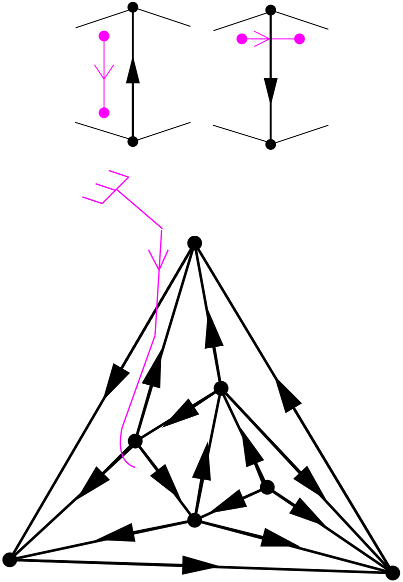

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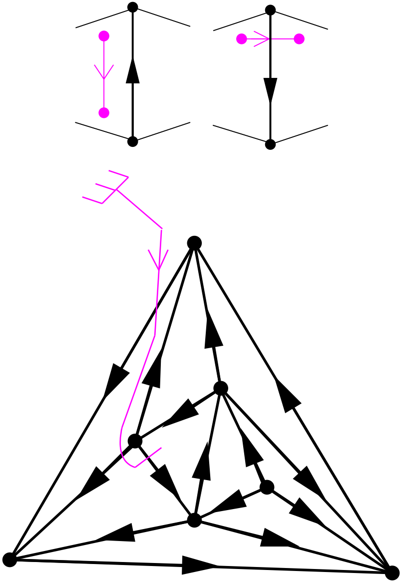

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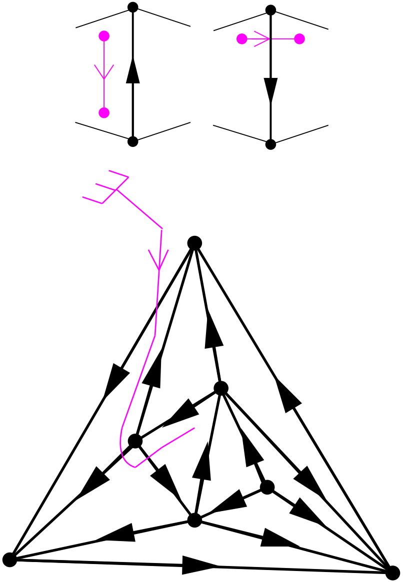

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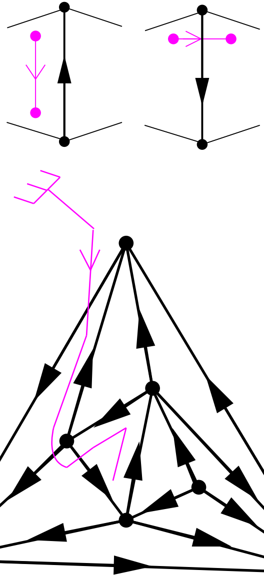

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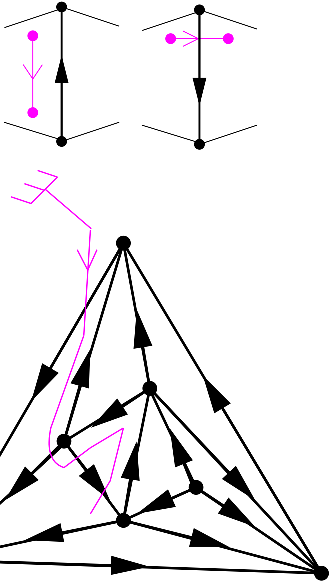

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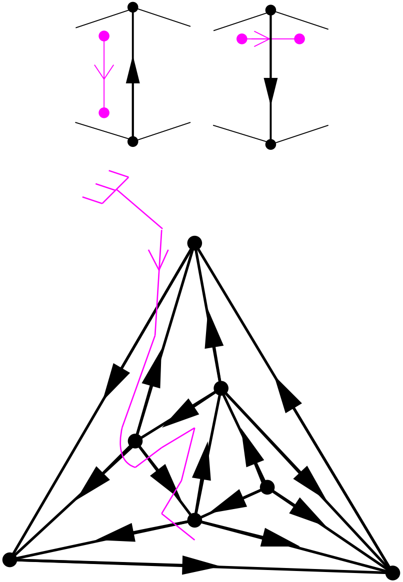

