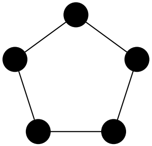
Kempe equivalence of colorings

Marthe Bonamy **Nicolas Bousquet**
Carl Feghali Matthew Johnson

JGA 2015

Graph recoloring



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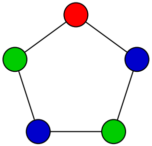

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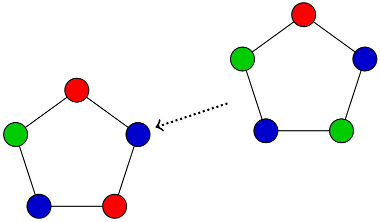

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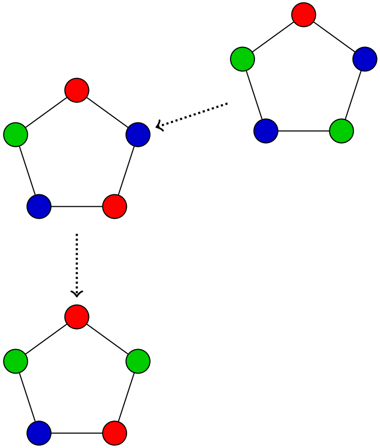

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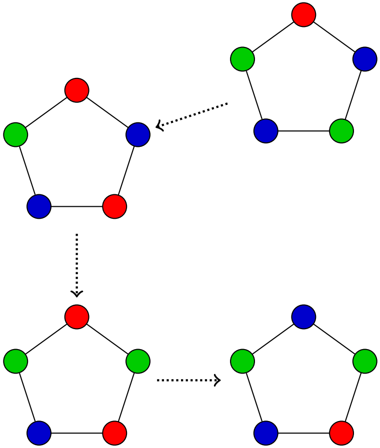

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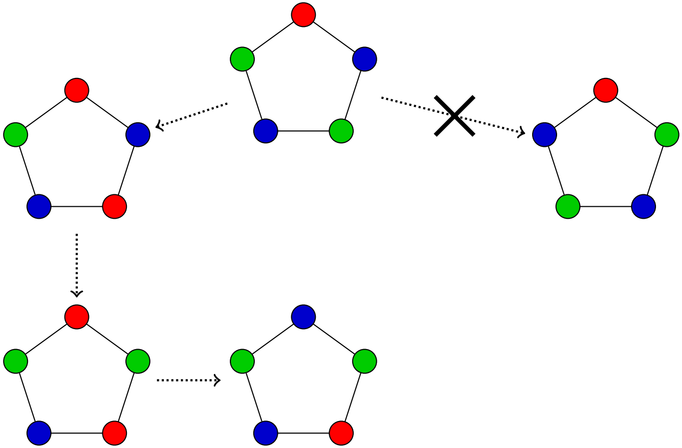
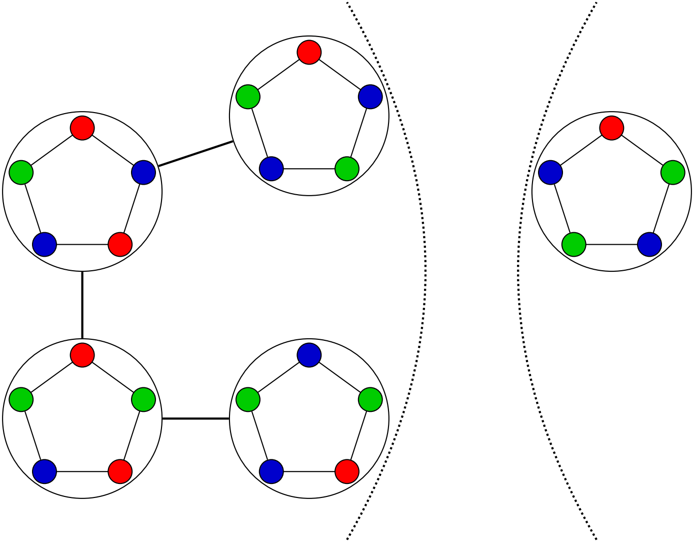


Graph recoloring

Solutions // Nodes. Most similar solutions // Neighbors.



