Multicut is FPT

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Joint work with: Jean Daligault, Stéphan Thomassé

FPT

A parameterized problem is *FPT* (Fixed Parameter Tractable) iff there is an algorithm which runs in time $Poly(n) \cdot f(k)$ for an instance of size n and of parameter k.

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- Binary tree of depth k : at most 2^k branches.

Courcelle's Theorem

Theorem (Courcelle)

All the problems definable in the Monadic Second Order Logic parameterized by the treewidth are FPT.

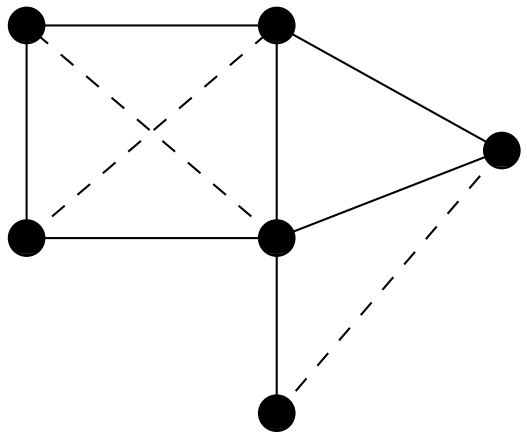
Multicut

Definition

Let $G = (V, E)$ be a graph and R be a set of pairs of vertices called *requests*. A subset E' of E is a Multicut iff for each pair $xy \in R$ there is no path from x to y in $G' = (V, E \backslash E')$.

Definition

A pair of vertices of R is called a *request*.
A vertex which is in a request is called a *terminal*.



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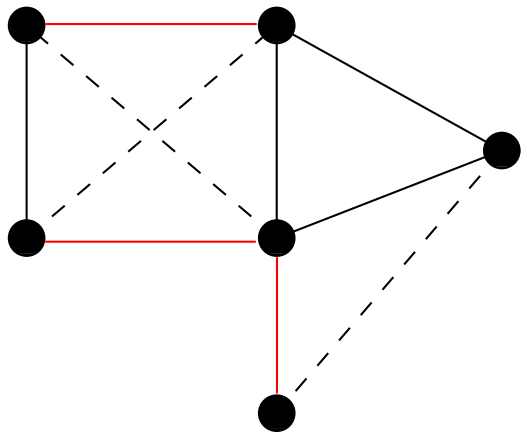

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Output : YES iff there exists a Multicut of (G, R) of size at most k.

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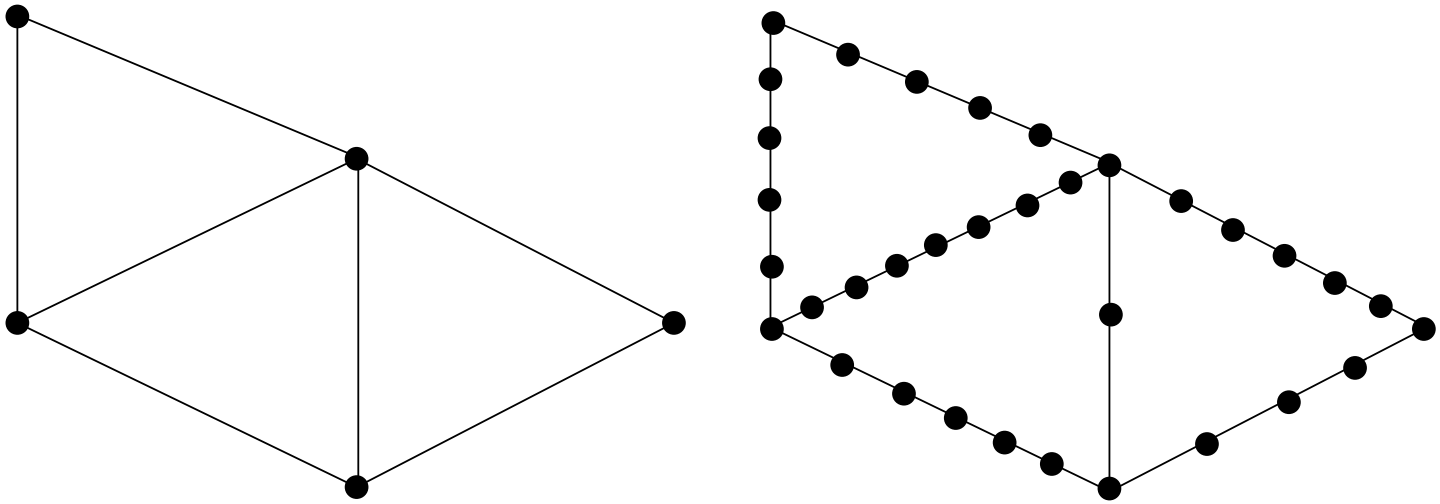
Theorem (B.,Daligault, Thomassé and Marx, Razgon)

Multicut parameterized by the size of the solution is FPT.