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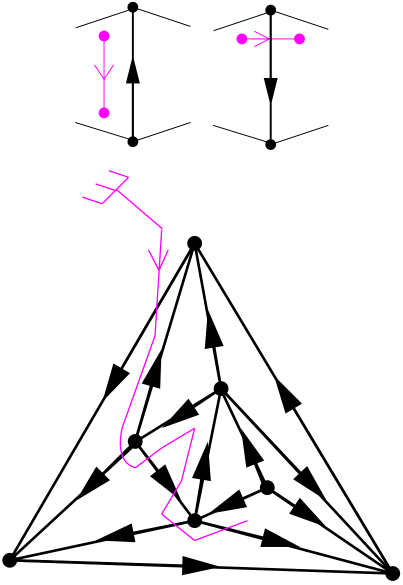

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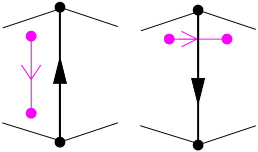

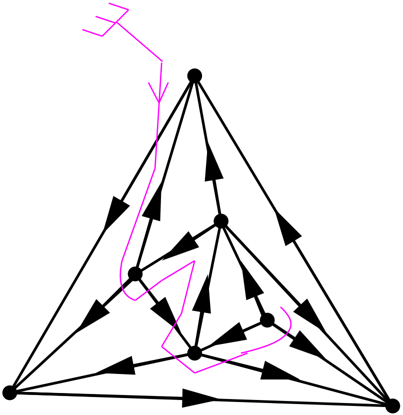

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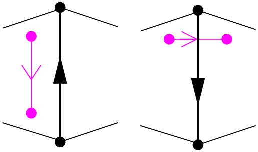

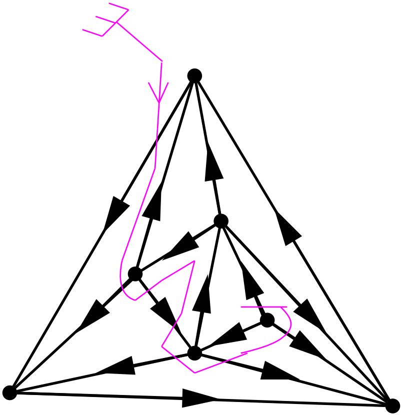

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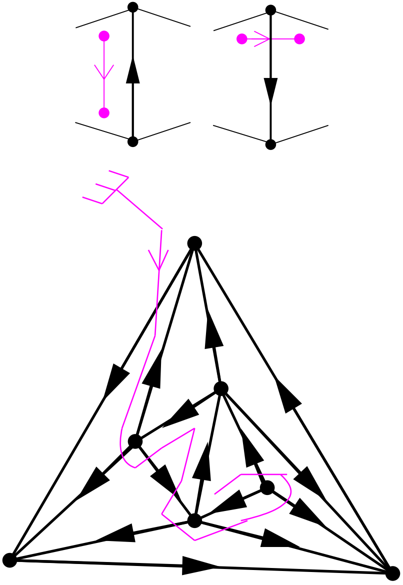

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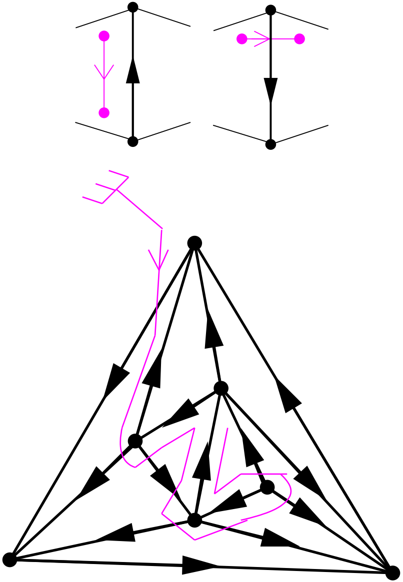

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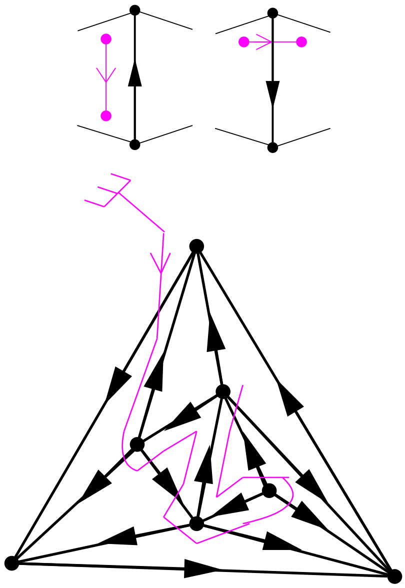

