Independent Set Reconfiguration via Token Sliding

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joint works with

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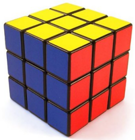
December 2020

Reconfiguration

A one-player game is a puzzle : one player makes a series of moves, trying to accomplish some goal.



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Giving my current position, can I reach a fixed target position ?

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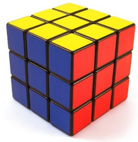

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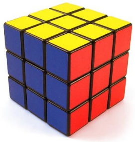

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Giving my current position, can I reach a fixed target position ?

- Reconfiguration introduced for colorings, satisfiability problems, dominating sets, cliques, list colorings, bases of matroids, boolean formulas...
- Applications to random sampling, bioinformatics...etc...

Main questions

- **Reachability problem.** Given two configurations, is it possible to transform the one into the other ?
- **Connectivity problem.** Given any pair of configurations, is it possible to transform the one into the other ?
- **Minimization.** Given two configurations, what is the length of a shortest sequence ?

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- **Algorithmics.** Can we efficiently solve these questions ? (In polynomial time, FPT-time...).

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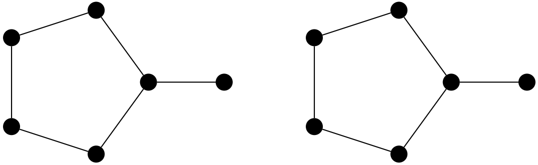

Definition (TS-sequence)

A TS-sequence $I_1, \ldots, I_\ell$ of independent sets is a sequence such that there exist $v \in I_{j+1}$ and $u \in I_j$ such that $I_{j+1} = I_j \cup \{v\} \setminus \{u\}$ and uv is an edge.

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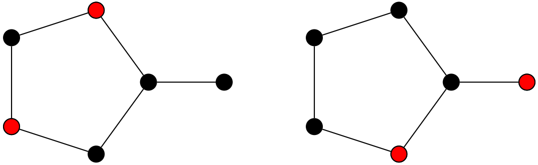

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We slide tokens along the edges in such a way the set remains an independent set at any step.

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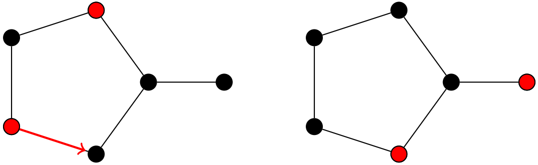

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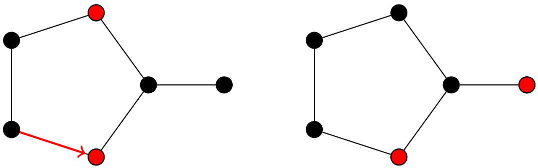

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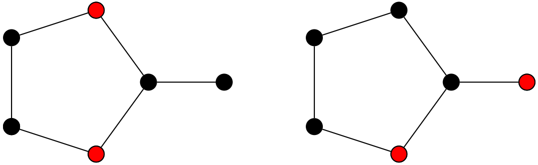

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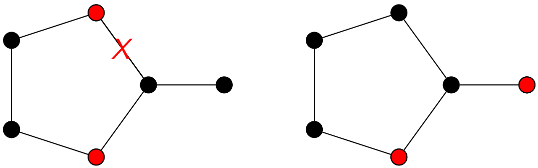

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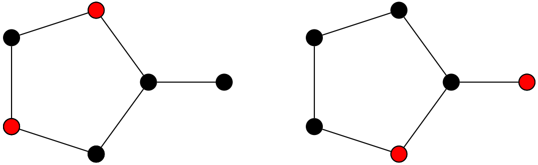

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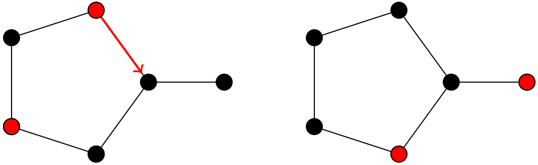

