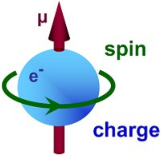
Graph Recoloring:
From statistical physics to graph theory

Nicolas Bousquet

JGA 2019
Bruxelles

Spin systems



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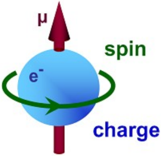

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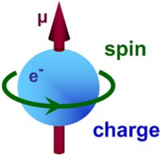

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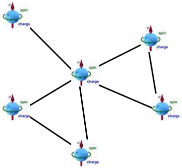

- An integer k being the number of states.
- An interaction $\{0, 1\}$ (symmetric) matrix modelizing the interaction between spins.
  - 0 = no interaction = no link.
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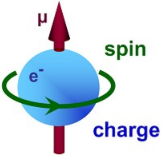

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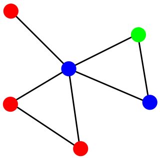

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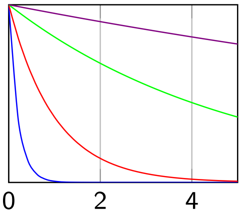
A spin configuration is a function $f : S \rightarrow \{1, \dots, k\}^n$.
$\Leftrightarrow$ A (non necessarily proper) graph coloring.



$T = 5, 1, 0.2, 0.05$

Antiferromagnetic Potts model

$H(\sigma)$: number of monochromatic edges.

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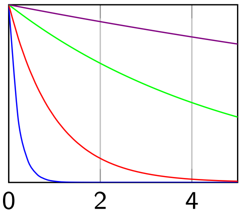
Edges with both endpoints of the same color.

Gibbs measure at fixed temperature T :

$$\nu_T(\sigma) = e^{-\frac{H(\sigma)}{T}}$$



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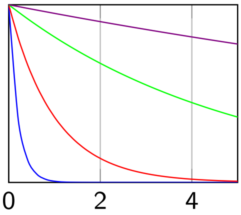

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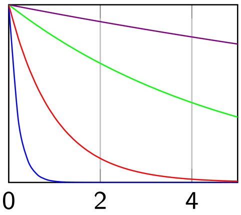

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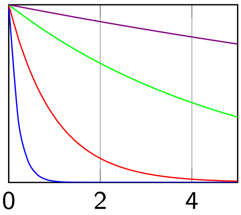

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Definition (Glauber dynamics)

Limit of a k-state Potts model when $T \to 0$.
$\Rightarrow$ Only **proper** colorings have positive measure.

Reconfiguration graph

Definition (k-Reconfiguration graph $\mathcal{C}_k(G)$ of G)

- Vertices : Proper k-colorings of G.
- Create an edge between any two k-colorings which differ on exactly one vertex.

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Definition :
The k-recoloring diameter is the diameter of $\mathcal{C}_k(G)$ (when connected).

Sampling spin configurations

In the statistical physics community, the following Monte Carlo Markov chain was proposed to sample a configuration :

- Start with an initial coloring c ;
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Remark :

The Markov chain is a random walk in the reconfiguration graph.

Convergence of Markov chains

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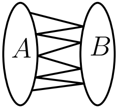

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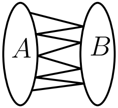

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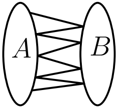

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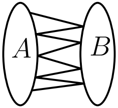

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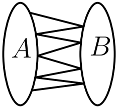

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Question :
How much time do we need to converge ?

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Mixing time $=$ number of steps needed to be “close” to the stationnary distribution.
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- Diameter of the Reconfiguration graph $= D$
  $\Rightarrow$ Mixing time $\geq 2 \cdot D$.
- Better lower bounds ? Look at the connectivity of the reconfiguration graph (e.g. bottleneck ratio).

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Conjecture

If $k \geq \Delta + 2$, the mixing time is $\mathcal{O}(n \log n)$.

Coupon collector problem

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- In each cereal box, there is a gift.
- There are n distinct gifts in total.
- Goal : get them all !

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Theorem

The expected number of steps needed to collect them all (if distribution iid) is $\Theta(n \log n)$.

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- [Hayes, Sinclair '05] There exist graphs for which the mixing time is $O(n)$.

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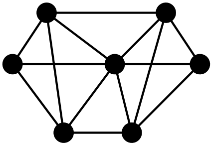
- If the answer is positive, how many steps do we need ?
  What is the diameter of the reconfiguration graph ?
- Can we effiently find a short transformation ?
  Can we find a path between two vertices of the reconfiguration graph in polynomial time ? in FPT time ?

Cereceda's conjecture

Conjecture (Cereceda '08)

The $(d+2)$-recoloring diameter of any d-degenerate graph is $\mathcal{O}(n^2)$.

A graph is d-degenerate if there exists an ordering $v_1, \ldots, v_n$ such that for every i, $|N(v_i) \cap \{v_{i+1}, \ldots, v_n\}| \leq d$.



