Cliques, stable sets and colorings

Nicolas Bousquet

(joint work with Marthe Bonamy, Aurélie Lagoutte and Stéphan Thomassé)

First definitions

- ω the maximum size of a clique.
- α the maximum size of a stable set.
- χ the chromatic number.

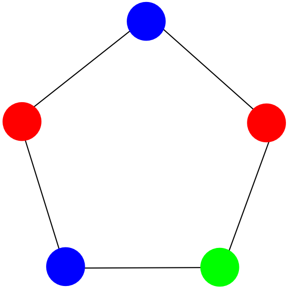
- P_k : induced path on k vertices.
- C_k : induced cycle on k vertices.
- class $=$ class closed under induced subgraphs.
- n : number of vertices of the graph.

Chromatic number and stable sets

Observation

$\chi \geq \frac{n}{\alpha}$.

A coloring is a partition of the vertex set into independent sets.



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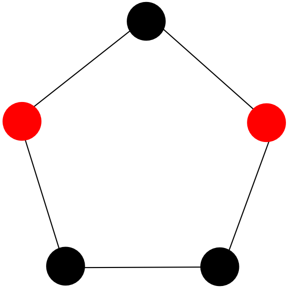

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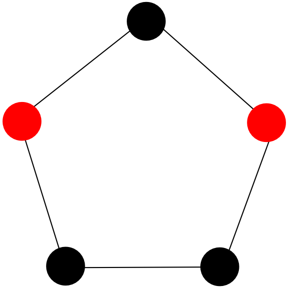
A coloring is a partition of the vertex set into independent sets. At least $\frac{n}{\alpha}$ colors are necessary since each color class has size at most α.



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Existence of an empty bipartite graph of size $\frac{n}{2c}$.

Question :

Reverse of these implications ?

- First implication : FALSE.
- Second implication : we only have a polynomial clique or a polynomial stable set.

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Observation : We always have $\omega \leq \chi$.

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Definition (χ-bounded)

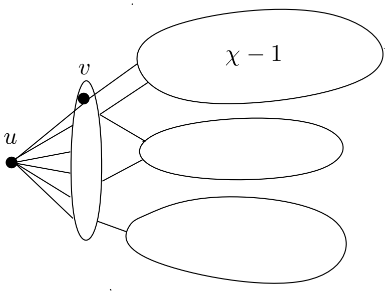
A class is χ-bounded if $\chi \leq f(\omega)$.

Example : Graphs with no P_k are χ-bounded (Gyárfás '87).

Gyarfás proof (for triangle-free graphs)

Take a vertex *u*.

- A connected component *X* of $G \setminus N(u)$ has chromatic number at least $\chi - 1$.



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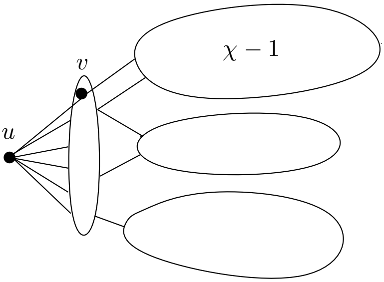

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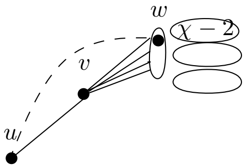

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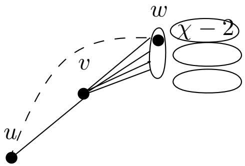

When the clique is unbounded, the function becomes exponential...

χ-bounded classes

- P_k-free graphs
- Star-free graphs
- Disk graphs
- Perfect graphs

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For χ-bounded classes of graphs, we try to find the best possible function f.

Conjecture (Gyárfás '87)

A graph with no copy of P_k has chromatic number at most $Poly(k, \omega)$.

Erdős-Hajnal and χ-boundedness

Conjecture (Erdős Hajnal '89)

A graph with no copy of P_k has a clique or a stable set of size n^ϵ.

Folklore

If a class $\mathcal{C}$ of graphs satisfies $\chi \leq \omega^c$ then $\mathcal{C}$ has a polynomial clique or stable set.

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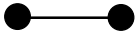
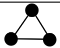
Proof :

- Either $\omega \geq n^{\frac{1}{2c}} \Rightarrow$ OK.
- Or $\omega \leq n^{\frac{1}{2c}} \Rightarrow \chi \leq \sqrt{n}$.
  So there is a stable set of size $\sqrt{n}$.

$\Rightarrow$ Polynomial χ-bounded stronger than Erdős-Hajnal.

