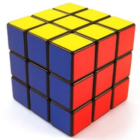
Independent Set Reconfiguration

Nicolas Bousquet

4 septembre 2024

Reconfiguration

A one-player game is a puzzle : one player makes a series of moves, trying to accomplish some goal.



Question :
Giving my current position, can I reach a fixed target position ?

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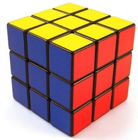

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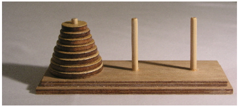

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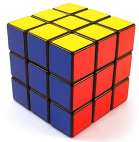

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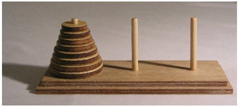

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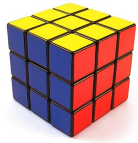

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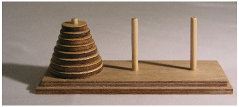
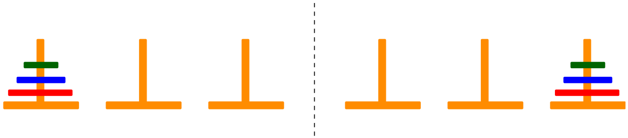


- Widely studied on graph problems in the last 15 years.
  Colorings, independent sets, dominating sets, cliques, list colorings, bases of matroids, CSP and boolean formulas...
- Important problems in random sampling, bioinformatics, discrete geometry, games...etc... for decades.

Focus on Hanoi tower



Goal :

Move disks from the first to the last rod moving one disk at every step.

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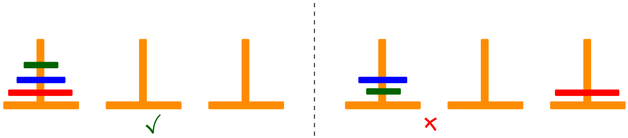

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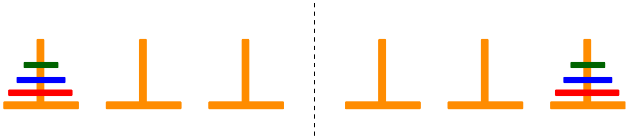

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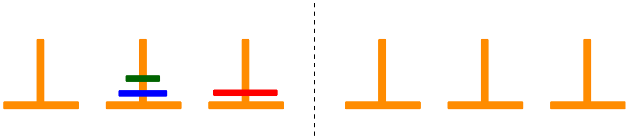

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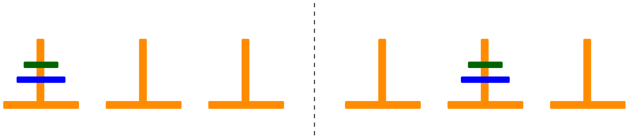

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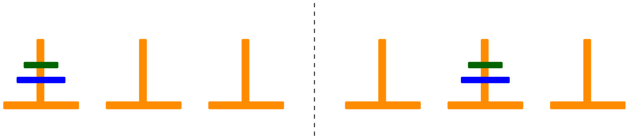

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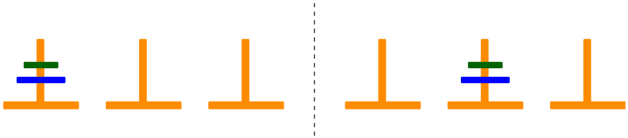

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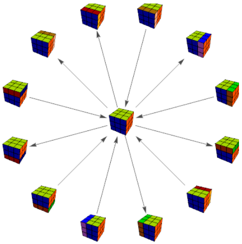
  Looks simple but computationally hard.
- Understandable because of symmetry.

  In what follows, symmetry / structure will vanish.

Configuration graph

Definition (Configuration graph $\mathcal{C}(I)$ of I)

- Vertices : Valid solutions of I.
- Create an edge between any two solutions if we can transform one into the other in one elementary step.



Reconfiguration diameter = Diameter of $\mathcal{C}(I)$ (when connected)

Main questions

- **Reachability problem.** Given two configurations, is it possible to transform the first into the other ?