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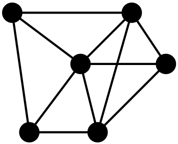

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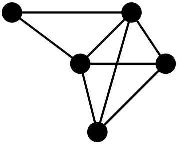

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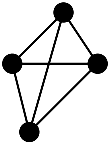

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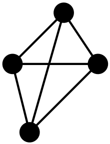

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- Delete a vertex of degree at most d.
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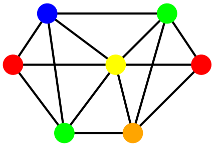

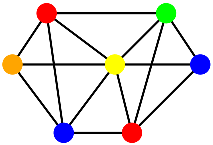

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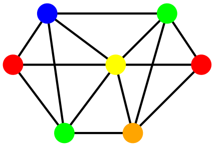

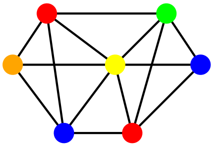

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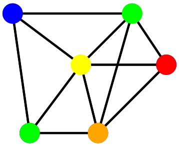
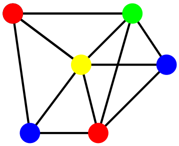

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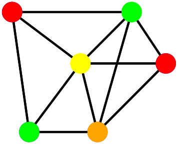

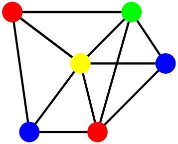

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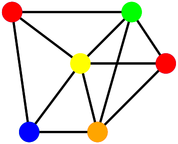

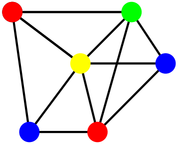

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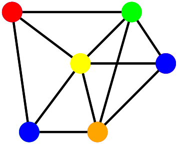

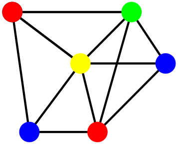

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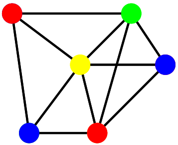

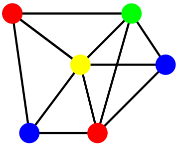

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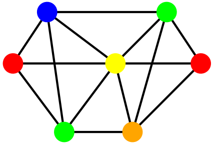

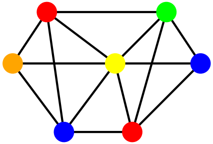

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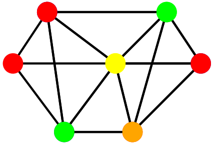

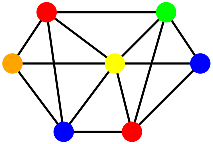

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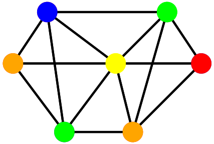

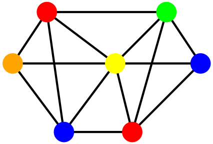

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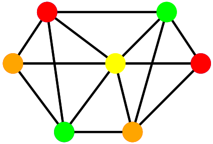

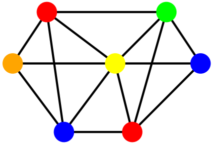

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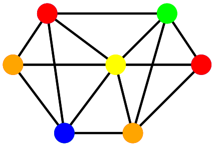

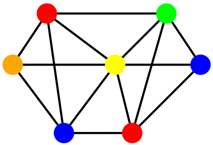

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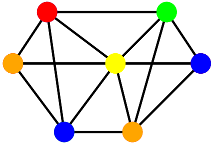

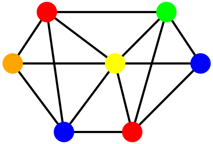

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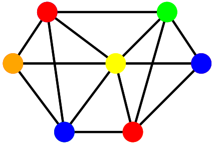

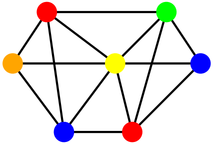

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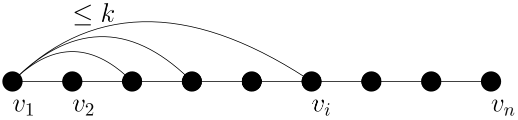

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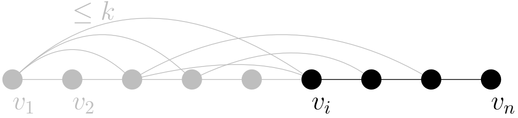

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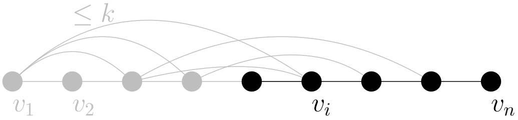

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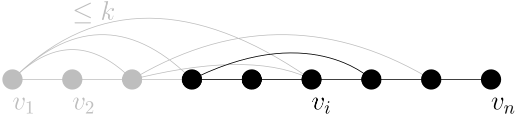

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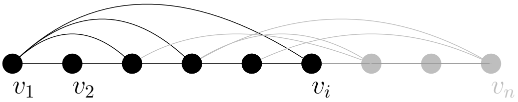

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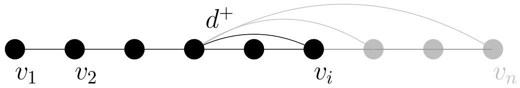