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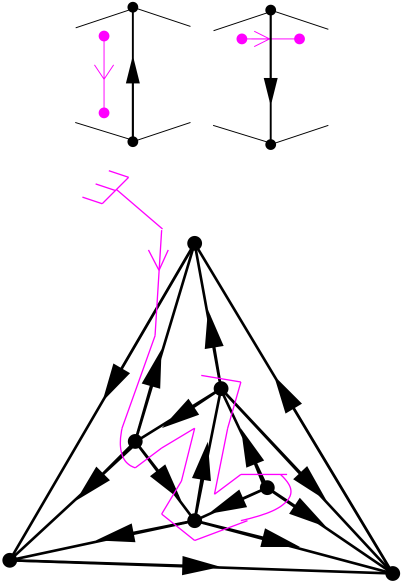

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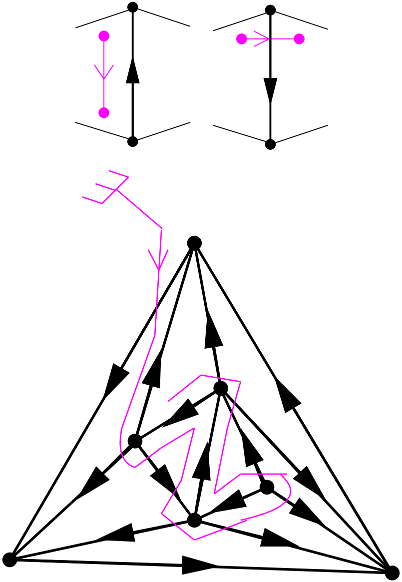

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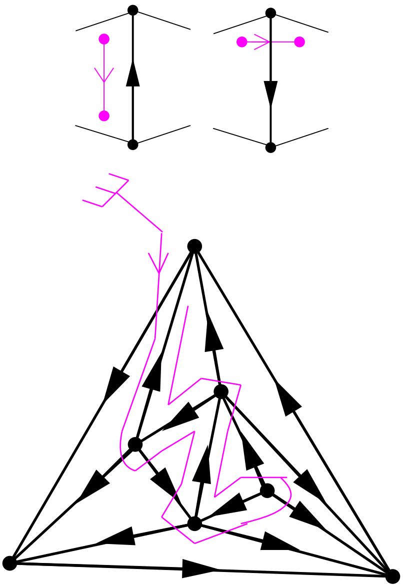

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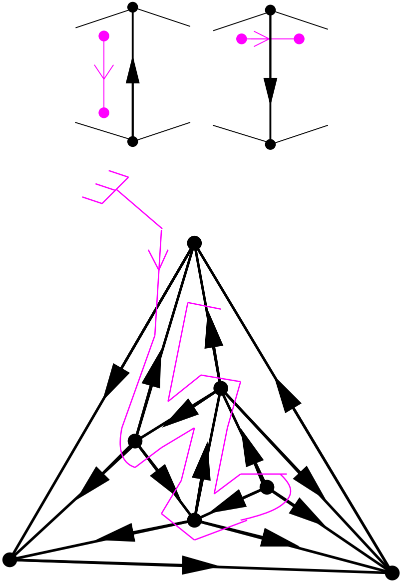

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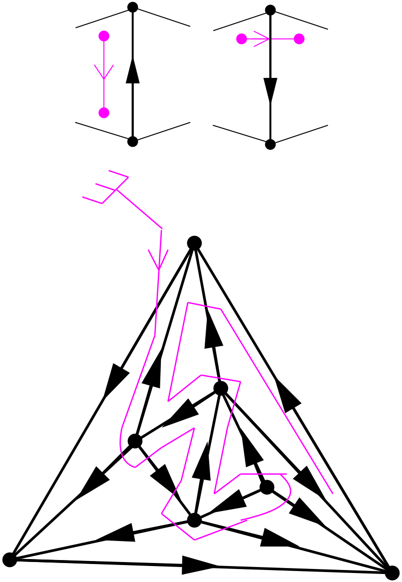