Motivations

- Obtain a 'random' coloring of a graph.

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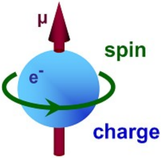
- Obtain lower bounds on the mixing time of a Markov chain.

Anti-ferromagnetic Potts Model



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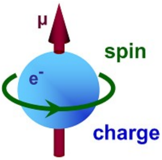

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Probability that a configuration appears is inversely proportional to the number of monochromatic edges divided by the temperature of the system.

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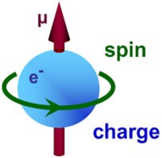

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Limit of a k-state Potts model when $T \to 0$.
$\Leftrightarrow$ All the k-colorings of G.

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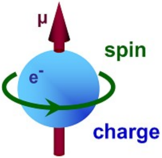

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- Generate all the possible states of a Glauber dynamics.

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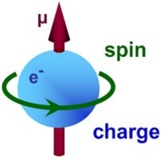

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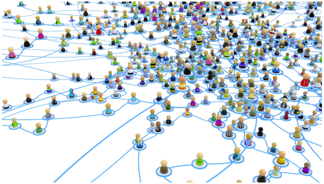
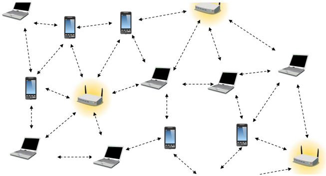
The physicists want to:

- Find the mixing time of Markov chains on Glauber dynamics.
  We need to recolor only one vertex at a time.
- Generate all the possible states of a Glauber dynamics.
  We have no constraint on the method.

Limit of the recoloring model

- In many applications, colors are interchangeable.





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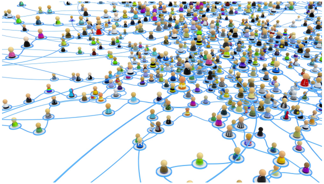

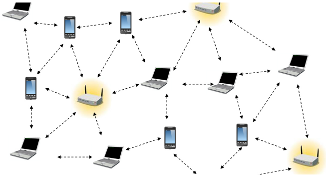

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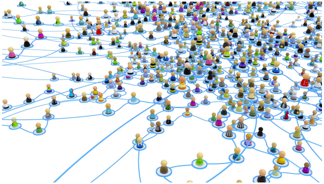
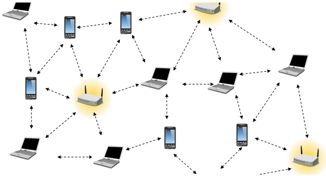
- Which type of actions ensures that the reconfiguration graph is connected?





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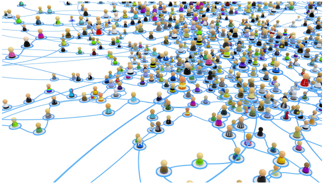
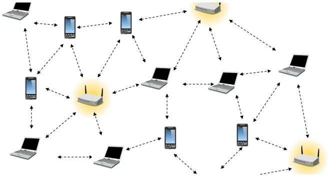
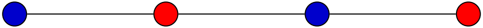
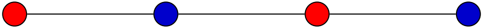


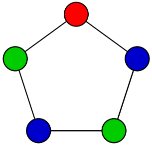


Idea: Recoloring vertices along a Kempe chain.





Kempe chains



Let a, b be two colors.

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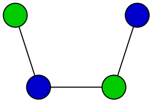

Let a, b be two colors.

- A connected component of the graph induced by the vertices colored by a or b is a Kempe chain.