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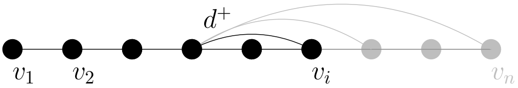

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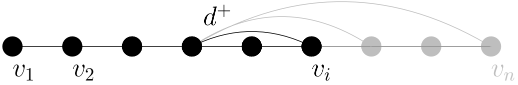

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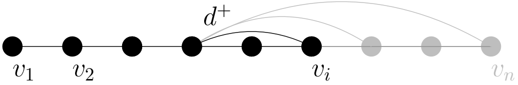

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Prove the Cerededa's conjecture for $d = 2$... and $\Delta = 4$!

[Feghali, Johnson, Paulusma '17] $d = 2$ and $\Delta = 3$ is true.

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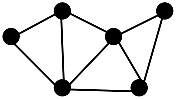
Definition :

G is chordal if any cycle of length ≥ 4 has a chord.

$\Leftrightarrow$ G can be prunned via simplicial vertices (vertices whose neighborhood is a clique).



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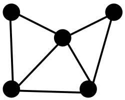

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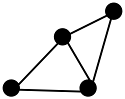

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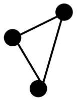

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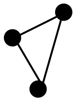

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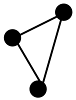

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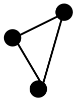

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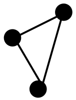

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- Chordal graphs with $\chi = 2$ are exactly trees
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- [Bonamy, B. '18] Can be extended to treewidth $\leq k$ with $k + 2$ colors.

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Proof based on some identification of vertices technique.

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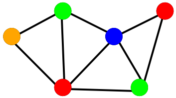

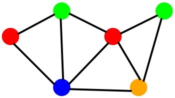

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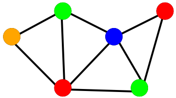

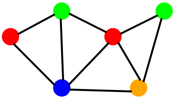

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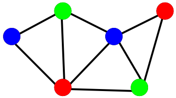

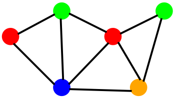

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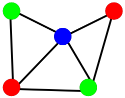

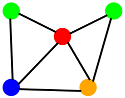

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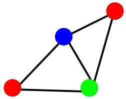

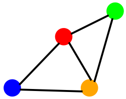

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- If $k \geq n+1$, we can obtain any coloring of K_n by recoloring every vertex ≤ 2 times.

Open problems

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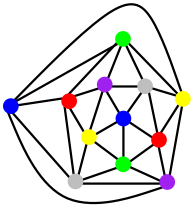

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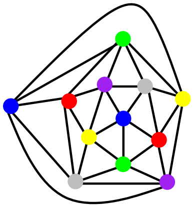

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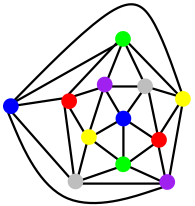

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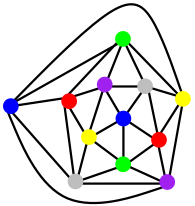

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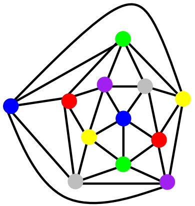

What happens if k increases ?

$\rightarrow$ Go to Valentin Bartier's talk.

Lower bound

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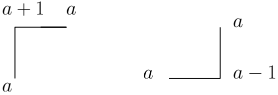

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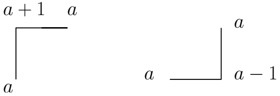

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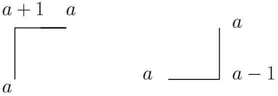

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Question : Find a (non trivial) lower bound for other graph classes ? Or when $k \geq d+2$?

Algorithmic aspects of reconfiguration

COLORING REACHABILITY (CR)
Input : A graph G, an integer k, two k-colorings c_1, c_2.
Output : YES iff c_1 can be transformed into c_2.

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For line graphs :
[Osawa et al. '18] PSPACE-complete if $k \geq 5$. Open for $k = 4$.

What does “close” mean ?

At the beginning of the talk (a long time ago...).

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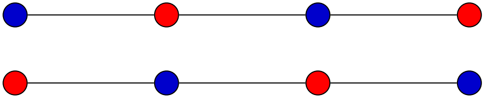
- Change the color of a Kempe chain !



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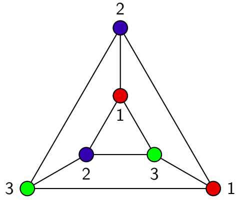

Counter-example proposed by Jan van den Heuvel (2013)

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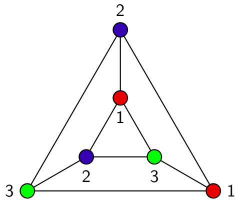
Theorem (Bonamy, B., Feghali, Johnson '19)

We can generate all the Δ-colorings of any graph but the 3-prism using Kempe chains.



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Thanks for your attention !