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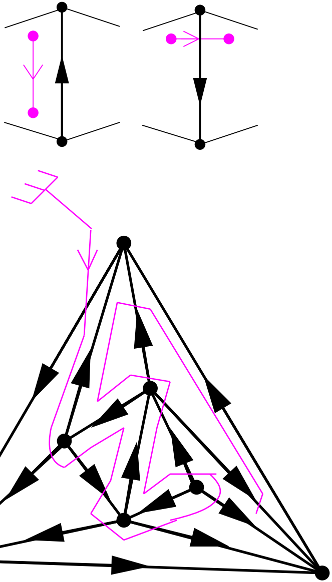

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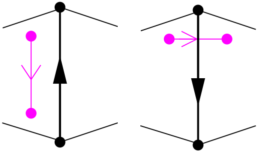

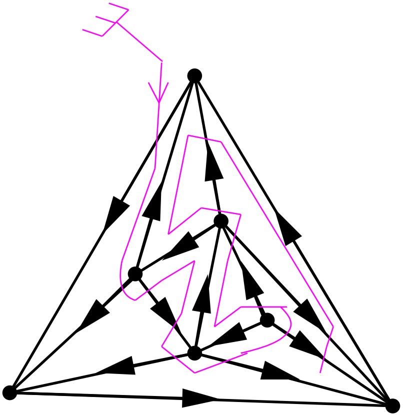

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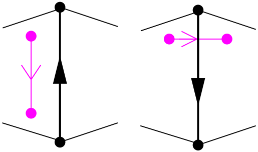

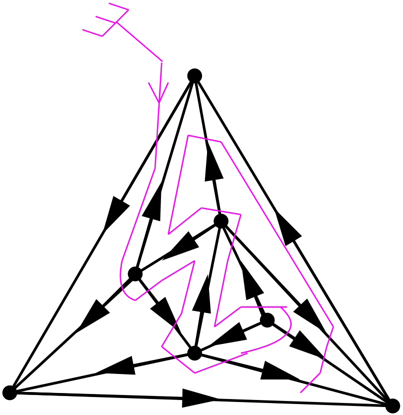

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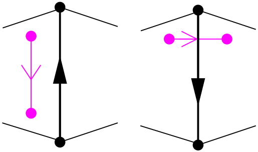

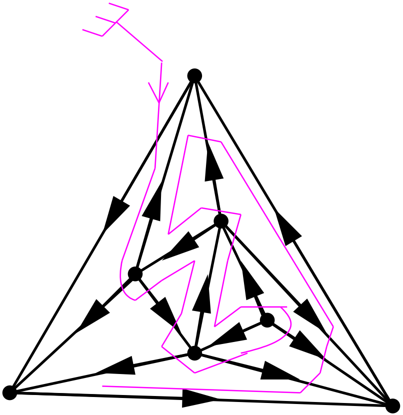

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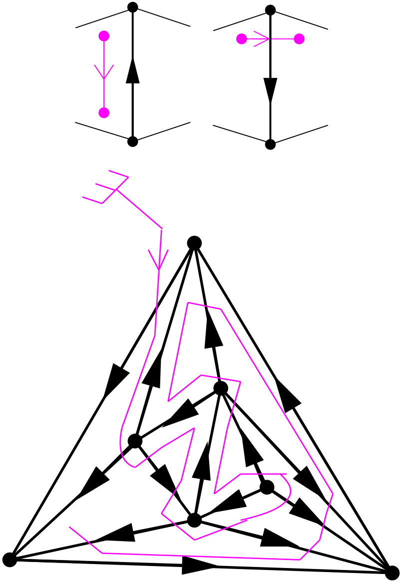

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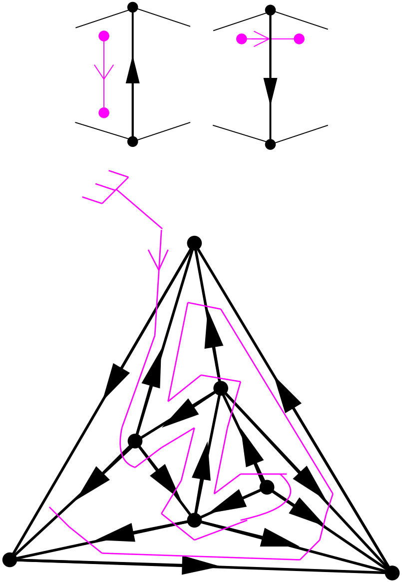

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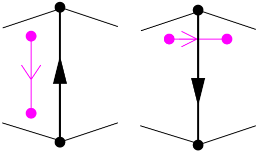

