Independent Set Reconfiguration
Which price for locality?

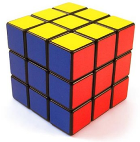
Nicolas Bousquet

joint works with
Valentin Bartier, Marthe Bonamy, Clément Dallard, Kyle Lomer, Amer Mouawad

CoRE 2021 - July 2021

Reconfiguration

A one-player game is a puzzle : one player makes a series of moves, trying to accomplish some goal.



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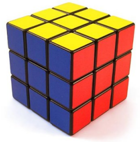

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- Applications to random sampling, bioinformatics, discrete geometry...etc...

Main questions

- **Reachability problem.** Given two configurations, is it possible to transform the one into the other ?
- **Connectivity problem.** Given any pair of configurations, is it possible to transform the one into the other ?
- **Minimization.** Given two configurations, what is the length of a shortest sequence ?

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Today : Reachability + Independent Set Reconfiguration.

Genesis of Independent Set Reconfiguration

- [Hopcroft, Schwartz, Sharir '83] Warehouseman's problem - Motion of rectangular robots in a grid.
  $\Rightarrow$ PSPACE-complete (but they need large robots).

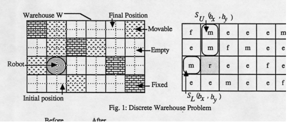

Fig. 1: Discrete Warehouse Problem

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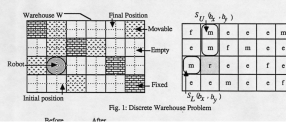

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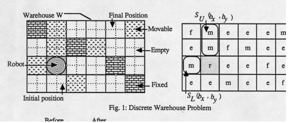

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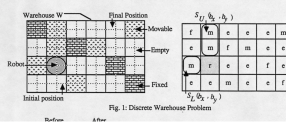

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> **Theorem** (Hearn, Demaine '05)
>
> Warehouseman's problem is PSPACE-complete for dominos shaped-robots.

What about other applications ?

For puzzles :

$\rightarrow$ The transformation is local.

What about other applications ?

For puzzles :

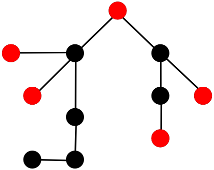
→ The transformation is local.

Other applications :

- Random sampling.
- Enumeration.
- Centralized distribution of tokens.

→ No locality needed.

Token Jumping vs Token Sliding



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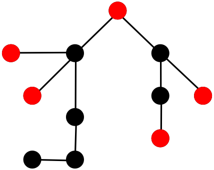

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Select one vertex of I and move it anywhere else.
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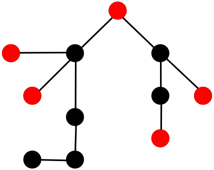

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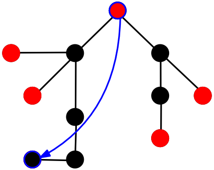

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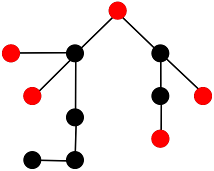

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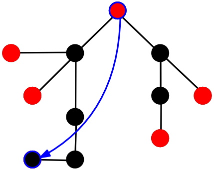

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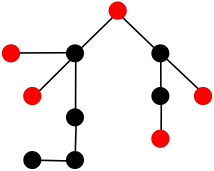

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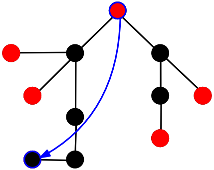

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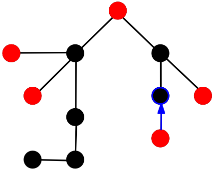

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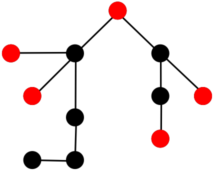

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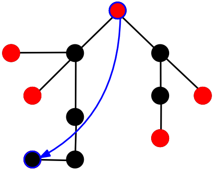

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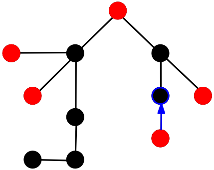

Question : What is the complexity of TS / TJ-REACHABILITY ?