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- **Minimization.** Given two configurations, what is the length of a shortest sequence ?
  What is the diameter of the configuration graph $\mathcal{C}(I)$?
- **Algorithmics.** Can we efficiently solve these questions ? (In polynomial time, FPT-time...).

Genesis of ISR

- [Hopcroft, Schwartz, Sharir '83] Warehouseman's problem - Motion of rectangular robots in a grid.
  ⇒ PSPACE-complete (but they need large robots).

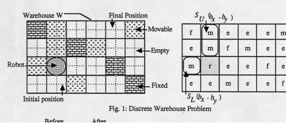

Fig. 1: Discrete Warehouse Problem

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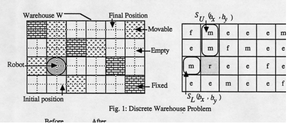

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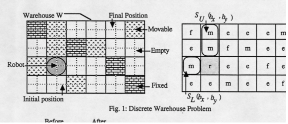

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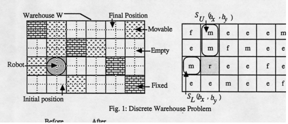

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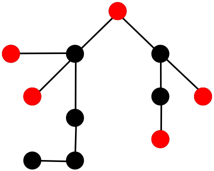
Theorem [Hearn, Demaine '04]

The problem is PSPACE-complete.

Token Jumping vs Token Sliding



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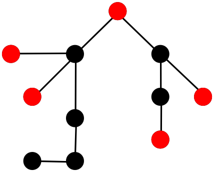

Token Jumping
Select one vertex of I and move it anywhere else.
(keeping an IS)

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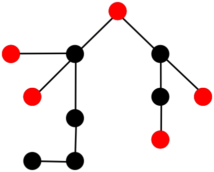

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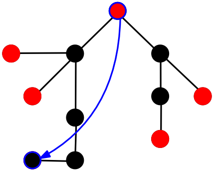

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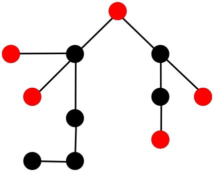

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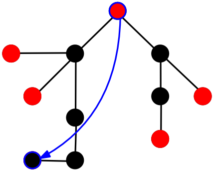

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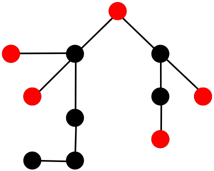

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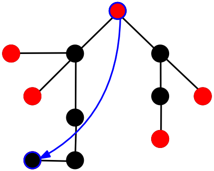

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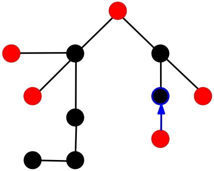

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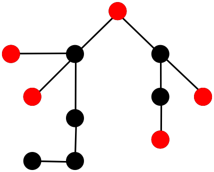

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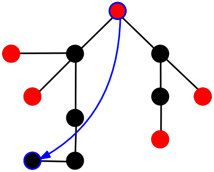

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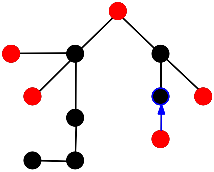

Question : What is the complexity of TS / TJ-Reachability ?

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Today :
Focus on parameterized algorithms.

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In this talk :
Parameter = size of the IS.

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Theorem (Bodlaender, Groenland, Swennenhuis '21)

TS and TJ-Reachability are XL-complete.

