Graph Recoloring: Many questions (and few answers)

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Reconfiguration graph

Definition (k-Reconfiguration graph $\mathcal{C}_k(G)$ of G)

- Vertices : Proper k-colorings of G.
- Create an edge between any two k-colorings which differ on exactly one vertex.

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Definition :
The k-recoloring diameter is the diameter of $\mathcal{C}_k(G)$ (when connected).

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- If the answer is positive, how many steps do we need ?
  What is the diameter of the reconfiguration graph ?

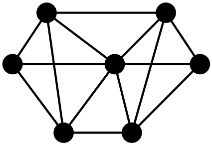
- Can we effiently find a short transformation ?
  Can we find a path between two vertices of the reconfiguration graph in polynomial time ? in FPT time ?

Cereceda's conjecture

Conjecture (Cereceda '08)

The $(d+2)$-recoloring diameter of any d-degenerate graph is $\mathcal{O}(n^2)$.

A graph is d-degenerate if there exists an ordering $v_1, \ldots, v_n$ such that for every i, $|N(v_i) \cap \{v_{i+1}, \ldots, v_n\}| \leq d$.



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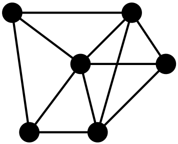

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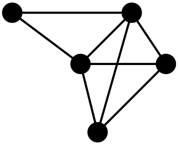

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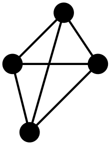

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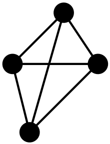

Theorem (Dyer et al. '06)

The $(d+2)$-recoloring diameter of any d-degenerate graph is at most 2^n.

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Theorem (B., Heinrich '19)

The $(d+2)$-recoloring diameter of any d-degenerate graph is $\mathcal{O}(n^{d+1})$.

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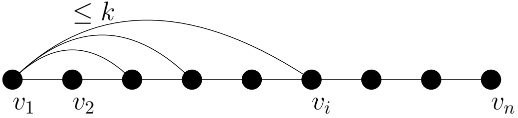

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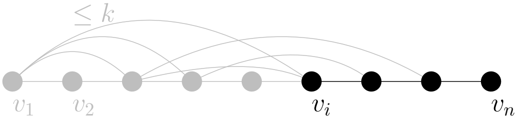

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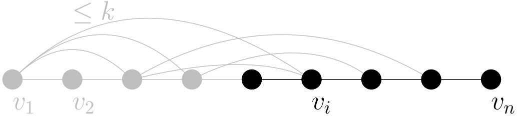

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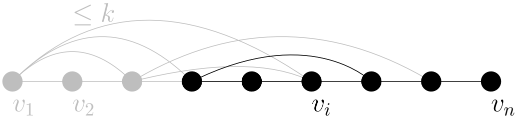

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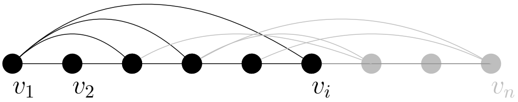

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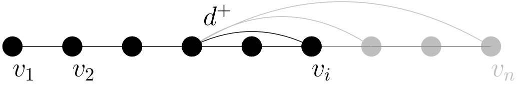

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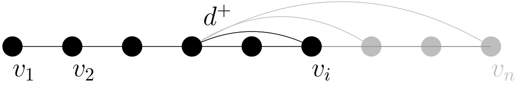

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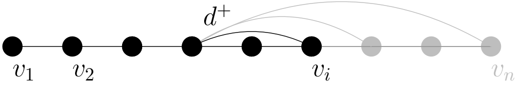

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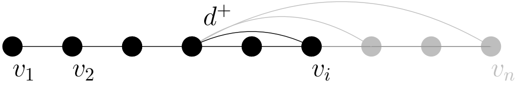

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Prove the Cerededa's conjecture for $d = 2$... and $\Delta = 4$!

[Feghali, Johnson, Paulusma '17] $d = 2$ and $\Delta = 3$ is true.

Graph classes

The conjecture has been verified for a few graph classes.

Known results :

- [Bonamy et al. '12] Chordal graphs.
- [Bousquet, Heinrich '19] Bipartite planar graphs.
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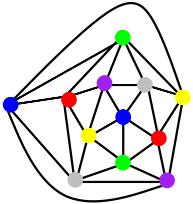
Planar graphs

- There are frozen 6-colorings.



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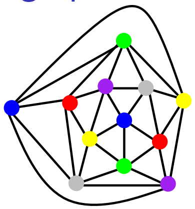

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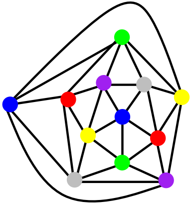

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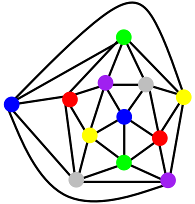

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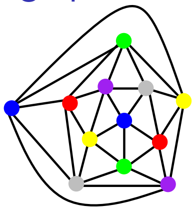

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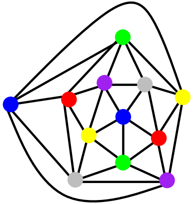

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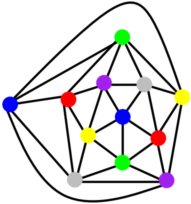

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- Graphs on surfaces ?
- Transformation between non-frozen 6-colorings ?
  [Feghali, Johnson, Paulusma] proved a similar result for Δ colorings.

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Open problems :

- Prove it for degenerate graphs. (Or disprove it !)
- Remove the dependency on Δ (and replace it by d).
- When can we remove any dependency on d ?

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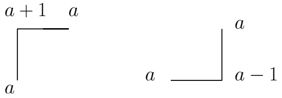

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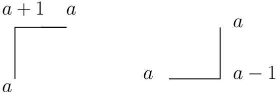

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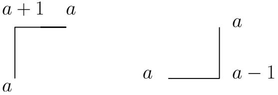

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Question : Find a (non trivial) lower bound for other graph classes? Or when $k \geq d + 2$?

Algorithmic aspects of reconfiguration

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Input : A graph G, an integer k, two k-colorings c_1, c_2.
Output : YES iff c_1 can be transformed into c_2.

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(PSPACE-complete for chordal graphs).

For line graphs :
[Osawa et al. '18] PSPACE-complete if $k \geq 5$. Open for $k = 4$.

Conclusion

More questions :

- A connected reconfiguration graph with exponential diameter.
- Understand better what "not connected" means.

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Thanks for your attention !