Subdivision of a graph



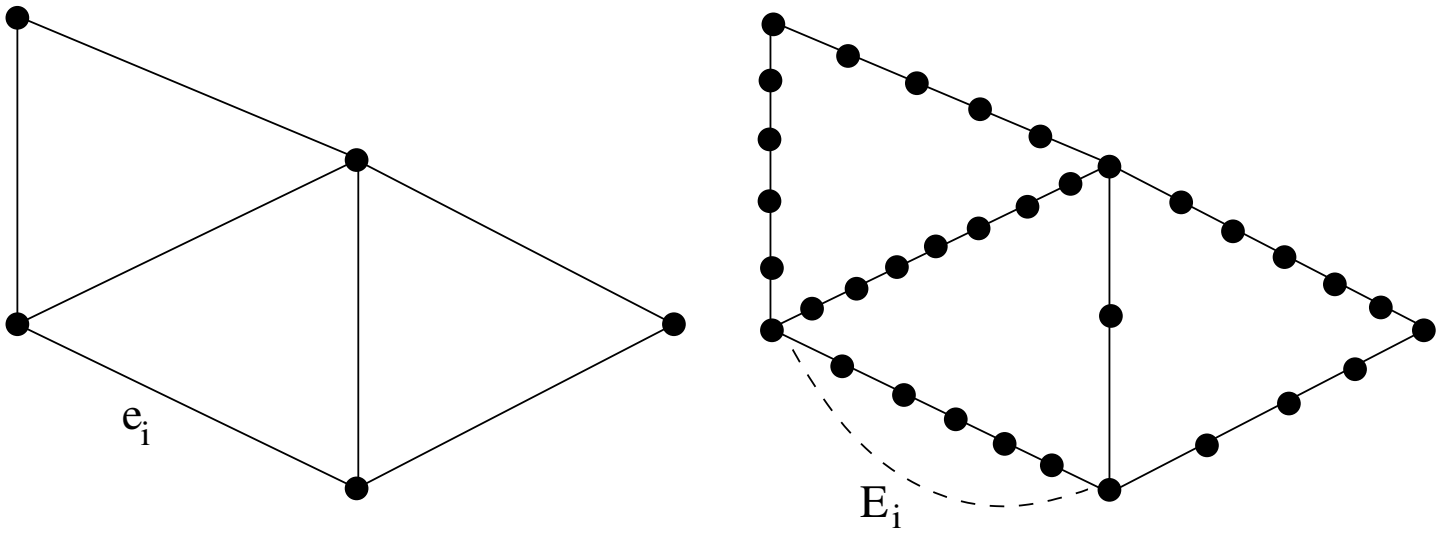
Definition

A graph H is a subdivision of a graph G iff H can be obtained by a subdivision of the edges of G.

Subdivided Multicut

Notation

We denote by E_i the set of edges of H associated to an edge e_i of G.



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We denote by E_i the set of edges of H associated to an edge e_i of G.

Subdivided Multicut

Input : A graph H which is a subdivision of a graph G with k edges. A set of requests with endpoints which do not belong to the same E_i.

Output : YES iff there is a multicut which selects exactly one edge in each E_i.

Subdivided Multicut

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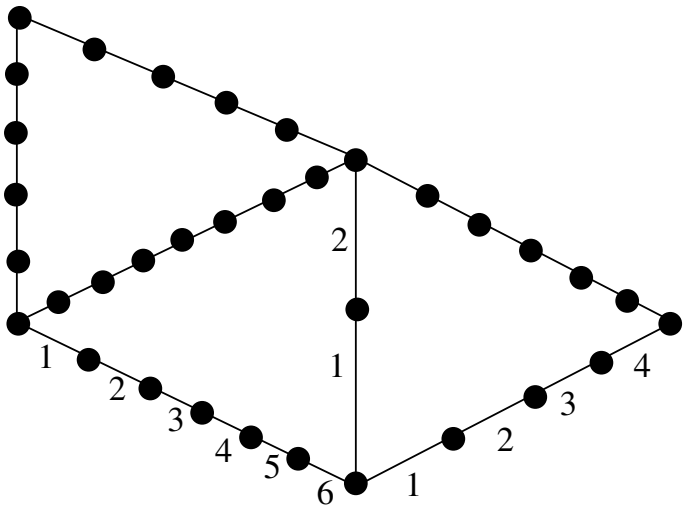
We denote by E_i the set of edges of H associated to an edge e_i of G.