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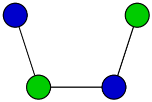

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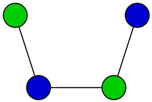

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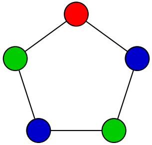
- Permuting the colors of a Kempe chain is a Kempe change.

Remark: If a component is reduced to a single vertex, then the Kempe change consists in recoloring one vertex.
$\Rightarrow$ Kempe changes generalize single vertex recolorings.

Kempe equivalence



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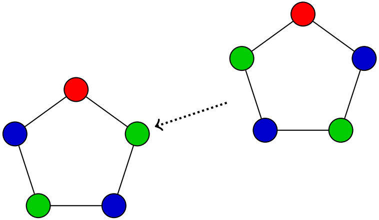

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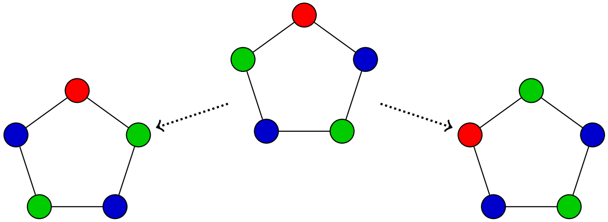

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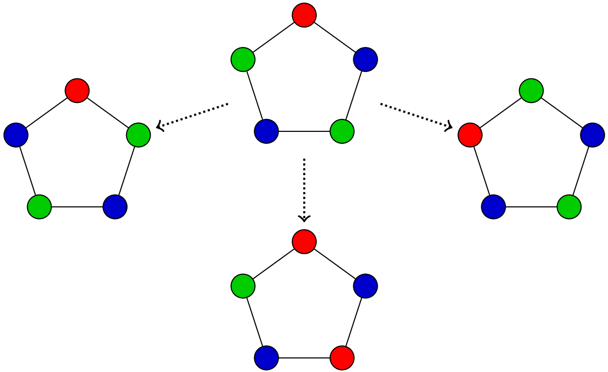

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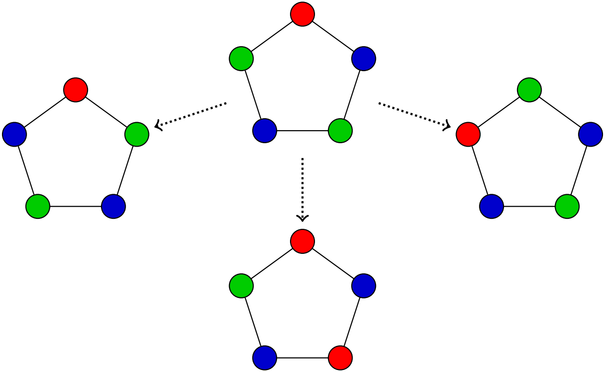

Definition (Kempe equivalent)

Two colorings are Kempe equivalent if we can transform the one into the other within a sequence of Kempe changes.

Mohar conjecture

Δ: Maximum degree of the graph

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Theorem (Brooks '41)

Every graph is Δ-colorable, except for cliques and odd cycles.

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Theorem (Las Vergnas, Meyniel '81): All the $(k+1)$-colorings of a k-degenerate graph are Kempe equivalent.

(k-degenerate: every subgraph contains a vertex of degree $\leq k$)

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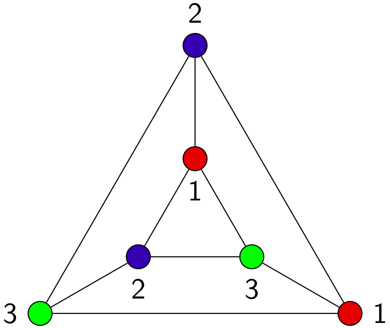
Results

The conjecture is false! (van den Heuvel '13)



(3-prism)

Results (2)

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All the k-colorings of a connected k-regular graph with $k \geq 4$ are Kempe equivalent.