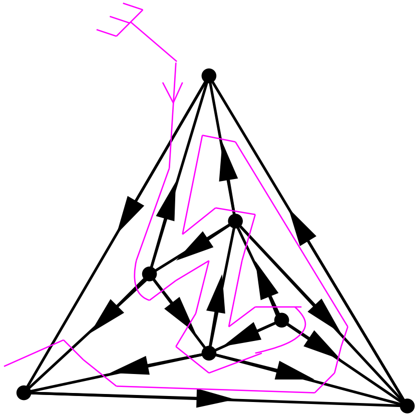

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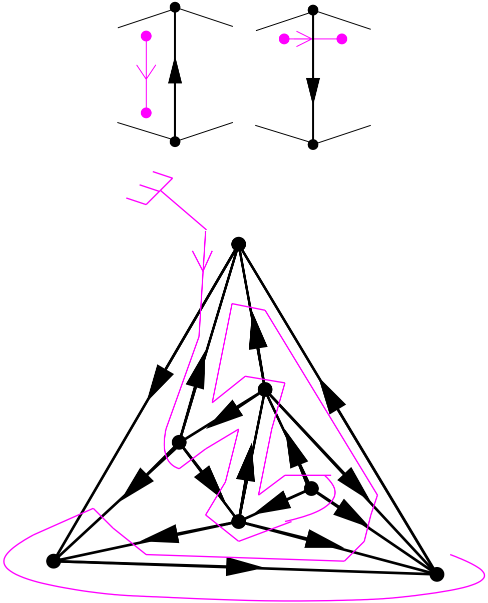

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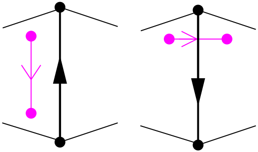
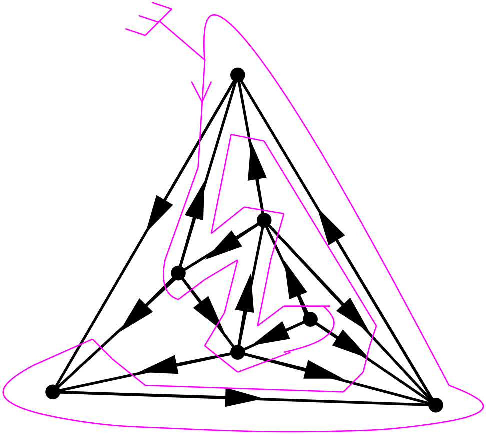
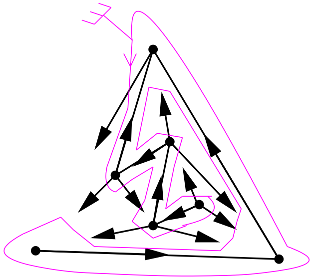




Bijection



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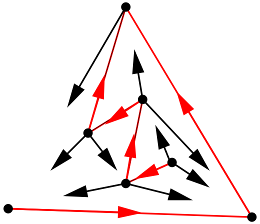

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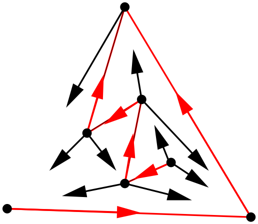

Using this red tree, the planar graph can be encoded using $3,25 \cdot n$ bits. (Poulalhon, Schaeffer '03)

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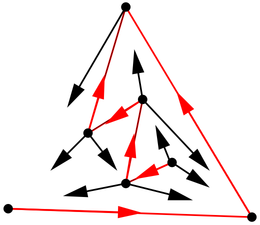

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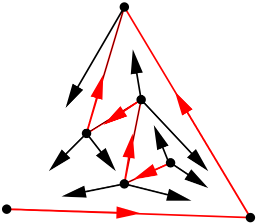
$\Rightarrow$ OPTIMAL !

Bijection



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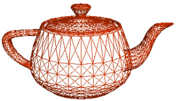
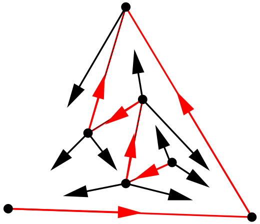


Theorem (Despré, Gonçalves, Levesque '15)

A similar result exists on the torus.

Bijection



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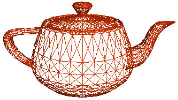

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Question: what about higher genus?

Other topics related to my work

- VC-dimension.
- Spectrum auctions.
- Flow-cuts problems.
- Voronoi diagram and Delaunay triangulations.

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Thanks for your attention