Theorem

Subdivided Multicut can be decided in polynomial time.

Proof



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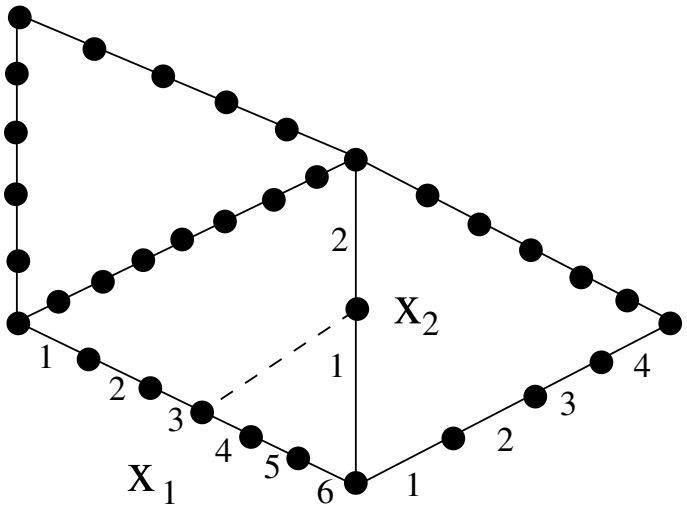

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- Number the edges of each E_i.
- Create a variable x_i for each E_i : the value of x_i refer to the edge selected by the Multicut.

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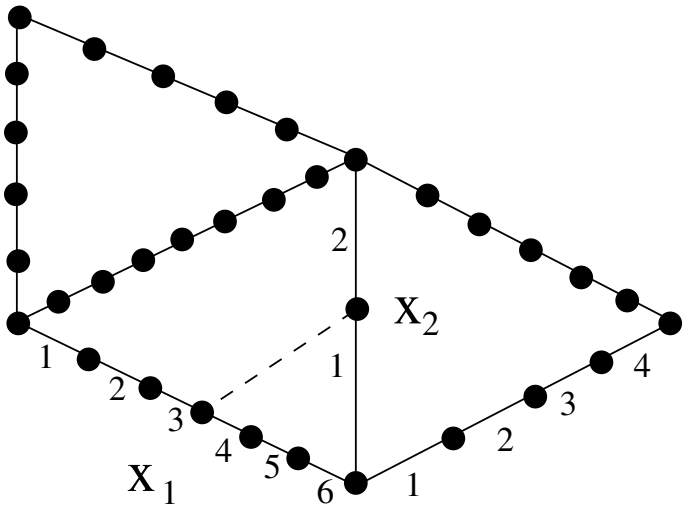

Proof :

- Number the edges of each E_i.
- Create a variable x_i for each E_i : the value of x_i refer to the edge selected by the Multicut.
- Encode the requests with 2-SAT constraints : $x_1 \leq 3 \Rightarrow x_2 \leq 1$.

Tansformation into a boolean instance

Objective

Transformation into a boolean instance.

- Create boolean instances "$x_i \geq k$" for each pair (i, k).

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- Create boolean instances "$x_i \geq k$" for each pair (i, k).
- Encode the requests in the same way : $x_1 \leq 3 \Rightarrow x_2 \leq 1$.
- Encode the constraints of the order :
  - $x_i \geq k \Rightarrow x_i \geq k - 1$.
  - $x_i \leq k \Rightarrow x_i \leq k + 1$.

Iterative Compression

Main idea

Compute a solution of an instance using a solution of a smaller instance.

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Compute a solution of an instance using a solution of a smaller instance.

Theorem

Let P be a problem parameterized by the number of vertices k in the solution. If the problem given a solution of size $k+1$ is FPT, then the problem is FPT.

Proof : By induction on the size of the graph.

- Solve the problem on a graph of size 1 : ok.
- Solve the problem on the graph restricted to $V \backslash v$: the solution on $V \backslash v$ plus v is a solution of size $k+1$.