Main lemma

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Let u, w, v be an induced P_3. All the colorings where u and v are colored alike are Kempe equivalent.

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Sketch:

- Identify u and v.
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- The resulting graph is $(\Delta - 1)$-degenerate.
- Δ-colorings of a $(\Delta - 1)$-degenerate graph are equivalent.

Consequence: If any coloring is equivalent to a coloring where u and v are colored alike, all the colorings are Kempe equivalent.

$$\begin{array}{ccc} \Delta\text{-coloring } \alpha & & \Delta\text{-coloring } \beta \\ \Downarrow & & \Uparrow \\ \Delta\text{-col. } \alpha' \text{ where } \alpha'(u) = \alpha'(v) & \Rightarrow & \Delta\text{-col. } \beta' \text{ where } \beta'(u) = \beta'(v) \end{array}$$

Sketch for the main result

Theorem (Bonamy, B., Feghali, Johnson '15)

All the colorings of a connected k-regular graph with $k \geq 4$ are Kempe equivalent.

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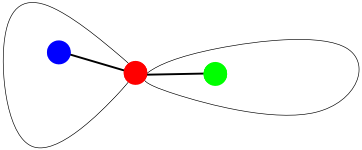

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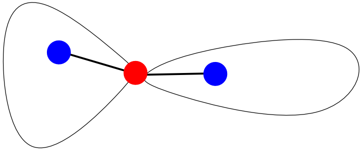

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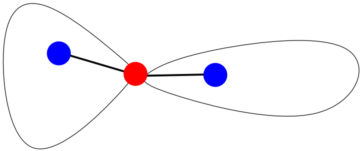
- If G is not 3-connected $\Rightarrow$ contradiction.
- If G does not have diameter at least 3 $\Rightarrow$ contradiction.

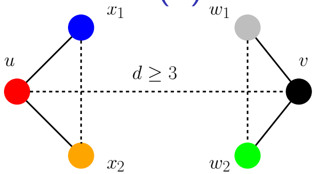
$\Rightarrow$ G is 3-connected of diameter ≥ 3.



Sketch (2)

So G is 3-connected of diameter ≥ 3.

- Let u, v at distance ≥ 3.
- Let w_1, w_2 in $N(u)$ s.t. $(w_1, w_2) \notin E$.
- Let x_1, x_2 in $N(v)$ s.t. $(x_1, x_2) \notin E$.



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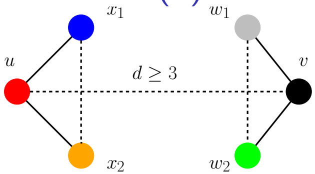

If:

(i) There exists a coloring s.t. w_1, w_2 are colored alike and x_1, x_2 are colored alike.

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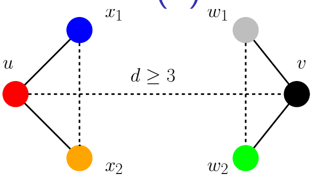

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Δ-coloring α $\downarrow$ Δ-col. α' s.t. $\alpha'(x_1) = \alpha'(x_2)$ $\searrow$

Δ col. γ s.t. $\gamma(w_1) = \gamma(w_2)$ and $\gamma(x_1) = \gamma(x_2)$

$\nearrow$ Δ-col. β' s.t. $\beta'(w_1) = \beta'(w_2)$ $\uparrow$ Δ-coloring β

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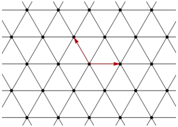
Question

Number of Kempe classes for the triangular lattice for $k = 5$?



Consequence in physics: Close the study of the Wang-Swendsen-Koteký algorithm for Glauber dynamics on triangular lattices.

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- Characterize the graphs for which all the $(\Delta - 1)$-colorings are Kempe equivalent.

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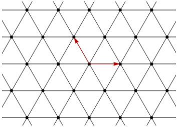

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Thanks for your attention!