Erdős-Hajnal conjecture

What is the value of $\max(\omega, \alpha)$ if some graph H is forbidden ?

	$\alpha = n$
	$\alpha \geq \sqrt{n \log n}$
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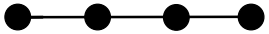	α or ω are at least $\sqrt{n}$

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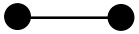
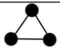
	$\alpha = n$
	$\alpha \geq \sqrt{n \log n}$
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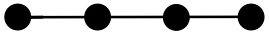	α or ω are at least $\sqrt{n}$

Conjecture (Erdős-Hajnal '89)

For every H, there exists $\epsilon > 0$ such that every H-free graph satisfies max$(\alpha, \omega) \geq n^{\epsilon}$.

On the importance of H

Lemma (Grimmet, Mc Diarmid '75)

Random graphs satisfy $\alpha, \omega = \mathcal{O}(\log n)$.

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Number of such sets $\approx n^{2 \log n} = 2^{2 \log^2 n}$.

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- Bull : Chudnovsky, Safra '08. ✓
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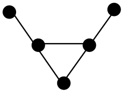

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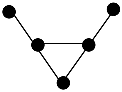

⇒ What happens if we enforce stronger conditions...
Idea : forbid a graph and its complement.

Erdős-Hajnal for paths and antipaths

Theorem (Chudnovsky, Zwols '11)

Graphs with no P_5 nor complement of P_6 have the Erdős-Hajnal property.

Theorem (Chudnovsky, Seymour '12)

Graphs with no P_5 nor complement of P_7 have the Erdős-Hajnal property.

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Structure of the proof :

1. Extract a sparse or a dense linear subgraph.
2. The graph contains an empty (or complete) linear bipartite subgraph.
3. Linear empty bipartite graph $\Rightarrow$ polynomial clique / stable set.

sparse = degree of each vertex $\leq \epsilon n$.
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Since the problem is the same up to complementation, we assume that there is a linear sparse subgraph.

Step 1 : sparse or dense subgraphs

Theorem (Rödl '86)

Every graph G satisfies one of the following conditions :

- G contains every graph on k vertices.
- G has a linear subset with average degree $\leq \epsilon$.
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Sketch of the proof :

- Apply Szemerédi's regularity lemma.
- Consider the graph of the partitions given by the Lemma.
- By Turán, there is a large clique which is "homogeneous", i.e. which only contains ϵ'-regular pairs.
- Every edge of this clique is of type : ϵ, $1 - \epsilon$, other.
- By Ramsey, there is a monochromatic clique : the conclusion depends on the color of the clique.

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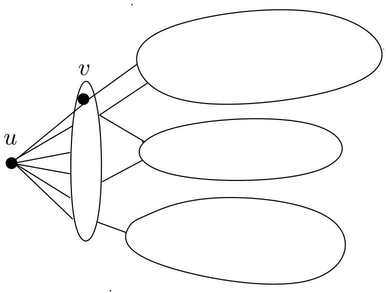

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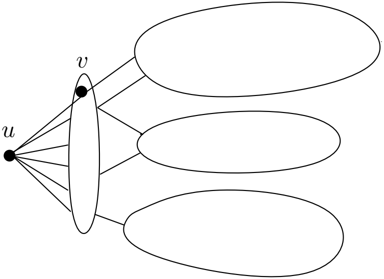

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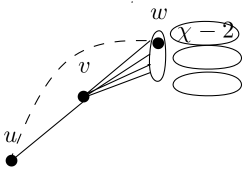

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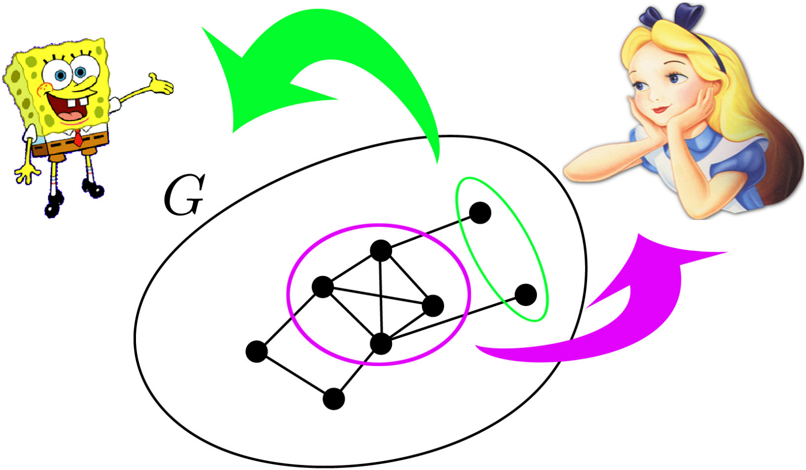
Remark :

Steps 1 and 3 hold as in the case of paths. But Step 2 is more involved...

