# Erdős-Hajnal for paths and cycles

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# First definitions

- $\omega$ the maximum size of a clique.
- $\alpha$ the maximum size of a stable set.
- $\chi$ the chromatic number.

- $P_k$ : induced path on $k$ vertices.
- $C_k$ : induced cycle on $k$ vertices.
- class $=$ class closed under induced subgraphs.

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**Question :**
Reverse of these implications ?

- First implication : FALSE.
- Second implication : we only have a polynomial clique or a polynomial stable set.

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**Conjecture (Erdős Hajnal '89)**

A graph with no copy of $P_k$ has a clique or a stable set of size $n^{\epsilon}$.

# Erdős-Hajnal and $\chi$-boundedness

**Folklore**

If a class $\mathcal{C}$ of graphs satisfies $\chi \leq \omega^c$ then $\mathcal{C}$ has a polynomial clique or stable set.

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**Proof :**

- Either $\omega \geq n^{\frac{1}{2c}} \Rightarrow$ OK.
- Or $\omega \leq n^{\frac{1}{2c}} \Rightarrow \chi \leq \sqrt{n}$.
  So there is a stable set of size $\sqrt{n}$.

$\Rightarrow$ Polynomial $\chi$-bounded stronger than Erdős-Hajnal.

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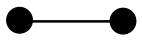
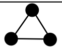
|  | $\alpha = n$ |
|  | $\alpha \geq \sqrt{n \log n}$ |
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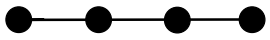
|  | $\alpha$ or $\omega$ are at least $\sqrt{n}$ |

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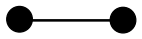
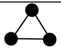
|  | $\alpha = n$ |
|  | $\alpha \geq \sqrt{n \log n}$ |
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**Conjecture (Erdős-Hajnal '89)**

For every $H$, there exists $\epsilon > 0$ such that every $H$-free graph satisfies $\max(\alpha, \omega) \geq n^{\epsilon}$.

# On the importance of $H$

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**Sketch of proof :**

Probability that a set of size $2 \log n$ is a clique $\approx (\frac{1}{2})^{2 \log^2 n}$

Number of such sets $\approx n^{2 \log n} = 2^{2 \log^2 n}$.

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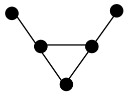
- Bull : Chudnovsky, Safra '08. ✓
- $P_5$, $C_5$ : widely open.



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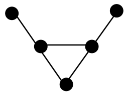

$\Rightarrow$ What happens if we enforce stronger conditions...

**Idea :** forbid a graph and its complement.

# Erdős-Hajnal for paths and antipaths

**Theorem (Chudnovsky, Zwols '11)**

Graphs with no $P_5$ nor complement of $P_6$ have the Erdős-Hajnal property.

**Theorem (Chudnovsky, Seymour '12)**

Graphs with no $P_5$ nor complement of $P_7$ have the Erdős-Hajnal property.

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**Structure of the proof :**

1. Extract a sparse or a dense linear subgraph.
2. The graph contains an empty (or complete) linear bipartite subgraph.
3. Linear empty bipartite graph $\Rightarrow$ polynomial clique / stable set.

sparse = degree of each vertex $\leq \epsilon n$.
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Since the problem is the same up to complementation, we assume that there is a linear sparse subgraph.

# Step 1 : sparse or dense subgraphs

**Theorem (Rödl '86)**

Every graph $G$ satisfies one of the following conditions :

- $G$ contains every graph on $k$ vertices.
- $G$ has a linear subset with average degree $\leq \epsilon$.
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**Sketch of the proof :**

- Apply Szemerédi's regularity lemma.
- Consider the graph of the partitions given by the Lemma.
- By Turán, there is a large clique which is "homogeneous", *i.e.* which only contains $\epsilon'$-regular pairs.
- Every edge of this clique is of type : $\epsilon$, $1 - \epsilon$, other.
- By Ramsey, there is a monochromatic clique : the conclusion depends on the color of the clique.

# Step 2 : extracting an empty (or complete) linear bipartite

Inspired from Gyárfás' proof that (triangle, $P_k$)-free graphs are $\chi$-bounded.

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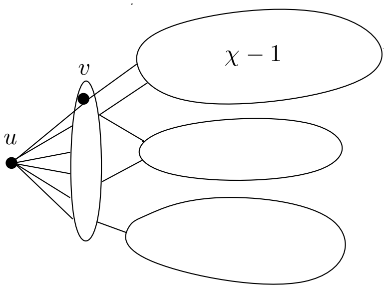

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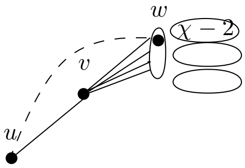

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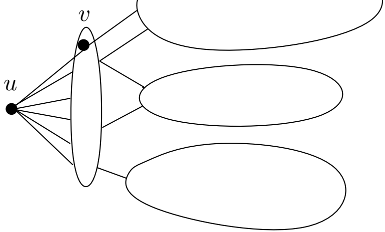

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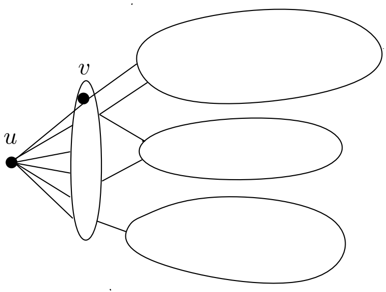

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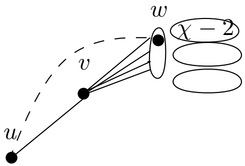

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- Apply induction on each part for finding a cograph of size $(\frac{n}{c})^{\epsilon}$.
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$\Rightarrow$ Every cograph has a clique or a stable set of size $\sqrt{n}$.

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**Remark :**

Steps 1 and 3 hold as in the case of paths. But Step 2 is more involved...

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**Lemma**

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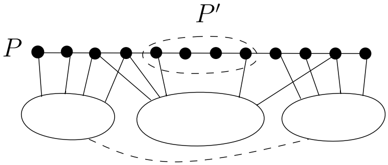


- Choose a subpath $P'$ of size $k$.
- No edge between vertices at the left and at the right of $P'$.
- By sparsity, not too many vertices on the middle.

# Step 2 : "Magic" tree

**Lemma**

Every graph contains a tree $T$ such that :

- Either a path of $T$ dominates $\frac{1}{4}$-th of the vertex set.
- Or there exists a linear empty bipartite graph.

**Hint of the proof :**

- Associate a "good" subset to each vertex.
- Put weight on the nodes of $T$ depending on the set associated to the vertex.
- Extract either a path of weight $1/4$ or two independent forests of weight $1/4$.

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**Question**

Find a class of graphs with linear empty bipartite graphs but no linear stable set.

# Thanks for your attention