Iterative compression

Theorem

If Multicut given a Vertex Multicut of size at most $k+1$ can be solved in $f(k) \cdot n^c$ then Multicut can be solved in $f(k) \cdot n^{c+1}$.

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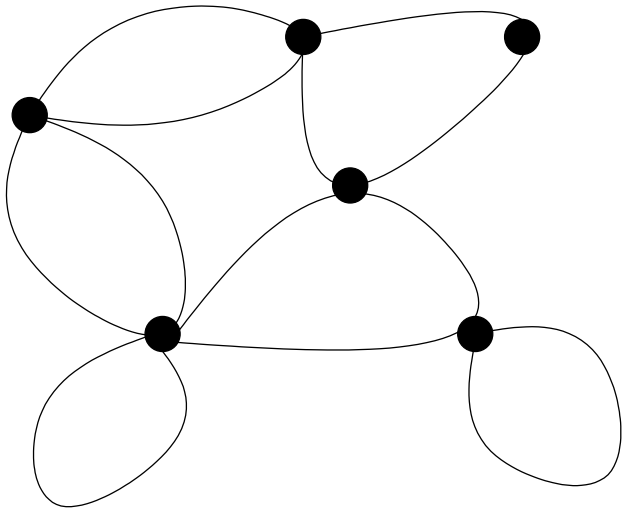
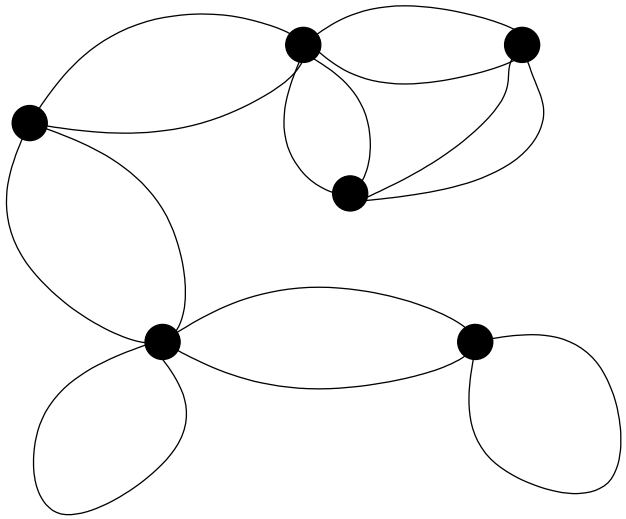
Proof by induction on the size of the graph :

- Solve the problem on the graph exept one vertex.
- One endpoint of each edge and the new vertex is a vertex multicut of size $k+1$.
- Solve the problem Multicut given a vertex multicut of size $k+1$.

Total time : $f(k) \cdot (n-1)^{c+1} + f(k) \cdot n^c \leq f(k) \cdot n^{c+1}$.



- We can assume that the vertices of the Vertex Multicut are separated by the Multicut.



- We can assume that the vertices of the Vertex Multicut are separated by the Multicut.
- We can assume that the components have one or two attachment vertices.

Left cuts

Let G be a connected graph and x be a vertex called root.

Definition

A *cut* S is a subset of vertices containing x.
The *border* Δ of a cut S is the set of edges with one endpoint in S.
We denote by δ its size.

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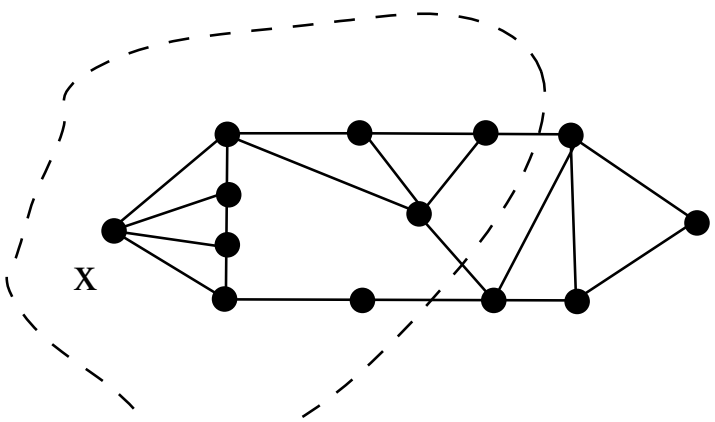