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Input : A graph G, two independent sets I, J.

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TS/TJ-REACHABILITY is PSPACE-complete.

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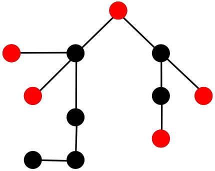

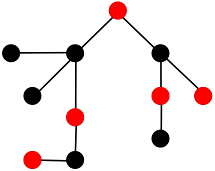

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TJ-Reachability is polynomial on trees.

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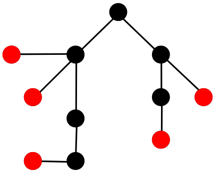

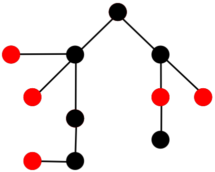

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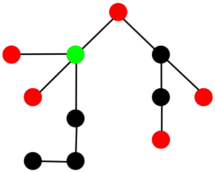

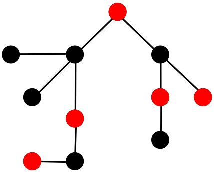

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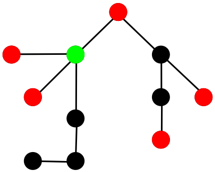

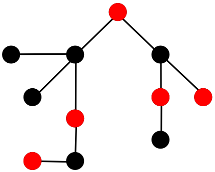

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- YES iff same number of vertices in each component of $T \setminus X$.

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- [Belmonte et al. '19] PSPACE-complete for split graphs.
  (split graph = $V = V_1 \cup V_2$ where V_1 induces a clique and V_2 a stable set)
- [Bonamy, B. '18] Polynomial for interval graphs.

Bounded treewidth graphs (and below)

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TS and TJ-Reachability are PSPACE-complete on bounded bandwidth graphs.

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- Graphs of treewidth at most 2 ?

  Outerplanar ? Series-parallel graphs ?

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[Bodlaender, Groenland, Swennenhuis '21+] XL-complete

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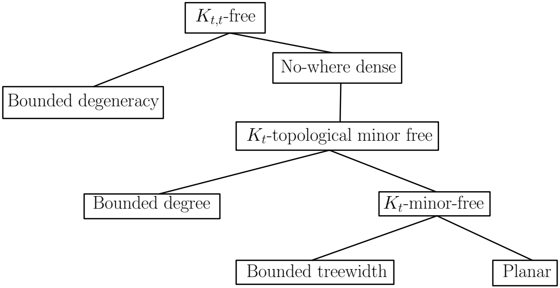

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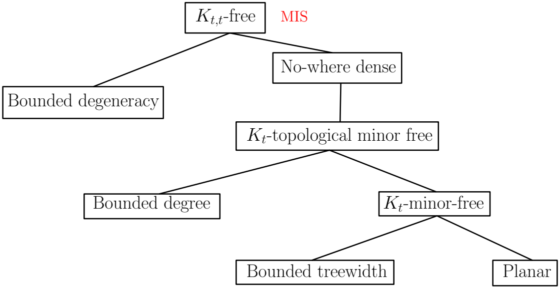

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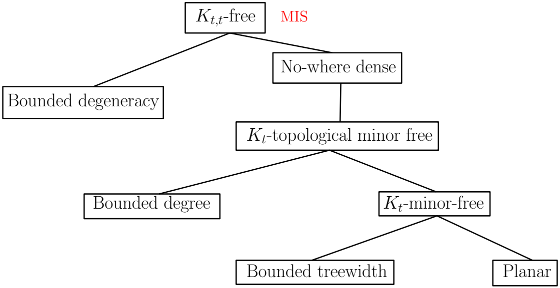

