Reconfiguration:
From statistical physics to graph theory

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Joint works with Marthe Bonamy, Carl Feghali,
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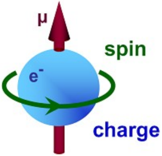
Journées Structures Discrètes
ENS Lyon

Spin systems



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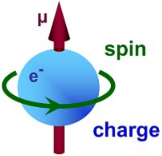

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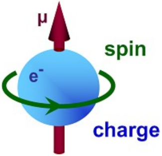

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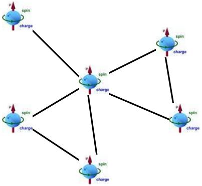
A spin configuration is a function $\sigma : S \to \{1, \ldots, k\}$.



- Interactions between spins in S are modelized via an interaction matrix.
- If coefficients are in $0 - 1$: representation with a graph :
  - $0 =$ no interaction $=$ no link.
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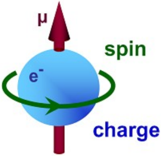

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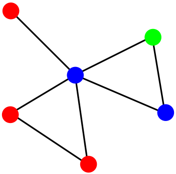

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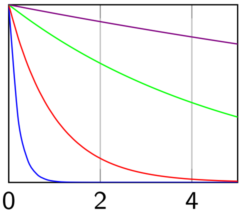
Spin configuration ⇒ (non necessarily proper) graph coloring.



$T = 5, 1, 0.2, 0.05$

Antiferromagnetic Potts model

$H(\sigma)$: number of monochromatic edges.

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Edges with both endpoints of the same color.

Gibbs measure at fixed temperature T :

$$\nu_T(\sigma) = e^{-\frac{H(\sigma)}{T}}$$

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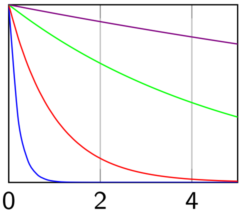

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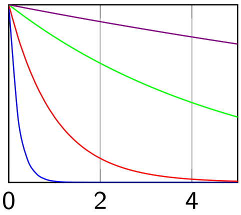
- Free to rescale, ν_T = probability distribution $\mathbb{P}$ on the colorings.



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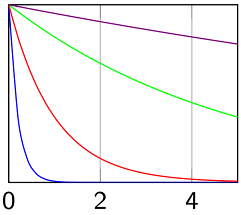

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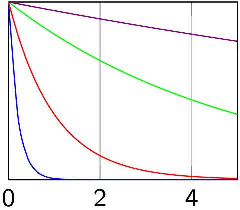

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Definition (Glauber dynamics)

Limit of a k-state Potts model when $T \to 0$.
$\Leftrightarrow$ Only **proper** colorings have positive measure.

Sampling spin configurations

In the statistical physics community, the following Monte Carlo Markov chain was proposed to sample a configuration :

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Questions :

- Can we generate any solution ?
- How much time do we need to "sample a solution almost at random" ?

Reconfiguration graph

Definition (k-Reconfiguration graph $\mathcal{C}_k(G)$ of G)

- Vertices : Proper k-colorings of G.
- Create an edge between any two k-colorings which differ on exactly one vertex.

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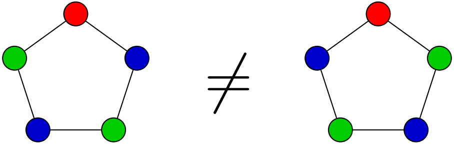

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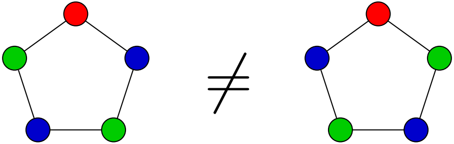

Remark 2.
All the k-colorings can be generated
$\Leftrightarrow$ The k-reconfiguration graph is connected.

Convergence of Markov chains

A Markov chain is irreducible if any solution can be reached from any other. $\Leftrightarrow$ The reconfiguration graph is connected.

A chain is aperiodic if there exists t_0 such that $Pr(X_t = a)$ is positive for every $t > t_0$ and every state a.

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Mixing time and Reconfiguration graph ?

- Diameter of the Reconfiguration graph = D
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- Better lower bounds ? Look at the connectivity of the reconfiguration graph.