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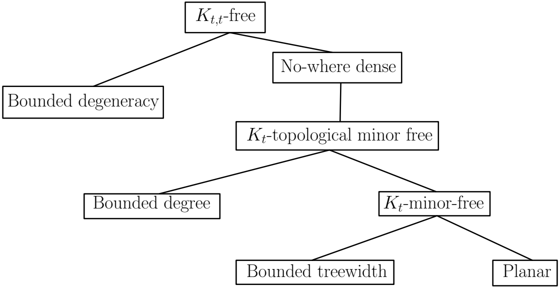

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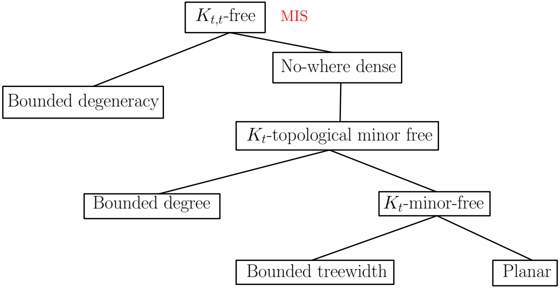

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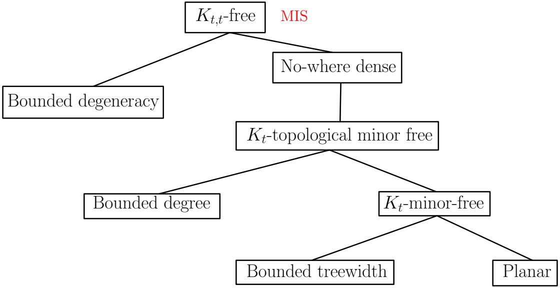

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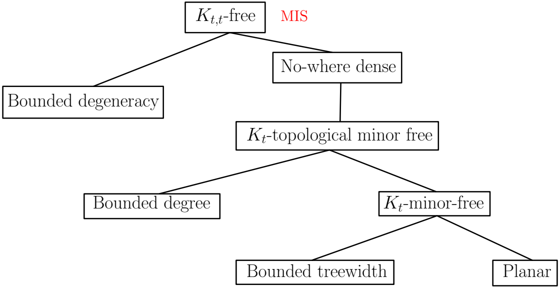

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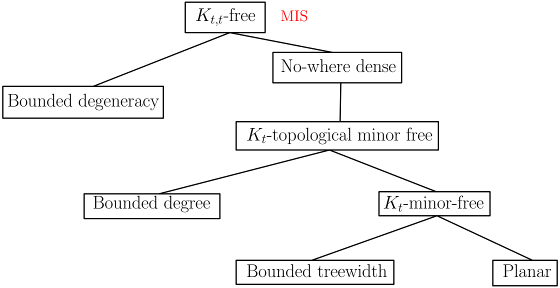

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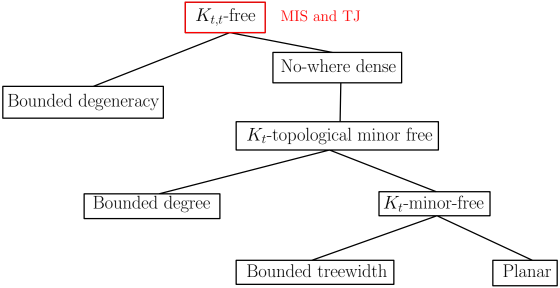

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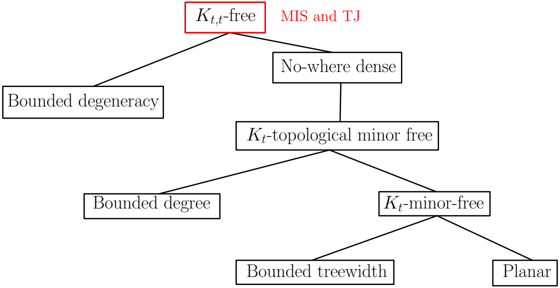

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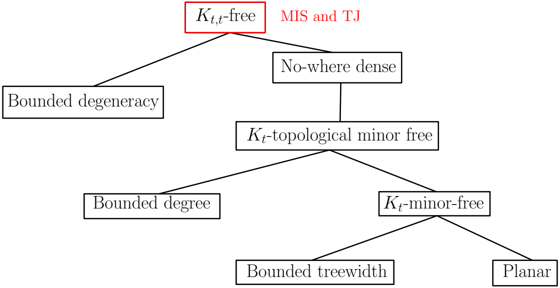

