Multicut is FPT

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1. Introduction
   - Parameterized complexity
   - Multicut
2. An FPT algorithm
   - Reducing the instance
   - Left cuts
   - Two different proofs
3. Conclusion

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A problem parameterized by k is *FPT* (Fixed Parameter Tractable) iff there is an algorithm which runs in time $Poly(n) \cdot f(k)$ for an instance of size n and of parameter k.

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Theorem (Courcelle)

All the problems definable in the Monadic Second Order Logic parameterized by the treewidth are FPT.

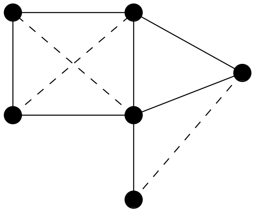
Multicut

Definition

Let $G = (V, E)$ be a graph and R be a set of pairs of vertices. A subset E' of E is a Multicut iff for each pair $xy \in R$ there is no path from x to y in $G' = (V, E \backslash E')$.

Definition

A pair of vertices of R is called a *request*.
A vertex which is in a request is called a *terminal*.



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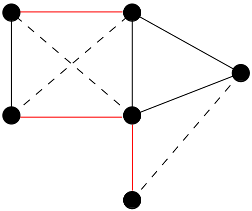

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Input : A graph G, a set of requests R, an integer k.
Output : YES iff there exists a Multicut of (G, R) of size at most k.

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Theorem (Gupta '03)

There is an algorithm running in polynomial time giving a solution of the Multicut problem of size at most OPT^2.

Approximation in FPT time

Theorem (Marx, Razgon '09)

There is an algorithm runnung in FPT-time which gives a solution of the Multicut problem of size at most $2 \cdot k$ or answer NO.

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Theorem (B.,Daligault, Thomassé and Marx, Razgon '11)

Multicut parameterized by the size of the solution is FPT.

Vertex Multicut

Theorem

If we can solve the Multicut Problem given a Vertex Multicut of size k^2 in FPT time, then we can solve the general problem in FPT time.

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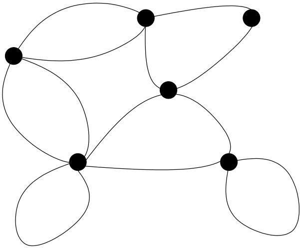
Proof :

Theorem (Gupta '03)

There is an algorithm running in polynomial time giving a solution of the Multicut problem of size at most OPT^2.

- If the solution given by Gupta is strictly greater than k^2, answer "No".
- Otherwise we have an edge Multicut of size at most k^2. Taking one endpoint of each edge gives a Vertex Multicut.

Some reductions



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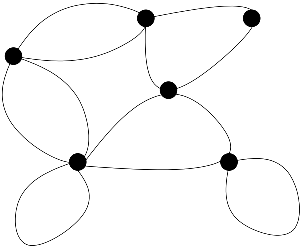

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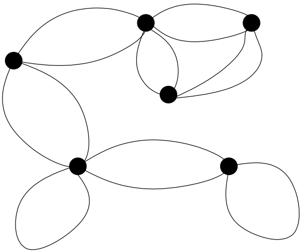

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- We can assume that the vertices of the Vertex Multicut are separated by the Multicut.
- We can assume that the components have one or two attachment vertices.

Left cuts

Let G be a connected graph and x be a vertex called root.

Definition

A *cut* S is a subset of vertices containing x.
The *border* Δ of a cut S is the set of edges with one endpoint in S.
We denote by δ its size.

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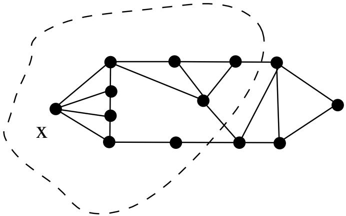
A *left cut* S such that if $T \subsetneq S$ then $\delta(T) > \delta(S)$.



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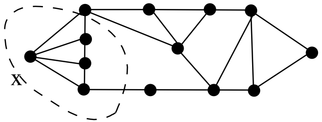

Indivisible left cuts

Definition

A cut is *indivisible* S iff $G \backslash S$ is connected.

Theorem

Let y be a vertex. There is a bounded number (in k) of indivisible left cuts of size at most k which separate x from y.

B.,Daligault, Thomassé's proof

Theorem

In a component with one attachment vertex, we can bound (in k) the number of terminal vertices.

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- The best way to use k edges of the Multicuy in a component is to select a left cut for a terminal vertex.
- For each vertex y, we have seen seen that there is a bounded number (in k) of indivisible left cuts of size at most k which separate x from y.
- In the solution, a union of indivisible left cut will be selected.

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Theorem

In a component with one attachment vertex, there is a bounded number (in k) of cuts such that if there is a solution of the Multicut problem, there is a solution using one of these cuts.

B., Daligault, Thomassé's proof

- Branch to choose a cut of $\mathcal{L}$ in each component with one attachment vertex.
- Branch (in FPT-time) to reduce the problem using the same kind of reductions for components with 2-attachment vertices.
- Solve the problem on the subdivision of a graph of size k.

Marx, Razgon's proof

Almost 2SAT problem

Input : n variables, a set of clauses of with two literals in each, an integer k.
Parameter : k
Output : YES iff there is an assignation of the variables which satisfy all exept at most k clauses.

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Theorem (Razgon, O'Sullivan '09)

Almost 2SAT is Fixed parameter tractable.

Marx, Razgon's proof

Isolated part

Given a solution E', the isolated part is empty iff each connected component of $G(E \backslash E')$ contains a vertex of the Vertex Multicut.

Marx, Razgon's proof

Isolated part

Given a solution E', the isolated part is empty iff each connected component of $G(E \backslash E')$ contains a vertex of the Vertex Multicut.

- When the isolated part is empty, the problem can be reduced to Almost 2SAT.
- The use of random sets with a strange distribution permits to reduce to the previous case.
- This procedure can be derandomized.

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Polynomial kernels

Definition

A problem parameterized by k has a polynomial kernel iff there is an algorithm running in polynomial time which transforms an instance (n, k) into an instance (n', k') such that :

- $n' \leq Poly(k)$ and $k' \leq k$.
- The new instance is positive iff the original instance is positive.

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Theorem (B.,Daligault, Thomassé, Yeo '09)

Multicut in trees parameterized by the size of the solution has a kernel of size $O(k^6)$.

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Open problem

Does the Multicut problem have a polynomial Kernel ?

Thanks for your attention

Questions ?