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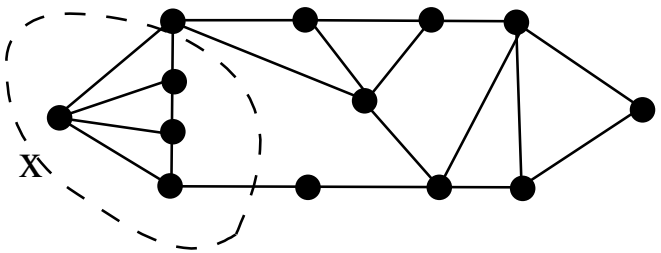

Indivisible left cuts

Definition

A cut is *indivisible* S iff $G \backslash S$ is connected.

Theorem

Let y be a vertex. There is a bounded number (in k) of indivisible left cuts of size at most k which separate x from y.

Active sets

Active sets

An active set $\mathcal{L}$ of edges a set such that if there is a solution of the Multicut problem, there is a solution which use only edges of $\mathcal{L}$.

Theorem

Let C be a component attached on x. There is an active set $\mathcal{L}(C)$ which have a bounded size.

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Theorem

Let $C_1,...,C_p$ be p disjoint components attached on x. Let $U_i = \bigcup_{k=1}^{i} C_i$. Then each U_i has a bounded active set $\mathcal{L}_i$ and $\mathcal{L}_j \cap U_i \subseteq \mathcal{L}_i$ if $i \leq j$.

Backbone

Let C be a xy connected component. A backbone is a path from x to y in C in which only one edge of the Multicut is deleted.

We can assume that :

- Each component has a backbone.
- At most $2\lambda - 1$ edges are deleted (where λ is the xy-connectivity).
- Each vertex of the backbone is a vertex cutset.

Dilworth's Theorem

Partial Order

A *partial order* is an acyclic transitive oriented graph.
A *chain* is a total order.

Antichain

An antichain of a partial order P is a subset of P with pairwise incomparable elements.

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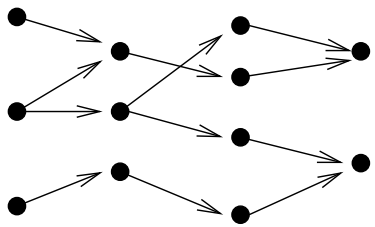

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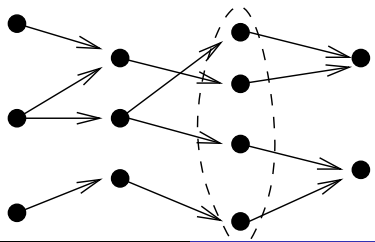
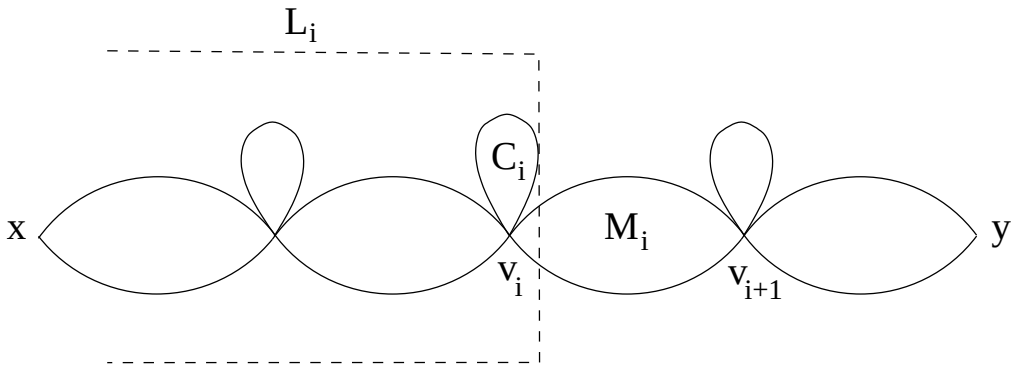
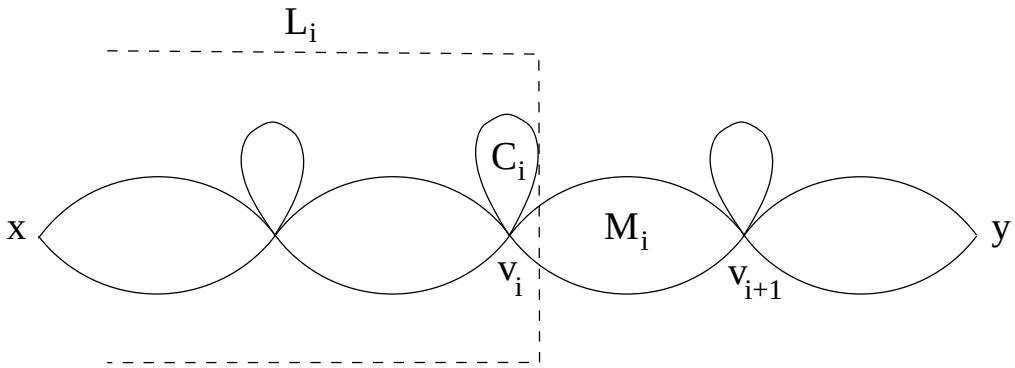