Main question in Statistical Physics

How many colors (in terms of the maximum degree Δ) do we need to ensure that the chain is rapidly mixing ?

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Conjecture

If $c \geq \Delta + 2$, the graph is $c(\Delta)$-mixing in time $\mathcal{O}(n \log n)$.

Path coupling

Informal definition

Two Markov chains (X_t, Y_t) are coupled if :

- X_t without knowning $Y_t = Y_t$ without knowing X_t.
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Theorem

If there exists a coupling defined **only** every X_t, Y_t that only differ on one vertex such that

$$\mathbb{E}(d(X_{t+1}, Y_{t+1})) < (1 - \frac{1}{n})$$

then the mixing time is $\mathcal{O}(n \log n)$.

$d(X, Y)$= Hamming distance
= number of vertices on which they differ

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Coupling :

If vertex u and color c are chosen in X_t, we make the same choice in Y_t.

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v = Unique vertex on which X_t and Y_t differ.

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Relation with enumeration

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An algorithm is polynomial delay if it enumerates all the solutions and the delay between two solutions is polynomial in n.

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Remark :

The algorithm might need an exponential space !

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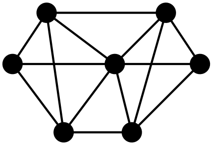
  What is the diameter of the reconfiguration graph ?
- Can we effiently find a short transformation (from an algorithmic point of view) ?
  Can we find a path between two vertices of the reconfiguration graph in polynomial time ?

Main question in CS

Conjecture (Cereceda)

The $(k+2)$-recoloring diameter of any k-degenerate graph is $\mathcal{O}(n^2)$.

A graph is k-degenerate if there exists an order $v_1, \ldots, v_n$ such that for every i, v_i has at most k neighbors after it in the order.



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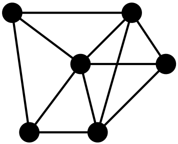

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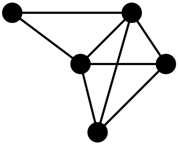

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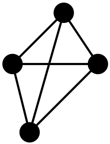

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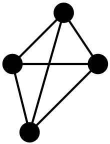

Theorem (Cereceda '07)

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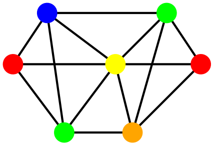

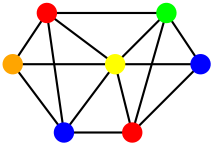

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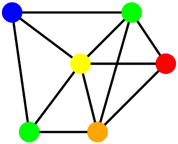

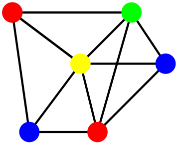

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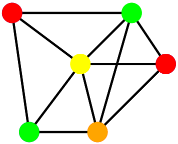

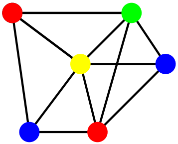

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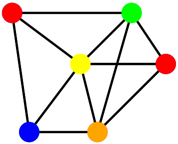

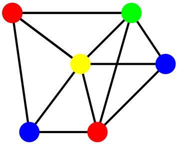

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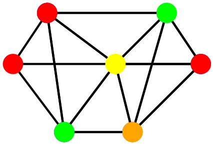

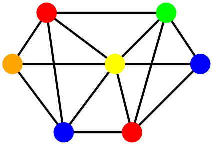

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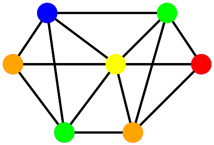

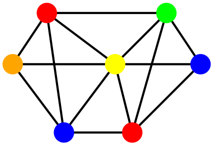

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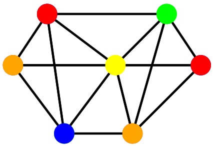

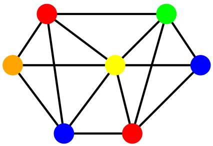

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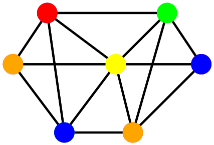

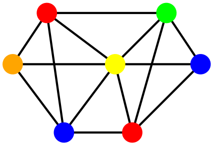

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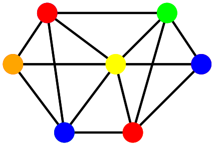
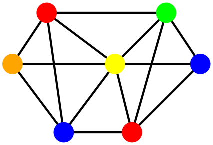
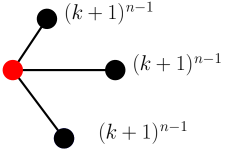




Number of recolorings?