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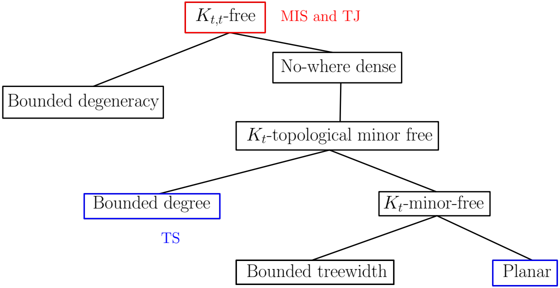

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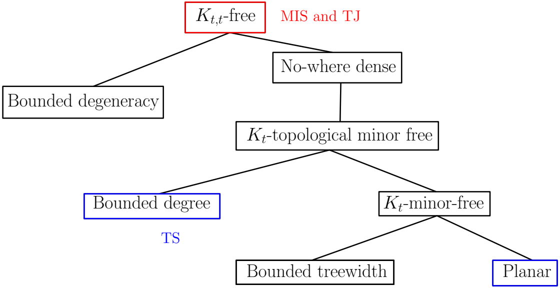

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- Chordal graphs of bounded ω.
- Graphs of girth ≥ 5.

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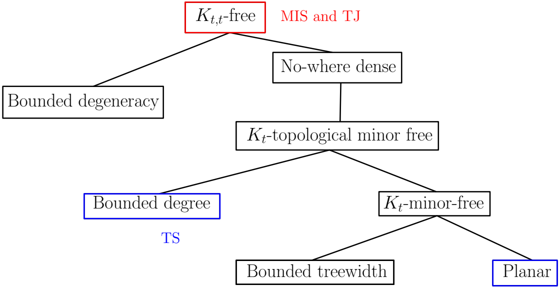

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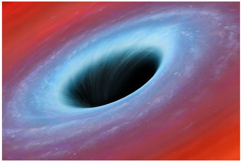
Galactic reconfiguration



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- might contain several tokens,
- might contain tokens even if they have tokens in their neighborhoods.

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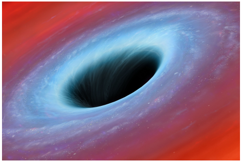

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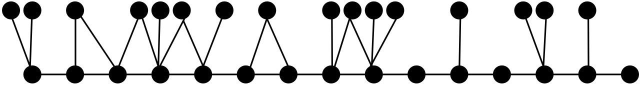
Reduction rule

If G admits a long enough geodesic path P with no token on it nor its neighborhood, then P can be collapsed into a single black hole vertex.



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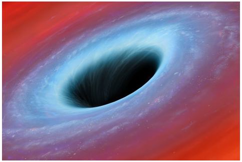

A galactic graph is a graph with special vertices called black holes that :

- might contain several tokens,
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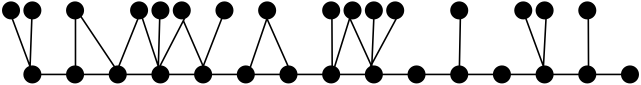
Reduction rule

If G admits a long enough geodesic path P with no token on it nor its neighborhood, then P can be collapsed into a single black hole vertex.



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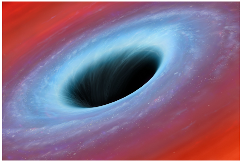

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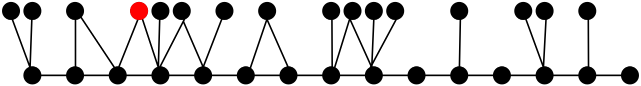

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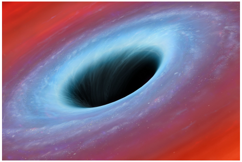

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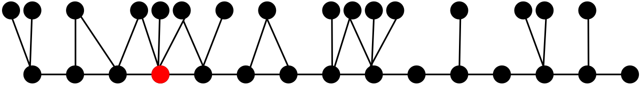

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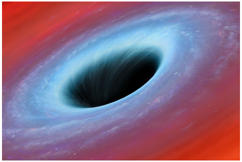