Clique vs Independent Set Problem



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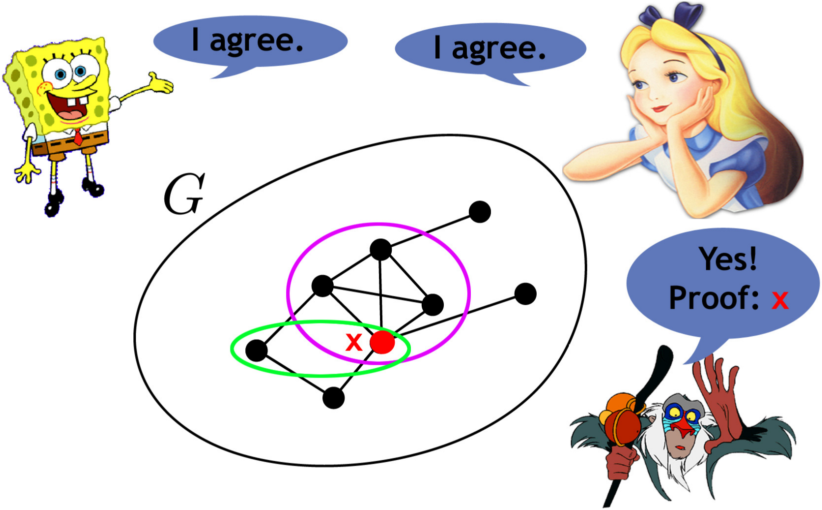

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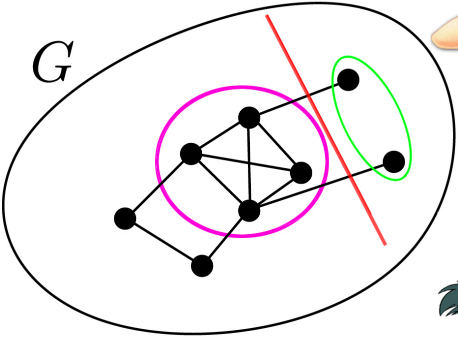

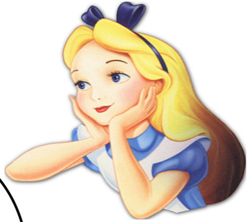

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Idea : Covering the Clique - Stable Set matrix with monochromatic rectangles.

CL-IS problem : Bounds

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Lower bound

There are some graphs with no CS-separator of size less than $n^{2-\epsilon}$.

Partial results : Random graphs

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Proof :
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A vertex v is in A with probability p and is in B otherwise.
$\Rightarrow$ Draw $\mathcal{O}(n^{5+\epsilon})$ such partitions.
W.h.p. there is a partition which separates C, S.

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Idea : $\mathcal{O}(|H|)$ vertices of the clique "simulate" the pair C,S.

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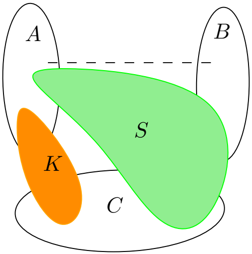

The case of P_k, $\overline{P_k}$-free graphs

Theorem

There is a polynomial CS-separator for P_k, $\overline{P_k}$-free graphs.

Proof :

- There exists a linear empty (or a complete) bipartite graph (A, B). Let C be the remaining vertices.
- Extend partitions of $A \cup C$ by putting B on the stable set side.
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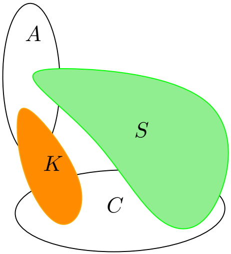

Conclusion

Questions

Does P_5 and/or C_5 have the Erdős-Hajnal property ?

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Conjecture (Gyárfás '87)

Graphs with no long cycle are χ-bounded.

Open even for triangle-free graphs.

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Question

Find a class of graphs with linear empty bipartite graphs (for every induced subgraph) but with no linear stable set.

Thanks for your attention