When do we need to recolor the leftmost vertex?

- Each time a neighbor is recolored.



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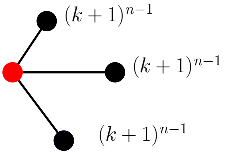

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- In the last round : $+1$ recoloring.

The total number of recolorings is at most $(k+1)^n$.

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Conjecture (Cereceda)

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For other values of k ?

d-degenerate

$d + 1$

Chordal graphs

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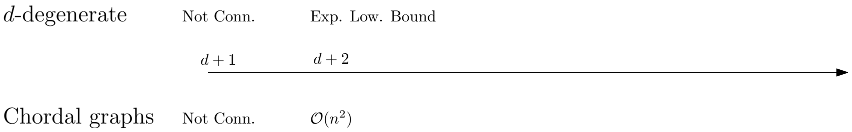

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	$d+1$	$d+2$	$2d+2$
d-degenerate	Not Conn.	Exp. Low. Bound	$\mathcal{O}(dn)$
Chordal graphs	Not Conn.	$\mathcal{O}(n^2)$	$\mathcal{O}(dn)$

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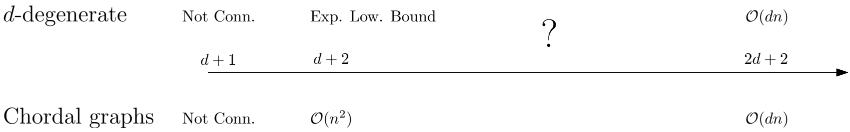

Lower bound

Theorem (Bonamy et al.)

The 3-recoloring diameter of the path P_n is $\Omega(n^2)$.

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Sketch of the proof

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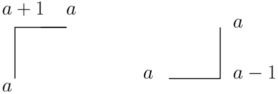

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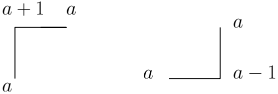
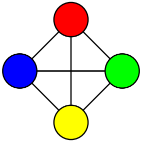


$\Rightarrow$ The surface is only modified by "one" at each step.

Claim : $\Omega(n^2)$ steps are needed to transform 123....123 into 132....132.

Going below $(\Delta + 2)$ colors

Impossible to sample - count $(\Delta + 1)$-colorings ?



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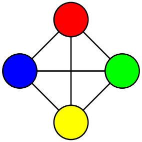

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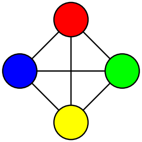

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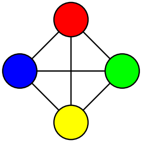

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- [Bonamy, B., Perarnau] The number of frozen $(\Delta + 1)$-colorings is exponentially smaller than the number of $(\Delta + 1)$-colorings (when Δ is small enough).

What does “close” mean ?

At the beginning of the talk (a long time ago...).

> *“At each step, change the color of a single vertex”*

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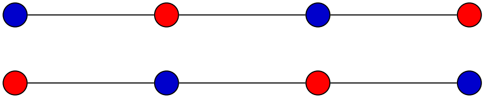

Recoloring via Kempe chains

Theorem (Mohar)

We can generate all the $(\Delta + 1)$-colorings using Kempe chains.

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Conjecture (Mohar)

We can generate all the Δ-colorings of any graph using Kempe chains.

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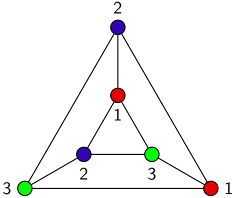
Theorem (Bonamy, B., Feghali, Johnson 2017+)

We can generate all the Δ-colorings of any graph except the 3-prism using Kempe chains.



Counter-example proposed by Jan van den Heuvel (2013)

Conclusion

Relation with other fields :

- Sampling. Connectivity ? Diameter ? Huge connectivity ?

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Thanks for your attention !