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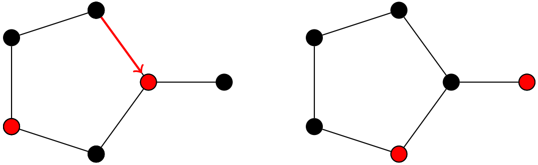

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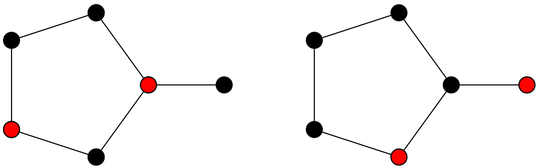

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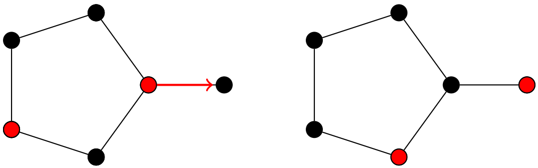

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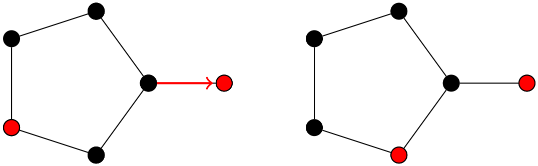

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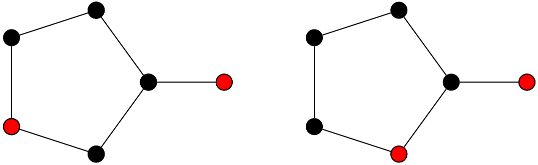

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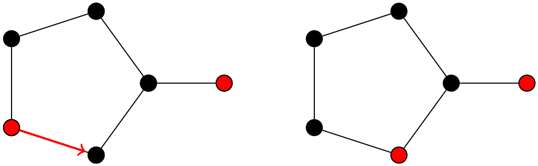

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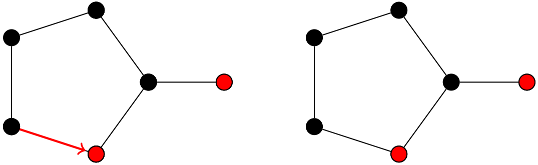

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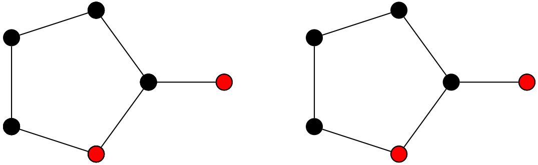

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- [Hopcroft, Schwartz, Sharir '83] Warehouseman's problem - Motion of rectangular robots in a grid.
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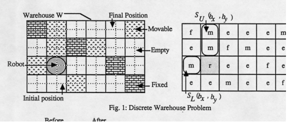

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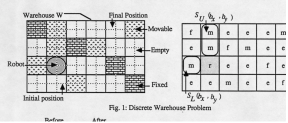

Fig. 1: Discrete Warehouse Problem

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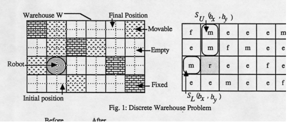

Fig. 1: Discrete Warehouse Problem

Question : What is the complexity of the Warehouseman problem for "dominos shaped" robots ?

TS-Reachability

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Input : A graph G, $k \in \mathbb{N}$, two independent sets I, J of size k.
Output : YES iff there exists a TS-sequence from I to J.

Theorem (Hearn, Demaine '05)

TS-Reachability is PSPACE-complete even restricted to planar graphs of maximum degree at most 3.

$\rightarrow$ On which graph class is the problem polynomial ?

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Polynomial time algorithms for :

- [Demaine et al.] Trees.
- [Kamiński, Medvedev, Milanič] Cographs.
- [Bonsma, Kamiński, Wrochna] Claw-free graphs.
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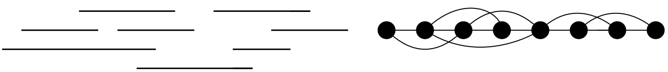
Can the TS-Reachability problem be decided on polynomial time on interval graphs ? On chordal graphs ?