Our goal

We want to reduce to the polynomial instance.



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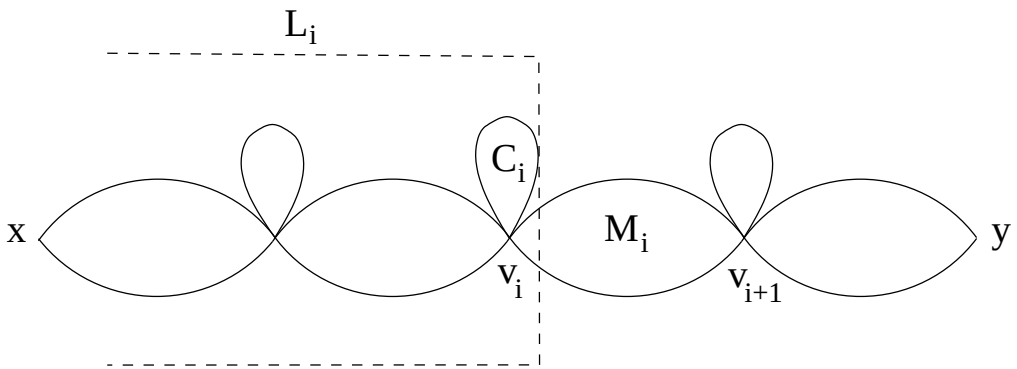
We can assume that there is no components C_i.

Application of Dilworth's Theorem



If the choosen edge in the backbone is $v_i v_{i+1}$, then :

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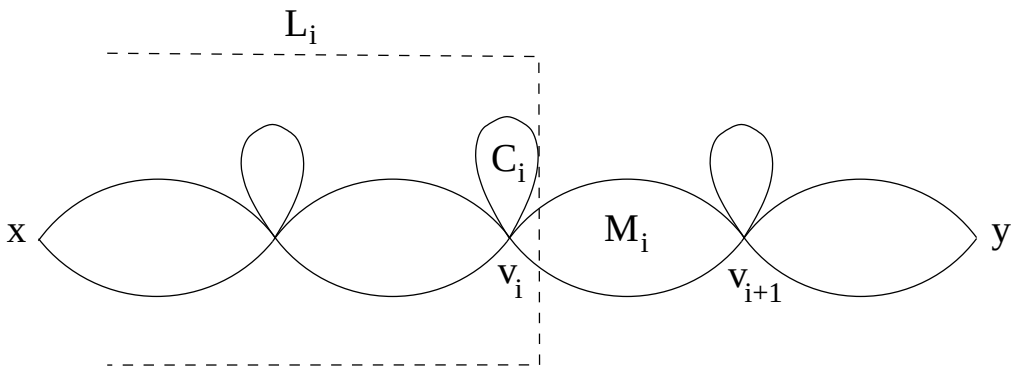

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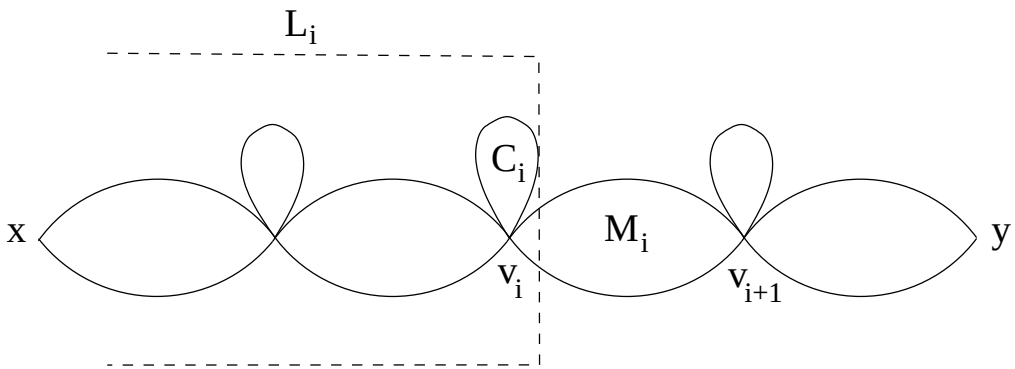

