Erdős-Hajnal for paths and cycles

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First definitions

- ω the maximum size of a clique.
- α the maximum size of a stable set.
- χ the chromatic number.

- P_k : induced path on k vertices.
- C_k : induced cycle on k vertices.
- class $=$ class closed under induced subgraphs.

Definition

A graph G is H-free if no induced subgraph of G is isomorphic to H.

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Question :
Reverse of these implications ?

- First implication : FALSE.
- Second implication : we only have a polynomial clique or a polynomial stable set.

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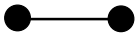
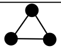
Folklore

If a class $\mathcal{C}$ of graphs satisfies $\chi \leq \omega^c$ then $\mathcal{C}$ has a polynomial clique or stable set.

Erdős-Hajnal conjecture

What is the value of $\max(\omega, \alpha)$ if some graph H is forbidden ?

	$\alpha = n$
	$\alpha \geq \sqrt{n \log n}$
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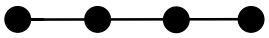	α or ω are at least $\sqrt{n}$

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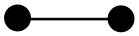
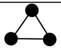
	$\alpha = n$
	$\alpha \geq \sqrt{n \log n}$
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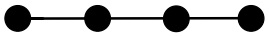	α or ω are at least $\sqrt{n}$

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For every H, there exists $\epsilon > 0$ such that every H-free graph satisfies $\max(\alpha, \omega) \geq n^{\epsilon}$.

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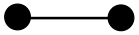
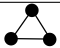
	$\alpha = n$
	$\alpha \geq \sqrt{n \log n}$
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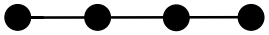	α or ω are at least $\sqrt{n}$

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Lemma (Grimmet, Mc Diarmid '75)

Random graphs satisfy $\alpha, \omega = \mathcal{O}(\log n)$.

Prime graphs

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If the Erdős-Hajnal conjecture holds for every prime graph H, then it holds for every graph.

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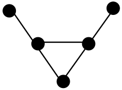
- Bull : Chudnovsky, Safra ’08. ✓
- P_5, C_5 : widely open.



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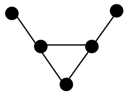

⇒ What happens if we enforce stronger conditions...

Idea : forbid a graph and its complement.

Our results (I)

Theorem (Chudnovsky, Zwols '11)

Graphs with no P_5 nor complement of P_6 have the Erdős-Hajnal property.

Theorem (Chudnovsky, Seymour '12)

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Theorem (B., Lagoutte, Thomassé '13)

Graphs with no P_k nor its complement have the Erdős-Hajnal property.

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Structure of the proof :

1. Extract a sparse or a dense linear subgraph.
2. The graph contains an empty (or complete) linear bipartite subgraph.
3. Linear empty bipartite graph $\Rightarrow$ polynomial clique / stable set.

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Since the problem is the same up to complementation, we assume that there is a linear sparse subgraph.

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Inspired from Gyárfás' proof that (triangle,P_k)-free graphs are χ-bounded.

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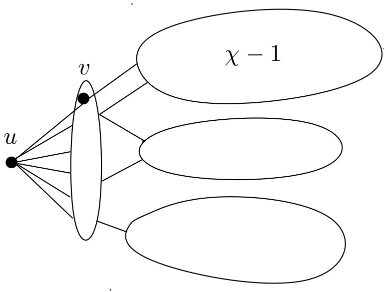

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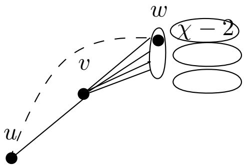

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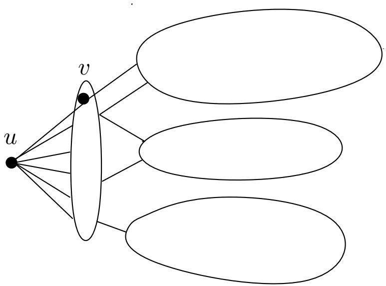

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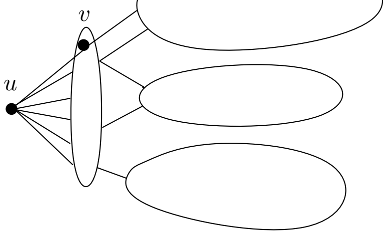

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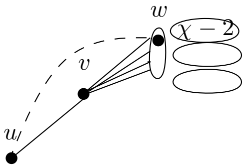

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- Find an empty or complete bipartite graph of size cn.
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Find a class of graphs with linear empty bipartite graphs but no linear stable set.

Thanks for your attention