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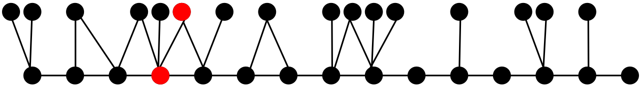

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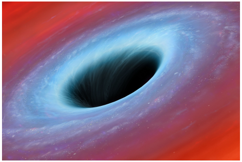

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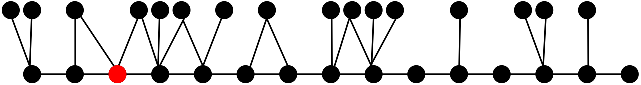

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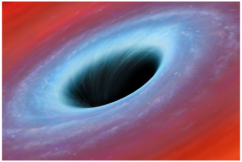

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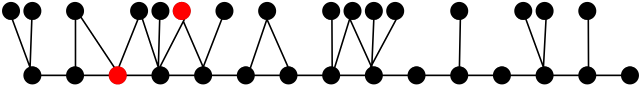

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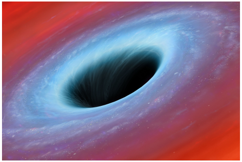

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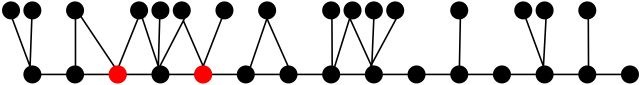

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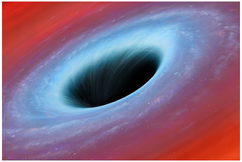

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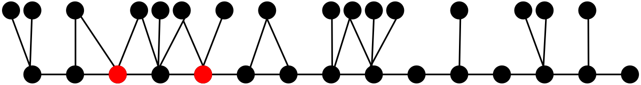
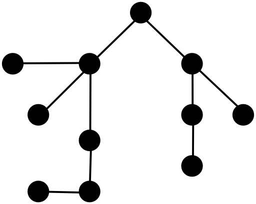


Consequences :

- FPT on bounded degree graphs.
- FPT on planar graphs.

Dominating Set Reconfiguration

A dominating set is a subset X of vertices such that $N[X] = V$.
$\Leftrightarrow$ A set of tokens whose (closed) neighborhood is V.



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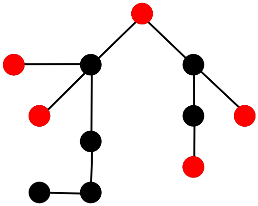
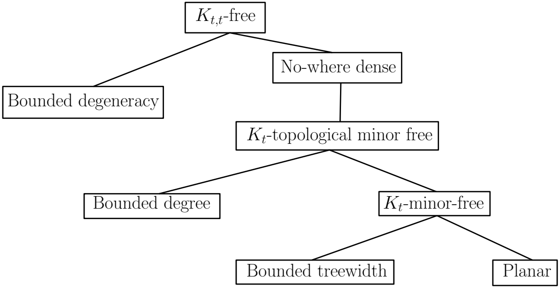
S, T are TS-adjacent (resp. TJ-adjacent) if T can be obtained from S by sliding a token along an edge (resp. jumping a token).



Parameterized results



- [Mouawad et al.'18] TJ-DSR is FPT on nowhere dense graphs.
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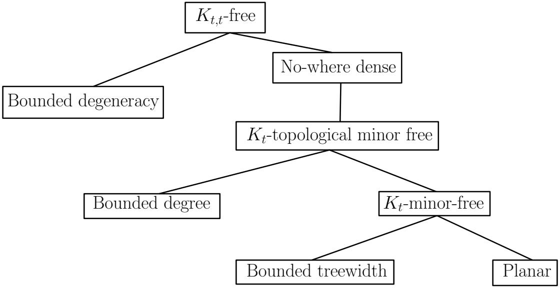


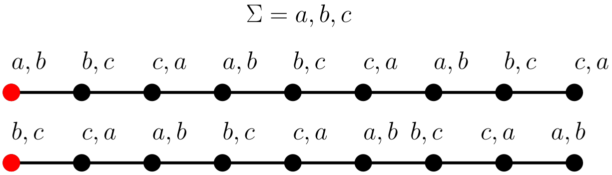
- [Mouawad et al.'18] TJ-DSR is FPT on nowhere dense graphs.
- [BDMMP'24+] TS-DSR is XL-complete on bounded treewidth graphs !