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The sets $\mathcal{L}_i$ satisfy $\mathcal{L}_j \cap L_i \subseteq \mathcal{L}_i$.

The order $\preceq$

Let $F_i \subseteq \mathcal{L}_i$ and $F_j \subseteq \mathcal{L}_j$ with $j \geq i$.
$F_i \preceq F_j$ iff $F_j \cap L_{i+1} \subseteq F_i$.

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Theorem

The order $\preceq$ can be covered by a bounded number of chains.

Application of Dilworth's Theorem

Proof : Let us prove by induction on k that the maximum size of an antichain is bounded.

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Assume that the antichain for cuts of size $k+1$ is unbounded. Let F_i be the elements of such an antichain. Enumerate the $F_i \subseteq \mathcal{L}_{t_i}$ of the antichain such that if $i < j$ then $t_i \leq t_j$.

- Since F_1 is not comparable with F_i, $F_i \cap \mathcal{L}_{t_1+1} \subsetneq F_1$.

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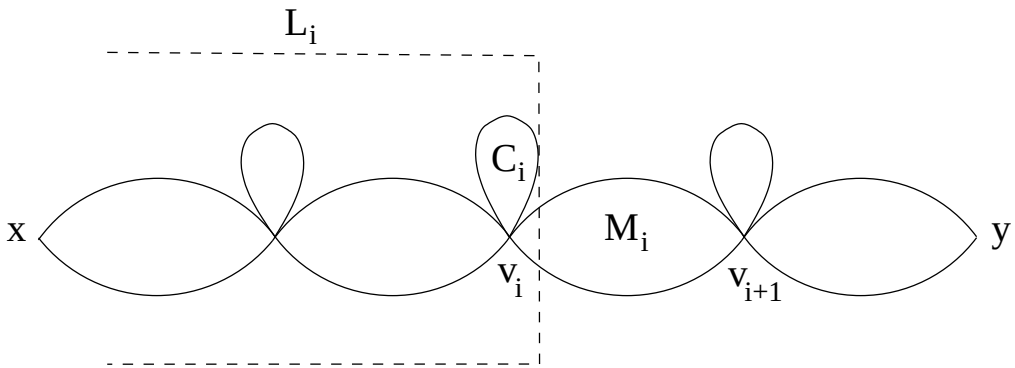
- Hence F_i has a vertex in $\mathcal{L}_{t_1+1}$.
- Since $\mathcal{L}_{t_1+1}$ has a bounded size, an arbitrarily large number of F_i share the same vertex.
- Apply the induction hypothesis on this set (the sets have size k since they share an edge).

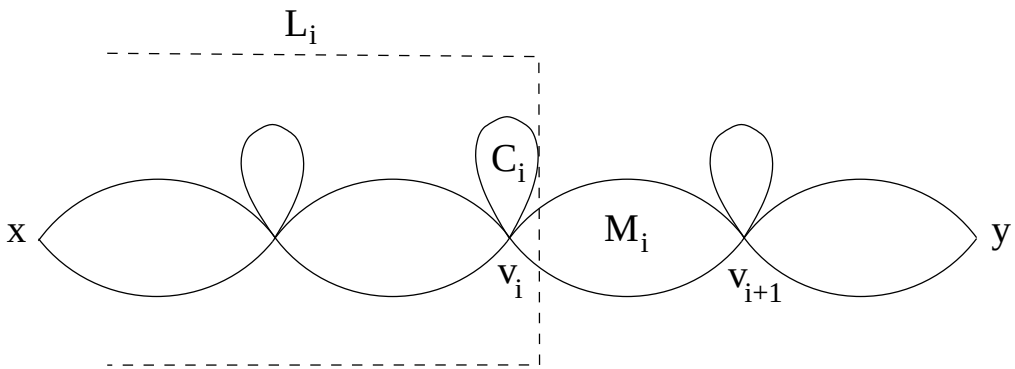
Projection of the requests



Consider a chain of P and a request from $y \in C_k$ passing through x then :

- When $i < k$, the request is cut.
- When $i \geq k$, the request is cut iff it is cut by $F_i \cap \mathcal{L}_k$.
- When the request becomes uncut, it is still uncut when i increases : indeed if $j > i$, $F_j \cap \mathcal{L}_i \subseteq F_i$.