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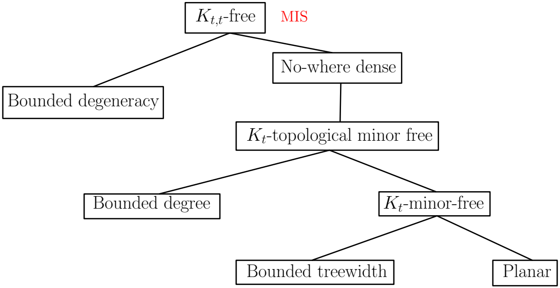

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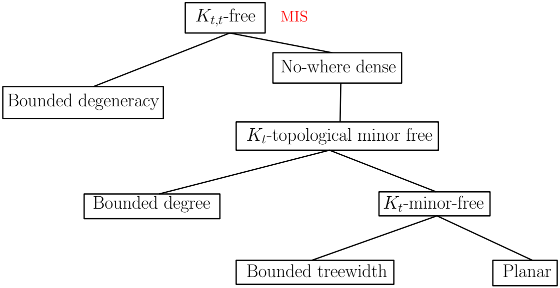

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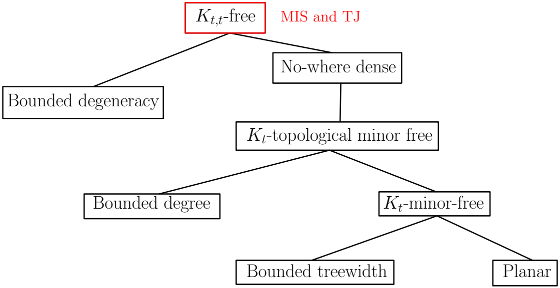

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- [B., Mary, Parreau '18] $K_{t,t}$-free graphs.

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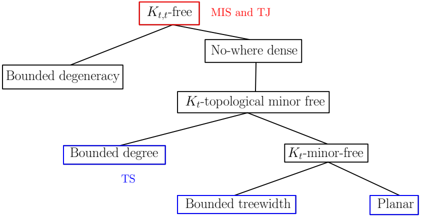

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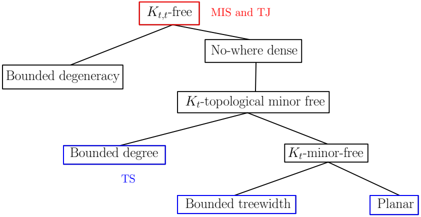
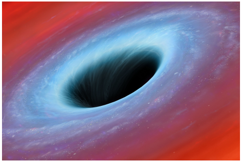


Galactic reconfiguration



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- might contain several tokens,
- might contain tokens while there are tokens in their neighborhood.

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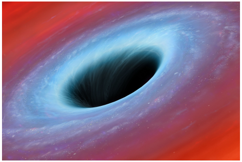

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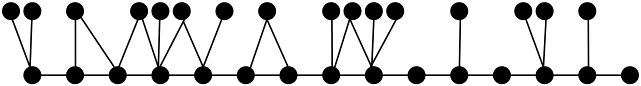
Reduction rule

If G admits a long enough geodesic path P with no token on it nor its neighborhood, then P can be collapsed into a single black hole vertex.



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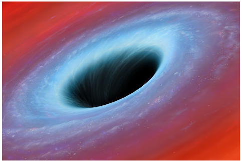

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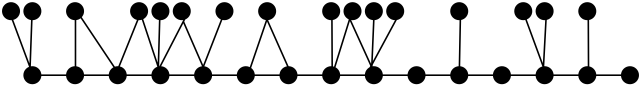

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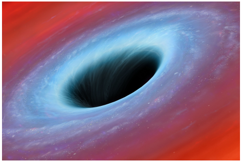

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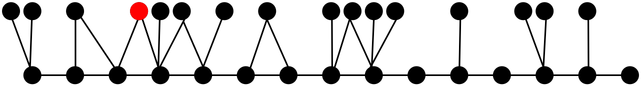

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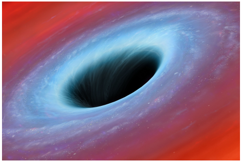