Remark :

- First reconfiguration problem hard on bounded treewidth graphs.
- First TS/TJ difference of behavior on sparse graphs.

Key ingredient - Tape Reconfiguration

- An alphabet Σ
- A collection of tapes where cells are labeled by $\subseteq \Sigma$.
- Lecture heads



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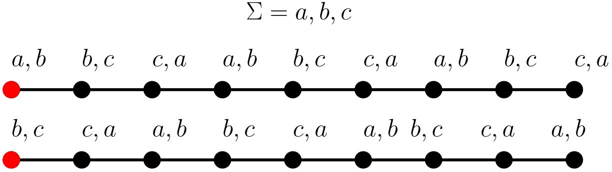

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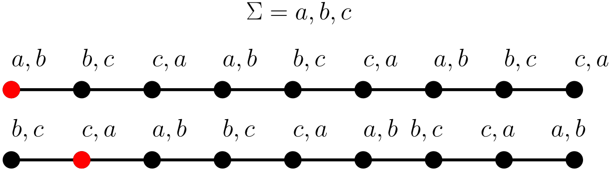

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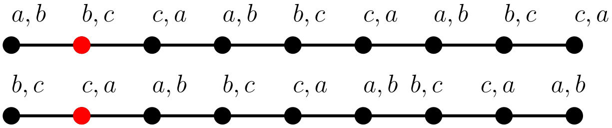

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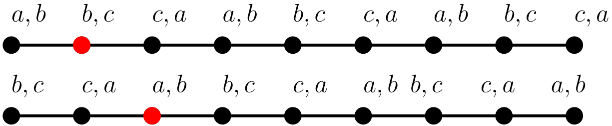

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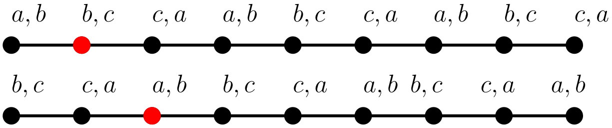
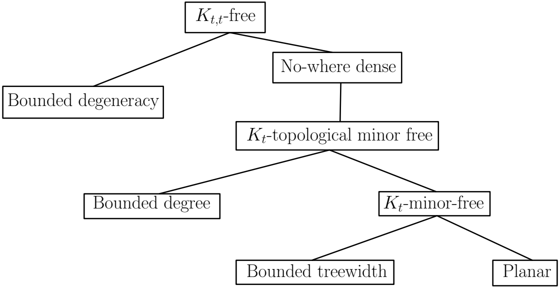


Theorem [BDMMP'24+] :
Tape Reconfiguration is XL-complete even on bounded treewidth instances.

Shortest path reconfiguration



Theorem [BGLM'24]
$W[1]$-hard even on bounded degenerate graphs (for TS and TJ).

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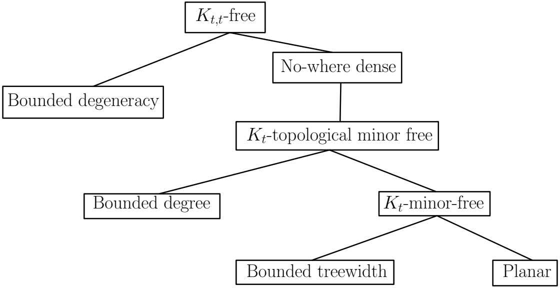

Theorem [BGLM'24]
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Questions :
- What about bounded treewidth ?
- Planar graphs ? (polytime for TJ).

What next ?

- Understand deeper the behavior of TS.
- Are TS locality and shortest path locality similar ?

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Thanks for your attention !