Projection of the requests



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If the request is cut before v_i and uncut after v_{i+1}, the request can be projected on v_{i+1}.

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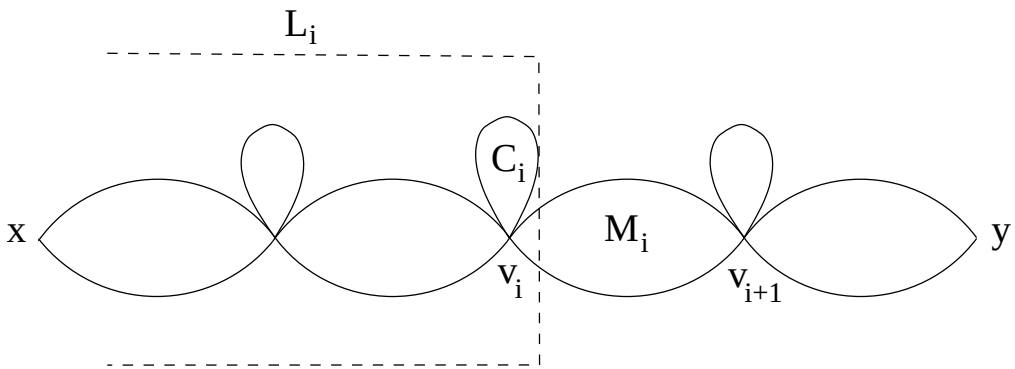

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Theorem

We can assume that no component is attached on the vertices v_i.

Comparison with Marx and Razgon's proof

Our positive points :

- We have a branching algorithm.
- Our proof is self-contained.

Their positive points :

- Their complexity is better : $O(2^{k^3})$.
- Their proof is written for Vertex-Multicut.

Multiflow

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Output :YES iff there are k edge-disjoint paths between pairs of vertices of R.

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Open problem

Is the Multiflow problem FPT parameterized by the size of the solution ?

Polynomial kernels

Definition

A problem parameterized by k has a polynomial kernel iff there is an algorithm running in polynomial time which transforms an instance (n, k) into an instance (n', k') such that :

- $n' \leq Poly(k)$ and $k' \leq k$.
- The new instance is positive iff the original instance is positive.

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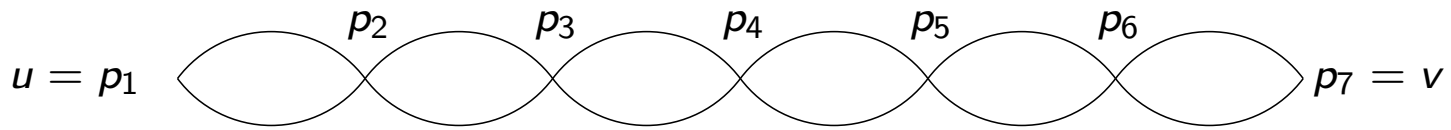
Does the Multicut problem have a polynomial Kernel ?

Another approach for Multicut

Open problem

Is it possible to encode a request by a bounded (in k) number of paths and to compute them in FPT-time ?



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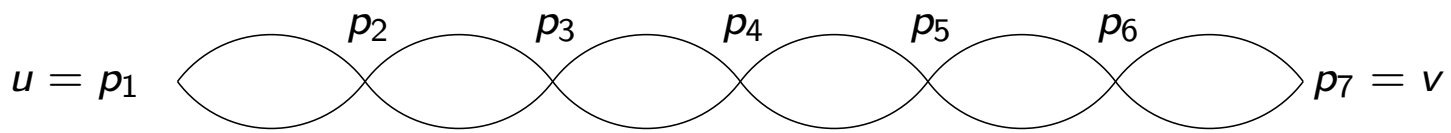

Hitting path problem

Input : A graph G, a set of paths $\mathcal{P}$, an integer k.
Output : A set of k edges which intersects all the paths.

Open problem

Is Hitting path parameterized by the size of the solution FPT ?

Thanks for your attention

Questions ?