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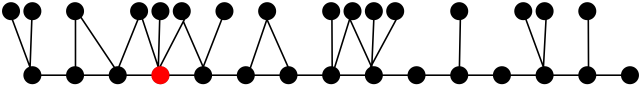
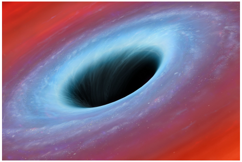


Galactic reconfiguration



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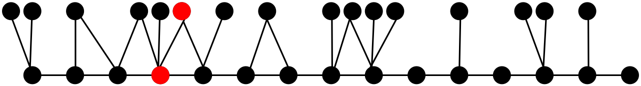

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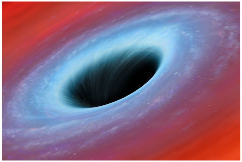

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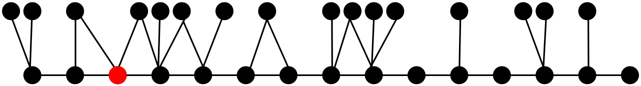

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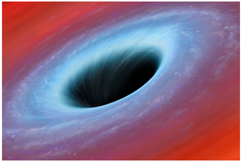

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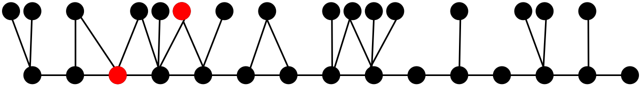

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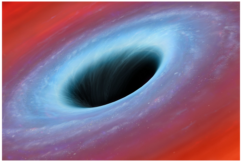

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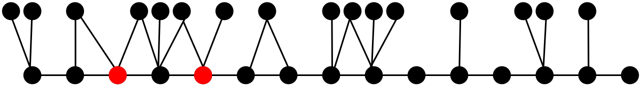

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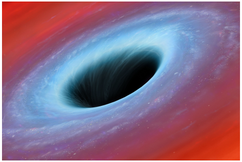

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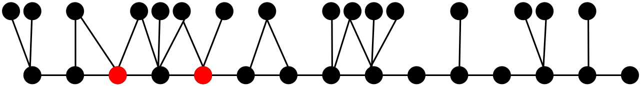
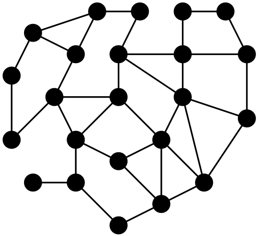


Consequences :

- FPT on bounded degree graphs.
- FPT on planar graphs.

Even more local ?



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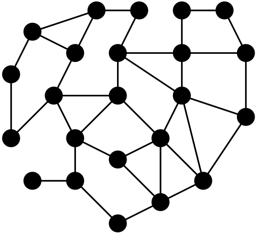

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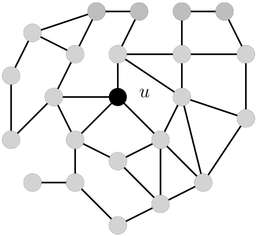

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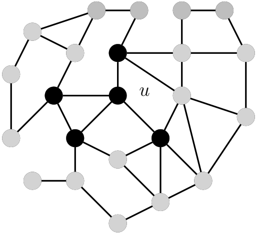

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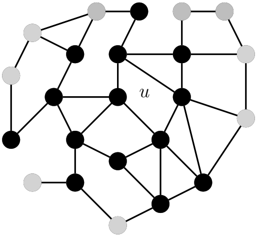

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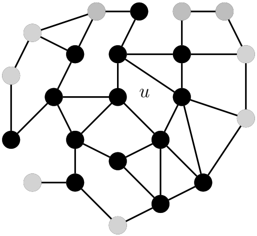

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What is the complexity ? Number of rounds.

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$\rightarrow$ Distributed Reconfiguration of Maximal Independent Sets, Censor-Hillel and Rabie (2019).

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Thanks for your attention !