Answers :

- [Bonamy, B. '18] YES on interval graphs.
- [Belmonte et al. '19] NO on split graphs.

(split graph = $V = V_1 \cup V_2$ where V_1 induces a clique and V_2 a stable set)

Interval graphs



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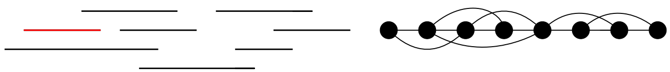
Remark :

A geometric representation can be obtained in polynomial time.

Interval graphs



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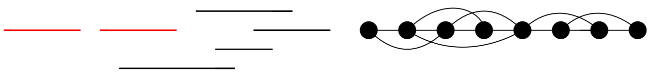

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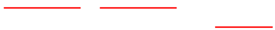

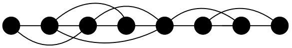

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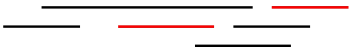
Today :

Decide if an independent set of size k can be transformed into the LIS.

First try : Naive Method



Lemma

I can be transformed into the LIS iff

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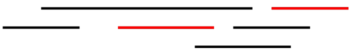

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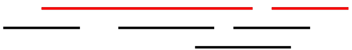

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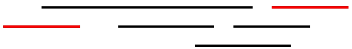

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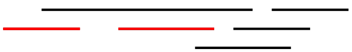

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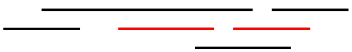

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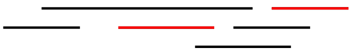

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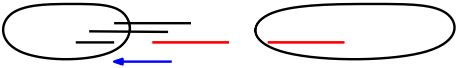

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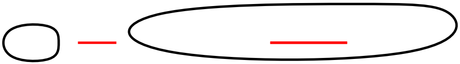

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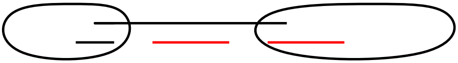

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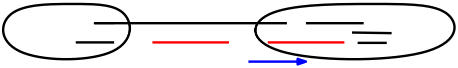

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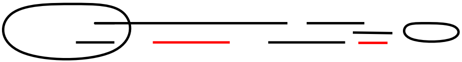

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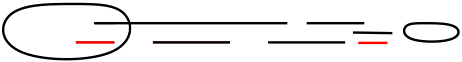

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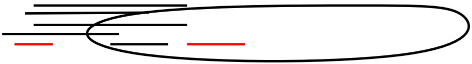

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- Otherwise we can't reach the LIS.

Questions

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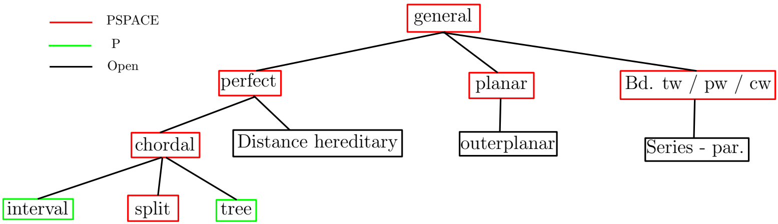

Matching Reconfiguration

A matching is a subset of edges that are pairwise endpoint disjoint.

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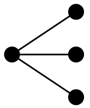

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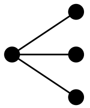

Question : For which H is TS-Independent Set Reconfiguration polynomial on H-free graphs ?

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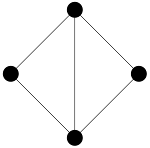

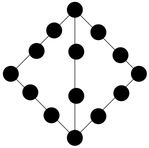

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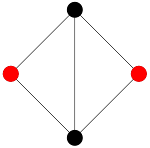

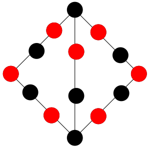

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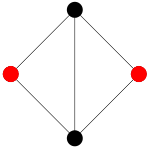

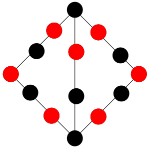

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- Start with a planar graph of maximum degree 3.
- Repeat the subdivision process $|H|$ times.

$\Rightarrow$ No copy of H if H has a vertex of degree ≥ 4 or a cycle or two vertices of degree ≥ 3.

$\mathbf{TS}_m$-Reachability

Input : A graph G, two independent sets I, J of maximum size.
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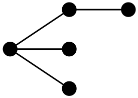

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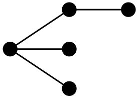

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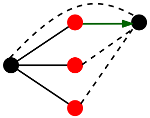

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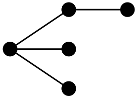

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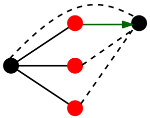

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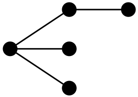

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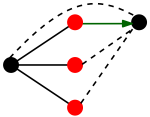

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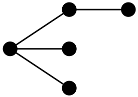

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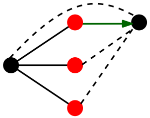

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- Complexity of TS-REACHABILITY on fork-free graphs ?
  Last case to completely characterize the complexity of TS-REACHABILITY on connected graphs.
- A few non connected graphs (on which we are currently working on).

Parameterized complexity

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Two research directions :

- Consider sparse graph classes (e.g. $|E| \leq c|V|$).
- Consider graph classes with "girth" restriction.

Restrictions on the possible cycles

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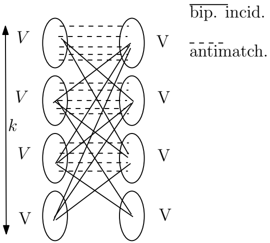

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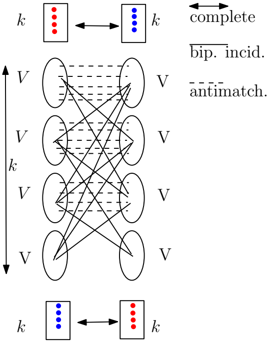

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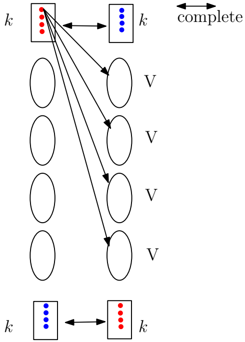

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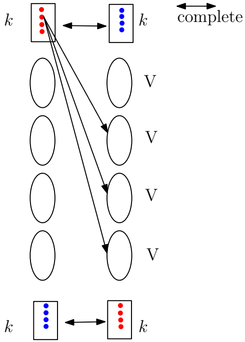

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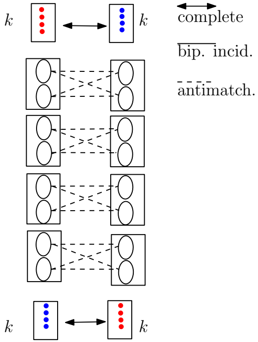

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High level idea :

- Bound the degree of the graph.
- [Fox Epstein et al.] Determine the frozen tokens is in P.
- [Fox Epstein et al.] Any reachable vertex can be reached via a sequence all the tokens but ≤ 1 slide $\leq$ once.

$\Rightarrow$ If IS at large distance, we can reach it.

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C_4 ? $\rightarrow$ NO !

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Questions :

- TS-REACHABILITY on graphs of girth ≥ 5 ? $\geq \ell$ for some fixed ℓ ?
- TS-REACHABILITY on even hole free graphs ?

  It would imply chordal graphs.

On going work

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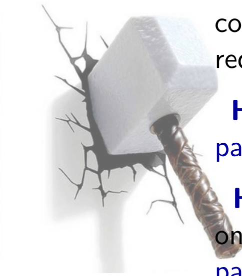

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Consequences : FPT algorithms for

- Bounded degree graphs.
- Planar graphs.
- Graphs of treewidth ≤ 4.
- Graphs of bounded treedepth.

What's next

For sparse graphs :

- Graphs of bounded treewidth ?
- Graph nowhere dense ?
- $K_{\ell,\ell}$-free graphs ?

For dense graphs :

- Chordal graphs ?
- Split graphs ?

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